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Research Paper

ON PEXIDER TYPE OF HILBERT C*-MODULE HIGHER DERIVATIONS

SAYED KHALIL EKRAMI

ABSTRACT. In this paper, we introduce the concept of pexider Hilbert C*-module higher $\{A_n, B_n, D_n\}$-derivations. Specifically, we focus on a Hilbert C*-module $\mathcal{M}$ and provide a comprehensive characterization of these pexider Hilbert C*-module higher $\{A_n, B_n, D_n\}$-derivations $\{\Phi_n\}_{n=0}^{\infty}$ on $\mathcal{M}$ in relation to pexider Hilbert C*-module $\{\alpha_n, \beta_n, \delta_n\}$-derivations $\{\varphi_n\}_{n=1}^{\infty}$ on $\mathcal{M}$. We demonstrate that for every pexider Hilbert C*-module higher $\{A_n, B_n, D_n\}$-derivation $\{\Phi_n\}_{n=0}^{\infty}$ on $\mathcal{M}$, there exists a unique sequence of pexider Hilbert C*-module $\{\alpha_n, \beta_n, \delta_n\}$-derivations $\{\varphi_n\}_{n=1}^{\infty}$ on $\mathcal{M}$ such that

$$\begin{cases} \varphi_n = \sum_{k=1}^{n} \left(\sum_{\sum_{j=1}^{k} r_j = n} (-1)^{k-1} \; r_1 \Phi_{r_1} \Phi_{r_2} \dots \Phi_{r_k} \right), \\ \alpha_n = \sum_{k=1}^{n} \left(\sum_{\sum_{j=1}^{k} r_j = n} (-1)^{k-1} \; r_1 A_{r_1} A_{r_2} \dots A_{r_k} \right), \\ \beta_n = \sum_{k=1}^{n} \left(\sum_{\sum_{j=1}^{k} r_j = n} (-1)^{k-1} \; r_1 B_{r_1} B_{r_2} \dots B_{r_k} \right), \\ \delta_n = \sum_{k=1}^{n} \left(\sum_{\sum_{j=1}^{k} r_j = n} (-1)^{k-1} \; r_1 D_{r_1} D_{r_2} \dots D_{r_k} \right), \end{cases}$$

for all positive integers n, where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{k} r_j = n$.

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∗Corresponding author

1. Introduction

In this paper, we introduce the concept of pexider Hilbert C*-module higher $\{A_n, B_n, D_n\}$-derivation. A Hilbert C*-module is a generalization of a Hilbert space in which the inner product takes its values in a C*-algebra [11]. Let $\mathcal{A}$ be a C*-algebra. An inner product $\mathcal{A}$-module is a complex linear space $\mathcal{M}$ which is a left $\mathcal{A}$-module with compatible scalar multiplication $\lambda(ax) = (\lambda a)x = a(\lambda x)$ $(\lambda \in \mathbb{C}, x \in \mathcal{M}, a \in \mathcal{A})$, together with an $\mathcal{A}$-valued inner product $(x, y) \mapsto \langle x, y\rangle : \mathcal{M} \times \mathcal{M} \to \mathcal{A}$ such that for each $x, y, z \in \mathcal{M}$, $\alpha, \beta \in \mathbb{C}$ and $a \in \mathcal{A}$,

(i) $\langle x, x\rangle \geq 0$ and the equality holds if and only if $x = 0$,
(ii) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$,
(iii) $\langle ax, y\rangle = a\langle x, y\rangle$,
(iv) $\langle x, y\rangle^* = \langle y, x\rangle$.

The condition (i) asserts that $\langle x, x\rangle$ is a positive element in C*-algebra $\mathcal{A}$. The conditions (ii) and (iv) impliy that the inner product is conjugate-linear in its second variable. Also the conditions (iii) and (iv) impliy that $\langle x, ay\rangle = \langle x, y\rangle a^*$ for all $x, y \in \mathcal{M}$ and $a \in \mathcal{A}$. An inner product $\mathcal{A}$-module $\mathcal{M}$ which is complete with respect to the norm $\|x\|_{\mathcal{M}} = \|\langle x, x\rangle\|_{\mathcal{A}}^{\frac{1}{2}}$ is called a Hilbert $\mathcal{A}$-module or a Hilbert C*-module over the C*-algebra $\mathcal{A}$. For example, every C*-algebra $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module under the $\mathcal{A}$-valued inner product $\langle a, b\rangle = ab^*$ $(a, b \in \mathcal{A})$. Every complex Hilbert space is a left Hilbert $\mathbb{C}$-module. The notion of a right Hilbert $\mathcal{A}$-module can be defined similarly.

Let $\mathcal{M}$ and $\mathcal{N}$ be Hilbert C*-modules over a C*-algebra $\mathcal{A}$. A mapping $T : \mathcal{M} \to \mathcal{N}$ is said to be adjointable, if there exists a mapping $S : \mathcal{N} \to \mathcal{M}$ such that

$$\langle T(x), y\rangle = \langle x, S(y)\rangle,$$

for all $x \in D_T \subseteq \mathcal{M}$, $y \in D_S \subseteq \mathcal{N}$. The unique mapping S is denoted by T^* and is called the adjoint of T. It is well known that any adjointable mapping $T : \mathcal{M} \to \mathcal{N}$ is $\mathcal{A}$-linear (that is $T(ax + \lambda y) = aT(x) + \lambda T(y)$ for all $x, y \in \mathcal{M}, a \in \mathcal{A}, \lambda \in \mathbb{C}$) and bounded.

A linear mapping $\psi : \mathcal{M} \to \mathcal{M}$ is called a *Hilbert C*-module derivation* on $\mathcal{M}$, if it satisfies the equation

$$\psi(\langle x, y\rangle z) = \langle \psi(x), y\rangle z + \langle x, \psi(y)\rangle z + \langle x, y\rangle \psi(z),$$

for all $x, y, z \in \mathcal{M}$. Note that very adjointable mapping $\psi : \mathcal{M} \to \mathcal{M}$ satisfying $\psi^* = -\psi$ is a Hilbert C*-module derivation. Infact if $\psi^* = -\psi$, then $\langle \psi(x), y\rangle z + \langle x, \psi(y)\rangle z = 0$ for all

$x, y, z \in \mathcal{M}$. Moreover

$$\begin{aligned}\big\langle \psi(\langle x,y\rangle z), w\big\rangle &= \big\langle \langle x,y\rangle z, \psi^*(w)\big\rangle \\ &= \big\langle x,y\rangle\langle z, \psi^*(w)\big\rangle \\ &= \big\langle x,y\rangle\langle \psi(z), w\big\rangle \\ &= \big\langle \langle x,y\rangle \psi(z), w\big\rangle,\end{aligned}$$

for all $x, y, z, w \in \mathcal{M}$ which implies that $\psi(\langle x,y\rangle z) = \langle x,y\rangle \psi(z)$ for all $x, y, z. \in \mathcal{M}$.

Example 1.1. Let $M_2(\mathbb{C})$ be the C*-algebra of 2×2 complex matrices. The linear mapping $\psi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ defined by $\psi(X) = \begin{bmatrix} x_{21} & x_{22} \\ -x_{11} & -x_{12} \end{bmatrix}$, for all $X \in M_2(\mathbb{C})$, is a Hilbert C*-module derivation on $M_2(\mathbb{C})$.

$$\varphi_n(\langle x,y\rangle z) = \sum_{\substack{i+j+k=n \\ 0\le i,j,k\le n}} \langle \varphi_i(x), \varphi_j(y)\rangle \varphi_k(z),$$

for all $x, y, z \in \mathcal{M}$ and each non-negative integer n. Trivially when $\{\varphi_n\}_{n=0}^{\infty}$ is a Hilbert C*-module higher derivation, φ_1 is a Hilbert C*-module derivation. Ekrami [5] showed that for any Hilbert C*-module higher derivation $\{\varphi_n\}_{n=0}^{\infty}$ on a Hilbert C*-module $\mathcal{M}$, with $\varphi_0 = I$, there exists a unique sequence of Hilbert C*-module derivations $\{\psi_n\}_{n=1}^{\infty}$ on $\mathcal{M}$ such that for each non-negative integer n, φ_n is a linear combination of product of terms of the sequence $\{\psi_n\}_{n=1}^{\infty}$. For more informations about derivations, higher derivations, Hilbert C*-module derivations, Hilbert C*-module higher derivations and related topics, the reader refer to [1, 4, 5, 6, 7, 2, 3, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18].

Let $\mathcal{M}$ be a Hilbert C*-module over a C*-algebra $\mathcal{A}$ and I be the identity mapping on $\mathcal{M}$. We introduce the concept of pexider Hilbert C*-module $\{\alpha, \beta, \delta\}$-derivation as a linear mapping $\varphi : \mathcal{M} \to \mathcal{M}$ satisfying the equation

$$\varphi(\langle x,y\rangle z) = \langle \alpha(x), y\rangle z + \langle x, \beta(y)\rangle z + \langle x,y\rangle \delta(z),$$

for all $x, y, z \in \mathcal{M}$ and the concept of pexider Hilbert C*-module higher $\{A_n, B_n, D_n\}$-derivation as a sequence of linear mappings $\{\Phi_n : \mathcal{M} \to \mathcal{M}\}_{n=0}^{\infty}$ with $\Phi_0 = I$, satisfying the equation

$$\Phi_n(\langle x,y\rangle z) = \sum_{\substack{i+j+k=n \\ 0\le i,j,k\le n}} \langle A_i(x), B_j(y)\rangle D_k(z),$$

for all $x, y, z \in \mathcal{M}$ and each non-negative integer n, in which $\{A_n\}_{n=0}^{\infty}, \{B_n\}_{n=0}^{\infty}$ and $\{D_n\}_{n=0}^{\infty}$ are sequences of linear mappings on $\mathcal{M}$ with $A_0 = B_0 = D_0 = I$. Then we give a characterization of pexider Hilbert C*-module higher $\{A_n, B_n, D_n\}$-derivations $\{\Phi_n\}_{n=0}^{\infty}$ on $\mathcal{M}$ in terms of pexider Hilbert C*-module $\{\alpha_n, \beta_n, \delta_n\}$-derivations $\{\varphi_n\}_{n=1}^{\infty}$ on $\mathcal{M}$. We show that for every pexider Hilbert C*-module higher $\{A_n, B_n, D_n\}$-derivation $\{\Phi_n\}_{n=0}^{\infty}$ on $\mathcal{M}$, there exists a unique sequence of pexider Hilbert C*-module $\{\alpha_n, \beta_n, \delta_n\}$-derivations $\{\varphi_n\}_{n=1}^{\infty}$ on $\mathcal{M}$ such that

$$\varphi_n = \sum_{k=1}^{n} \Big(\sum_{\sum_{j=1}^{k} r_j = n} (-1)^{k-1}\ r_1 \Phi_{r_1} \Phi_{r_2} \ldots \Phi_{r_k} \Big),$$

for all positive integers n, where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{k} r_j = n$.

2. The Results

In this section, we define pexider Hilbert C*-module derivations, which are linear mappings characterized by their action on inner products. We also introduce higher-order pexider Hilbert C*-module derivations, represented by sequences of linear mappings, and establish their relationships through inductive constructions.

Definition 2.1. Let α, β, δ be linear mappings from $\mathcal{M}$ into $\mathcal{M}$. A linear mapping $\varphi : \mathcal{M} \to \mathcal{M}$ is called a *pexider Hilbert C*-module* $\{\alpha, \beta, \delta\}$*-derivation* on $\mathcal{M}$, if it satisfies the equation

$$\varphi(\langle x, y\rangle z) = \langle \alpha(x), y\rangle z + \langle x, \beta(y)\rangle z + \langle x, y\rangle \delta(z),$$

for all $x, y, z \in \mathcal{M}$.

Definition 2.2. A sequence of linear mappings $\{\Phi_n : \mathcal{M} \to \mathcal{M}\}_{n=0}^{\infty}$ with $\Phi_0 = I$, is called a *pexider Hilbert C*-module higher* $\{A_n, B_n, D_n\}$*-derivation* on $\mathcal{M}$, if there exist three sequences of linear mappings $\{A_n\}_{n=0}^{\infty}, \{B_n\}_{n=0}^{\infty}$ and $\{D_n\}_{n=0}^{\infty}$ on $\mathcal{M}$ with $A_0 = B_0 = D_0 = I$ satisfying the equation

$$\Phi_n(\langle x, y\rangle z) = \sum_{\substack{i+j+k=n \\ 0 \le i,j,k \le n}} \langle A_i(x), B_j(y)\rangle D_k(z),$$

for all $x, y, z \in \mathcal{M}$ and each non-negative integer n.

Example 2.3. Let φ be a pexider Hilbert C*-module $\{\alpha, \beta, \delta\}$-derivation on $\mathcal{M}$. Then by induction we have $\frac{\varphi^n}{n!}(\langle x, y\rangle z) = \sum_{i+j+k=n} \langle \frac{\alpha^i}{i!}(x), \frac{\beta^j}{j!}(y)\rangle \frac{\delta^k}{k!}(z)$ for all $x, y, z \in \mathcal{M}$ and each non-negative integer n. Thus if we define the sequences of linear mappings $\{\Phi_n\}_{n=0}^{\infty}, \{A_n\}_{n=0}^{\infty}, \{B_n\}_{n=0}^{\infty}$ and $\{D_n\}_{n=0}^{\infty}$ on $\mathcal{M}$ by $\Phi_n = \frac{\varphi^n}{n!}, A_n = \frac{\alpha^n}{n!}, B_n = \frac{\beta^n}{n!}$ and $D_n = \frac{\delta^n}{n!}$, then $\{\Phi_n\}_{n=0}^{\infty}$ is a pexider Hilbert C*-module higher $\{A_n, B_n, D_n\}$-derivation on $\mathcal{M}$.

Before proving the results of the paper, we would like to give the following discussion that makes clear the process of characterizing of a pexider Hilbert C*-module higher $\{A_n, B_n, D_n\}$-derivation by a sequence of pexider Hilbert C*-module $\{\alpha_n, \beta_n, \delta_n\}$-derivations. Let $\{\Phi_n : \mathcal{M} \to \mathcal{M}\}_{n=0}^{\infty}$ be a pexider Hilbert C*-module higher $\{A_n, B_n, D_n\}$-derivation on $\mathcal{M}$. So

$\Phi_n(\langle x,y\rangle z)=\sum_{\substack{i+j+k=n\\0\le i,j,k\le n}}\langle A_i(x),B_j(y)\rangle D_k(z)$ for all $x,y,z\in\mathcal{M}$ and each non-negative integer n. Since $\Phi_0=A_0=B_0=D_0=I$, it follows that $\Phi_1(\langle x,y\rangle z)=\langle A_1(x),y\rangle z+\langle x,B_1(y)\rangle z+\langle x,y\rangle D_1(z)$ for all $x,y,z\in\mathcal{M}$. Putting $\varphi_1=\Phi_1,\alpha_1=A_1,\beta_1=B_1$ and $\delta_1=D_1$, it follows that φ_1 is a pexider Hilbert C*-module $\{\alpha_1,\beta_1,\delta_1\}$-derivation on $\mathcal{M}$. Therefore we have

$$\begin{aligned}\Phi_1^2(\langle x,y\rangle z) &= \Phi_1\Big(\langle A_1(x),y\rangle z+\langle x,B_1(y)\rangle z+\langle x,y\rangle D_1(z)\Big)\\ &= \langle A_1^2(x),y\rangle z+\langle A_1(x),B_1(y)\rangle z+\langle A_1(x),y\rangle D_1(z)\\ &\quad+\langle A_1(x),B_1(y)\rangle z+\langle x,B_1^2(y)\rangle z+\langle x,B_1(y)\rangle D_1(z)\\ &\quad+\langle A_1(x),y\rangle D_1(z)+\langle x,B_1(y)\rangle D_1(z)+\langle x,y\rangle D_1^2(z),\end{aligned}$$

which imlies that

$$\begin{aligned}\Phi_1^2(\langle x,y\rangle z) &= \langle A_1^2(x),y\rangle z+\langle x,B_1^2(y)\rangle z+\langle x,y\rangle D_1^2(z)\\ &\quad+2\langle A_1(x),B_1(y)\rangle z+2\langle A_1(x),y\rangle D_1(z)+2\langle x,B_1(y)\rangle D_1(z),\end{aligned}$$

for all $x,y,z\in\mathcal{M}$. On the other hand

$$\begin{aligned}\Phi_2(\langle x,y\rangle z) &= \langle A_2(x),y\rangle z+\langle x,B_2(y)\rangle z+\langle x,y\rangle D_2(z)\\ &= \langle A_1(x),B_1(y)\rangle z+\langle A_1(x),y\rangle D_1(z)+\langle x,B_1(y)\rangle D_1(z),\end{aligned}$$

for all $x,y,z\in\mathcal{M}$. Using the last two equations, we get

$$(2\Phi_2-\Phi_1^2)(\langle x,y\rangle z)=\langle(2A_2-A_1^2)(x),y\rangle z+\langle x,(2B_2-B_1^2)(y)\rangle z+\langle x,y\rangle(2D_2-D_1^2)(z),$$

for all $x,y,z\in\mathcal{M}$. If we put $\varphi_2=2\Phi_2-\Phi_1^2,\alpha_2=2A_2-A_1^2,\beta_2=2B_2-B_1^2$ and $\delta_2=2D_2-D_1^2$, then we get $\varphi_2(\langle x,y\rangle z)=\langle\alpha_2(x),y\rangle z+\langle x,\beta_2(y)\rangle z+\langle x,y\rangle\delta_2(z)$ for all $x,y,z\in\mathcal{M}$ which means that φ_2 is a pexider Hilbert C*-module $\{\alpha_2,\beta_2,\delta_2\}$-derivation on $\mathcal{M}$. Similarly, we can show that, putting $\varphi_3=3\Phi_3-2\Phi_2\Phi_1-\Phi_1\Phi_2+\Phi_1^3,\alpha_3=3A_3-2A_2A_1-A_1A_2+A_1^3,\beta_3=3B_3-2B_2B_1-B_1B_2+B_1^3$ and $\delta_3=3D_3-2D_2D_1-D_1D_2+D_1^3$, it follows that φ_3 is a pexider Hilbert C*-module $\{\alpha_3,\beta_3,\delta_3\}$-derivation on $\mathcal{M}$. Thus we can inductively construct a sequence $\{\varphi_i\}_{i=1}^{\infty}$ of pexider Hilbert C*-module $\{\alpha_i,\beta_i,\delta_i\}$-derivations characterizing a pexider Hilbert C*-module higher $\{A_i,B_i,D_i\}$-derivation $\{\Phi_i\}_{i=0}^{\infty}$ with $\Phi_0=A_0=B_0=D_0=I$.

Theorem 2.4. *Let $\{\Phi_n\}_{n=0}^{\infty}$ be a pexider Hilbert C*-module higher $\{A_n,B_n,D_n\}$-derivation on $\mathcal{M}$ with $\varphi_0=I$. Then there exists a sequence $\{\varphi_n\}_{n=1}^{\infty}$ of pexider Hilbert C*-module $\{\alpha_n,\beta_n,\delta_n\}$-derivations on $\mathcal{M}$ such that*

$$(1)\qquad\begin{cases}(n+1)\Phi_{n+1}=\sum_{\ell=0}^{n}\varphi_{\ell+1}\Phi_{n-\ell},\\ (n+1)A_{n+1}=\sum_{\ell=0}^{n}\alpha_{\ell+1}A_{n-\ell},\\ (n+1)B_{n+1}=\sum_{\ell=0}^{n}\beta_{\ell+1}B_{n-\ell},\\ (n+1)D_{n+1}=\sum_{\ell=0}^{n}\delta_{\ell+1}D_{n-\ell},\end{cases}$$

for each non-negative integer n.

Proof. We use induction on n to prove the result. Since for $n=0$ we have

$$\begin{aligned}\Phi_1(\langle x,y\rangle z) &= \langle A_1(x),B_0(y)\rangle D_0(z)+\langle A_0(x),B_1(y)\rangle D_0(z)+\langle A_0(x),B_0(y)\rangle D_1(z)\\ &= \langle A_1(x),y\rangle z+\langle x,B_1(y)\rangle z+\langle x,y\rangle D_1(z),\end{aligned}$$

for all $x,y,z\in\mathcal{M}$. Thus if we put $\varphi_1=\Phi_1,\alpha_1=A_1,\beta_1=B_1$ and $\delta_1=D_1$, then φ_1 is a pexider Hilbert C*-module $\{\alpha_1,\beta_1,\delta_1\}$-derivation on $\mathcal{M}$. Moreover the equations (1) hold for $n=0$.

As induction assumption, suppose that for each $m=1,\dots,n$, the mapping φ_m defined in (1) is a pexider Hilbert C*-module $\{\alpha_m,\beta_m,\delta_m\}$-derivation on $\mathcal{M}$ and the equations (1) hold for all $m=0,1,\dots,n-1$. This means

$$\begin{cases}(m+1)\Phi_{m+1}=\sum_{\ell=0}^{m}\varphi_{\ell+1}\Phi_{m-\ell},\\ (m+1)A_{m+1}=\sum_{\ell=0}^{m}\alpha_{\ell+1}A_{m-\ell},\\ (m+1)B_{m+1}=\sum_{\ell=0}^{m}\beta_{\ell+1}B_{m-\ell},\\ (m+1)D_{m+1}=\sum_{\ell=0}^{m}\delta_{\ell+1}D_{m-\ell},\end{cases}$$

which implies that

$$\begin{cases}\varphi_{m+1}=(m+1)\Phi_{m+1}-\sum_{\ell=0}^{m-1}\varphi_{\ell+1}\Phi_{m-\ell},\\ \alpha_{m+1}=(m+1)A_{m+1}-\sum_{\ell=0}^{m-1}\alpha_{\ell+1}A_{m-\ell},\\ \beta_{m+1}=(m+1)B_{m+1}-\sum_{\ell=0}^{m-1}\beta_{\ell+1}B_{m-\ell},\\ \delta_{m+1}=(m+1)D_{m+1}-\sum_{\ell=0}^{m-1}\delta_{\ell+1}D_{m-\ell},\end{cases}$$

for all $m=0,1,\dots,n-1$. We show that $\varphi_{n+1}=(n+1)\Phi_{n+1}-\sum_{\ell=0}^{n-1}\varphi_{\ell+1}\Phi_{n-\ell}$ is a pexider Hilbert C*-module $\{\alpha_{n+1},\beta_{n+1},\delta_{n+1}\}$-derivation on $\mathcal{M}$. For all $x,y,z\in\mathcal{M}$ we have

$$\begin{aligned}\varphi_{n+1}(\langle x,y\rangle z) &= (n+1)\Phi_{n+1}(\langle x,y\rangle z)-\sum_{\ell=0}^{n-1}\varphi_{\ell+1}\Phi_{n-\ell}(\langle x,y\rangle z)\\ &= (n+1)\sum_{\substack{i+j+k=n+1\\0\le i,j,k\le n+1}}\langle A_i(x),B_j(y)\rangle D_k(z)\\ &\quad -\sum_{\ell=0}^{n-1}\varphi_{\ell+1}\Big(\sum_{\substack{p+q+r=n-\ell\\0\le p,q,r\le n-\ell}}\langle A_p(x),B_q(y)\rangle D_r(z)\Big)\\ &= \sum_{\substack{i+j+k=n+1\\0\le i,j,k\le n+1}}(n+1)\langle A_i(x),B_j(y)\rangle D_k(z)\\ &\quad -\sum_{\ell=0}^{n-1}\varphi_{\ell+1}\Big(\sum_{\substack{p+q+r=n-\ell\\0\le p,q,r\le n-\ell}}\langle A_p(x),B_q(y)\rangle D_r(z)\Big).\end{aligned}$$

Since $i+j+k=n+1$ and for each $k=1,\ldots,n$, φ_k is a pexider Hilbert C*-module $\{\alpha_k,\beta_k,\delta_k\}$-derivation, we have

$$\begin{aligned}
&\varphi_{n+1}(\langle x,y\rangle z)\\
&=\sum_{\substack{i+j+k\\=n+1}}(i+j+k)\langle A_i(x),B_j(y)\rangle D_k(z)-\sum_{\ell=0}^{n-1}\sum_{\substack{p+q+r\\=n-\ell}}\varphi_{\ell+1}\Big(\langle A_p(x),B_q(y)\rangle D_r(z)\Big)\\
&=\sum_{\substack{i+j+k\\=n+1}}\langle iA_i(x),B_j(y)\rangle D_k(z)+\sum_{\substack{i+j+k\\=n+1}}\langle A_i(x),jB_j(y)\rangle D_k(z)+\sum_{\substack{i+j+k\\=n+1}}\langle A_i(x),B_j(y)\rangle kD_k(z)\\
&\quad-\sum_{\ell=0}^{n-1}\sum_{\substack{p+q+r\\=n-\ell}}\Big(\langle\alpha_{\ell+1}A_p(x),B_q(y)\rangle D_r(z)+\langle A_p(x),\beta_{\ell+1}B_q(y)\rangle D_r(z)\\
&\quad+\langle A_p(x),B_q(y)\rangle\delta_{\ell+1}D_r(z)\Big).
\end{aligned}$$

If we put

$$\begin{aligned}
K_1&=\sum_{\substack{i+j+k\\=n+1}}\langle iA_i(x),B_j(y)\rangle D_k(z)-\sum_{\ell=0}^{n-1}\sum_{\substack{p+q+r\\=n-\ell}}\langle\alpha_{\ell+1}A_p(x),B_q(y)\rangle D_r(z),\\
K_2&=\sum_{\substack{i+j+k\\=n+1}}\langle A_i(x),jB_j(y)\rangle D_k(z)-\sum_{\ell=0}^{n-1}\sum_{\substack{p+q+r\\=n-\ell}}\langle A_p(x),\beta_{\ell+1}B_q(y)\rangle D_r(z),\\
K_3&=\sum_{\substack{i+j+k\\=n+1}}\langle A_i(x),B_j(y)\rangle kD_k(z)-\sum_{\ell=0}^{n-1}\sum_{\substack{p+q+r\\=n-\ell}}\langle A_p(x),B_q(y)\rangle\delta_{\ell+1}D_r(z),
\end{aligned}$$

then $\varphi_{n+1}(\langle x,y\rangle z)=K_1+K_2+K_3$. Now we compute K_1, K_2 and K_3. We have

$$K_1=\sum_{\substack{i+j+k=n+1\\0\le i,j,k\le n+1}}\langle iA_i(x),B_j(y)\rangle D_k(z)-\sum_{\ell=0}^{n-1}\sum_{\substack{i+j+k=n-\ell\\0\le i,j,k\le n-\ell}}\langle\alpha_{\ell+1}A_i(x),B_j(y)\rangle D_k(z).$$

In the second summation, we have $0\le\ell\le n-1$,$0\le i,j,k\le n-\ell$, $i+j+k+\ell=n$ and $\ell\neq n$. Thus if we put $i+\ell=r$, then $r+j+k=n$ and so

$$\begin{aligned}
K_1&=\sum_{i+j+k=n+1}\langle iA_i(x),B_j(y)\rangle D_k(z)-\sum_{r+j+k=n}\sum_{\substack{\ell=0,\\ \ell\neq n}}^{r}\langle\alpha_{\ell+1}A_{r-\ell}(x),B_j(y)\rangle D_k(z)\\
&=\sum_{i+j+k=n}\langle(i+1)A_{i+1}(x),B_j(y)\rangle D_k(z)-\sum_{i+j+k=n}\sum_{\substack{\ell=0,\\ \ell\neq n}}^{i}\langle\alpha_{\ell+1}A_{i-\ell}(x),B_j(y)\rangle D_k(z)
\end{aligned}$$

$$= \sum_{\substack{i+j+k=n,\\ i\neq n}} \langle (i+1)A_{i+1}(x), B_j(y)\rangle D_k(z) - \sum_{\substack{i+j+k=n,\\ i\neq n}} \sum_{\substack{\ell=0,\\ \ell\neq n}}^{i} \langle \alpha_{\ell+1}A_{i-\ell}(x), B_j(y)\rangle D_k(z)$$

$$+\langle (n+1)A_{n+1}(x), y\rangle z - \sum_{\ell=0}^{n-1} \langle \alpha_{\ell+1}A_{n-\ell}(x), y\rangle z$$

$$= \sum_{\substack{i+j+k=n,\\ i\neq n}} \Big\langle \Big((i+1)A_{i+1} - \sum_{\substack{\ell=0,\\ \ell\neq n}}^{i} \alpha_{\ell+1}A_{i-\ell}\Big)(x), B_j(y)\Big\rangle D_k(z)$$

$$+\Big\langle \Big((n+1)A_{n+1} - \sum_{\ell=0}^{n-1} \alpha_{\ell+1}A_{n-\ell}\Big)(x), y\Big\rangle z.$$

Since for all $i = 0, 1, \dots, n-1$, $(i+1)A_{i+1} = \sum_{\ell=0}^{i} \alpha_{\ell+1}A_{i-\ell}$, we have

$$K_1 = \Big\langle \Big((n+1)A_{n+1} - \sum_{\ell=0}^{n-1} \alpha_{\ell+1}A_{n-\ell}\Big)(x), y\Big\rangle z = \langle \alpha_{n+1}(x), y\rangle z.$$

Similarly we can show that $K_2 = \langle x, \beta_{n+1}(y)\rangle z$ and $K_3 = \langle x, y\rangle \delta_{n+1}(z)$. Therefore

$$\varphi_{n+1}(\langle x, y\rangle z) = K_1 + K_2 + K_3 = \langle \alpha_{n+1}(x), y\rangle z + \langle x, \beta_{n+1}(y)\rangle z + \langle x, y\rangle \delta_{n+1}(z),$$

for all $x, y, z \in \mathcal{M}$. That is, φ_{n+1} is a pexider Hilbert C*-module $\{\alpha_{n+1}, \beta_{n+1}, \delta_{n+1}\}$-derivation on $\mathcal{M}$. This completes the proof. □

Theorem 2.5. *Let $\{\Phi_n\}_{n=0}^{\infty}$ be a pexider Hilbert C*-module higher $\{A_n, B_n, D_n\}$-derivation on $\mathcal{M}$ with $\Phi_0 = A_0 = B_0 = D_0 = I$. Then there exists a sequence $\{\varphi_n\}_{n=1}^{\infty}$ of pexider Hilbert C*-module $\{\alpha_n, \beta_n, \delta_n\}$-derivations on $\mathcal{M}$ such that*

$$(2)\qquad \begin{cases} \varphi_n = \sum_{k=1}^{n} \Big(\sum_{\sum_{j=1}^{k} r_j = n} (-1)^{k-1}\ r_1 \Phi_{r_1}\Phi_{r_2}\dots\Phi_{r_k}\Big), \\ \alpha_n = \sum_{k=1}^{n} \Big(\sum_{\sum_{j=1}^{k} r_j = n} (-1)^{k-1}\ r_1 A_{r_1}A_{r_2}\dots A_{r_k}\Big), \\ \beta_n = \sum_{k=1}^{n} \Big(\sum_{\sum_{j=1}^{k} r_j = n} (-1)^{k-1}\ r_1 B_{r_1}B_{r_2}\dots B_{r_k}\Big), \\ \delta_n = \sum_{k=1}^{n} \Big(\sum_{\sum_{j=1}^{k} r_j = n} (-1)^{k-1}\ r_1 D_{r_1}D_{r_2}\dots D_{r_k}\Big), \end{cases}$$

where the inner summation is taken over all positive integers r_j with $\sum_{j=1}^{k} r_j = n$.

Proof. By Theorem 2.4, for pexider Hilbert C*-module higher $\{A_n, B_n, D_n\}$-derivation $\{\Phi_n\}_{n=0}^{\infty}$, there exists a sequence of pexider Hilbert C*-module $\{\alpha_n, \beta_n, \delta_n\}$-derivations $\{\varphi_n\}_{n=1}^{\infty}$ such that the equations (1) hold for each non-negative integer n. So we get

$$(3)\qquad \begin{cases} \varphi_{n+1} = (n+1)\Phi_{n+1} - \sum_{k=0}^{n-1} \varphi_{k+1}\Phi_{n-k}, \\ \alpha_{n+1} = (n+1)A_{n+1} - \sum_{k=0}^{n-1} \alpha_{k+1}A_{n-k}, \\ \beta_{n+1} = (n+1)B_{n+1} - \sum_{k=0}^{n-1} \beta_{k+1}B_{n-k}, \\ \delta_{n+1} = (n+1)D_{n+1} - \sum_{k=0}^{n-1} \delta_{k+1}D_{n-k}, \end{cases}$$

for all $n \in \mathbb{N}$. Now we use induction on n. For $n = 1$ we have $\varphi_1 = \Phi_1$. Suppose that φ_k is defined for $k \leq n$ as equation (2). For $k = n+1$ we have

$$\begin{aligned}
\sum_{k=1}^{n+1} & \Big(\sum_{\sum_{j=1}^{k} r_j = n+1} (-1)^{k-1}\ r_1 \Phi_{r_1} \Phi_{r_2} \dots \Phi_{r_k} \Big) \\
&= (n+1)\Phi_{n+1} + \sum_{k=2}^{n+1} \Big(\sum_{\sum_{j=1}^{k} r_j = n+1} (-1)^{k-1}\ r_1 \Phi_{r_1} \Phi_{r_2} \dots \Phi_{r_k} \Big) \\
&= (n+1)\Phi_{n+1} - \sum_{k=1}^{n} \Big(\sum_{\sum_{j=1}^{k} r_j = n+1-r_{k+1}} (-1)^{k-1}\ r_1 \Phi_{r_1} \Phi_{r_2} \dots \Phi_{r_k} \sum_{r_{k+1}=1}^{n-(k-1)} \Phi_{r_{k+1}} \Big) \\
&= (n+1)\Phi_{n+1} - \sum_{k=1}^{n} \sum_{i=1}^{n-(k-1)} \Big(\sum_{\sum_{j=1}^{k} r_j = n+1-i} (-1)^{k-1}\ r_1 \Phi_{r_1} \Phi_{r_2} \dots \Phi_{r_k} \Big) \Phi_i \\
&= (n+1)\Phi_{n+1} - \sum_{i=1}^{n} \sum_{k=1}^{n-(i-1)} \Big(\sum_{\sum_{j=1}^{k} r_j = n-(i-1)} (-1)^{k-1}\ r_1 \Phi_{r_1} \Phi_{r_2} \dots \Phi_{r_k} \Big) \Phi_i \\
&= (n+1)\Phi_{n+1} - \sum_{i=1}^{n} \varphi_{n-(i-1)} \Phi_i = (n+1)\Phi_{n+1} - \sum_{i=0}^{n-1} \varphi_{n-i} \Phi_{i+1} \\
&= (n+1)\Phi_{n+1} - \sum_{k=0}^{n-1} \varphi_{k+1} \Phi_{n-k} = \varphi_{n+1}.
\end{aligned}$$

Similarly we can prove the other equations of (2). □

Example 2.6. Using Theorem 2.5, the five terms of $\{\varphi_n\}$ are

$$\begin{aligned}
\varphi_1 &= \Phi_1, \\
\varphi_2 &= 2\Phi_2 - \Phi_1^2, \\
\varphi_3 &= 3\Phi_3 - 2\Phi_2\Phi_1 - \Phi_1\Phi_2 + \Phi_1^3, \\
\varphi_4 &= 4\Phi_4 - 3\Phi_3\Phi_1 - 2\Phi_2^2 - \Phi_1\Phi_3 + 2\Phi_2\Phi_1^2 + \Phi_1\Phi_2\Phi_1 + \Phi_1^2\Phi_2 - \Phi_1^4, \\
\varphi_5 &= 5\Phi_5 - 4\Phi_4\Phi_1 - 3\Phi_3\Phi_2 - 2\Phi_2\Phi_3 - \Phi_1\Phi_4 + 3\Phi_3\Phi_1^2 + \Phi_1\Phi_3\Phi_1 + \Phi_1^2\Phi_3 \\
&\quad + 2\Phi_2^2\Phi_1 + 2\Phi_2\Phi_1\Phi_2 + \Phi_1\Phi_2^2 - 2\Phi_2\Phi_1^3 - \Phi_1\Phi_2\Phi_1^2 - \Phi_1^2\Phi_2\Phi_1 - \Phi_1^3\Phi_2 + \Phi_1^5.
\end{aligned}$$

Theorem 2.7. *Let $\{\varphi_n\}_{n=1}^{\infty}$ be a sequence of pexider Hilbert C^*-module $\{\alpha_n, \beta_n, \delta_n\}$-derivations on $\mathcal{M}$. Then there is a pexider Hilbert C^*-module higher $\{A_n, B_n, D_n\}$-derivation*

$\{\Phi_n\}_{n=0}^{\infty}$ with $\Phi_0 = A_0 = B_0 = D_0 = I$ on $\mathcal{M}$ such that

$$(4)\qquad \begin{cases} \Phi_n = \sum_{k=1}^{n} \Big(\sum_{\sum_{j=1}^{k} r_j = n} \big(\prod_{j=1}^{k} \frac{1}{r_j+\ldots+r_k} \big) \ \varphi_{r_1} \ldots \varphi_{r_k} \Big), \\ A_n = \sum_{k=1}^{n} \Big(\sum_{\sum_{j=1}^{k} r_j = n} \big(\prod_{j=1}^{k} \frac{1}{r_j+\ldots+r_k} \big) \ \alpha_{r_1} \ldots \alpha_{r_k} \Big), \\ B_n = \sum_{k=1}^{n} \Big(\sum_{\sum_{j=1}^{k} r_j = n} \big(\prod_{j=1}^{k} \frac{1}{r_j+\ldots+r_k} \big) \ \beta_{r_1} \ldots \beta_{r_k} \Big), \\ D_n = \sum_{k=1}^{n} \Big(\sum_{\sum_{j=1}^{k} r_j = n} \big(\prod_{j=1}^{k} \frac{1}{r_j+\ldots+r_k} \big) \ \delta_{r_1} \ldots \delta_{r_k} \Big), \end{cases}$$

for each non-negative integer n, where the inner summations is taken over all positive integers r_j with $\sum_{j=1}^{k} r_j = n$.

Proof. First, we show that if Φ_n, A_n, B_n, D_n are defined by (4) for all $n \in \mathbb{N}$ and $\Phi_0 = A_0 = B_0 = D_0 = I$, then $\{\Phi_n\}_{n=0}^{\infty}, \{A_n\}_{n=0}^{\infty}, \{B_n\}_{n=0}^{\infty}$ and $\{D_n\}_{n=0}^{\infty}$ satisfy the recursive relations (1) of Theorem 2.4.

Note that if $r_1 + r_2 + \cdots + r_k = n+1$, then $(n+1) \prod_{j=1}^{k} \frac{1}{r_j+\ldots+r_k} = \prod_{j=2}^{k} \frac{1}{r_j+\ldots+r_k}$. Now we have

$$\begin{aligned} (n+1)\Phi_{n+1} &= \sum_{k=2}^{n+1} \Big(\sum_{\sum_{j=1}^{k} r_j = n+1} (n+1) \prod_{j=1}^{k} \frac{1}{r_j + \ldots + r_k} \varphi_{r_1} \varphi_{r_2} \ldots \varphi_{r_k} \Big) + \varphi_{n+1} \\ &= \sum_{k=2}^{n+1} \Big(\sum_{r_1=1}^{n-(k-2)} \varphi_{r_1} \sum_{\sum_{j=2}^{k} r_j = n+1-r_1} \prod_{j=2}^{k} \frac{1}{r_j + \ldots + r_k} \ \varphi_{r_2} \ldots \varphi_{r_k} \Big) + \varphi_{n+1} \\ &= \sum_{k=2}^{n+1} \Big(\sum_{i=1}^{n-(k-2)} \varphi_{i} \sum_{\sum_{j=2}^{k} r_j = n+1-i} \prod_{j=2}^{k} \frac{1}{r_j + \ldots + r_k} \ \varphi_{r_2} \ldots \varphi_{r_k} \Big) + \varphi_{n+1} \\ &= \sum_{k=1}^{n} \Big(\sum_{i=1}^{n-(k-1)} \varphi_{i} \sum_{\sum_{j=2}^{k+1} r_j = n+1-i} \prod_{j=2}^{k+1} \frac{1}{r_j + \ldots + r_{k+1}} \ \varphi_{r_2} \ldots \varphi_{r_{k+1}} \Big) + \varphi_{n+1} \\ &= \sum_{i=1}^{n} \varphi_i \sum_{k=1}^{n-(i-1)} \Big(\sum_{\sum_{j=1}^{k} r_j = n-(i-1)} \prod_{j=1}^{k} \frac{1}{r_j + \ldots + r_k} \ \varphi_{r_1} \ldots \varphi_{r_k} \Big) + \varphi_{n+1} \\ &= \sum_{i=1}^{n} \varphi_i \Phi_{n-(i-1)} + \varphi_{n+1} \\ &= \sum_{i=0}^{n-1} \varphi_{i+1} \Phi_{n-i} + \varphi_{n+1} \\ &= \sum_{\ell=0}^{n} \varphi_{\ell+1} \Phi_{n-\ell}, \end{aligned}$$

for each non-negative integer n. Similarly we can prove the other equations of (4).

Now, we use induction on n, to show that $\{\Phi_n\}_{n=0}^{\infty}$ is a pexider Hilbert C*-module higher $\{A_n, B_n, D_n\}$-derivation. For $n = 0$, we have $\varphi_0 = I$. Suppose that $\Phi_r(\langle x, y \rangle z) =$

$\sum_{i+j+k=r}\langle A_i(x), B_j(y)\rangle D_k(z)$ for all $r \leq n$. Then for $n+1$ we have

$$\begin{aligned}
(n+1)\Phi_{n+1}(\langle x,y\rangle z) &= \sum_{\ell=0}^{n}\varphi_{\ell+1}\Phi_{n-\ell}(\langle x,y\rangle z)\\
&= \sum_{\ell=0}^{n}\varphi_{\ell+1}\sum_{i+j+k=n-\ell}(\langle A_i(x), B_j(y)\rangle D_k(z))\\
&= \sum_{\ell=0}^{n}\sum_{i+j+k=n-\ell}\Big(\langle \alpha_{\ell+1}A_i(x), B_j(y)\rangle D_k(z)\\
&\quad +\langle A_i(x), \beta_{\ell+1}B_j(y)\rangle D_k(z) + \langle A_i(x), B_j(y)\rangle \delta_{\ell+1}D_k(z)\Big)\\
&= \sum_{\ell=0}^{n}\sum_{i+j+k=n-\ell}\Big(\langle \alpha_{\ell+1}A_i(x), B_j(y)\rangle D_k(z)\Big)\\
&\quad +\sum_{\ell=0}^{n}\sum_{i+j+k=n-\ell}\Big(\langle A_i(x), \beta_{\ell+1}B_j(y)\rangle D_k(z)\Big)\\
&\quad +\sum_{\ell=0}^{n}\sum_{i+j+k=n-\ell}\Big(\langle A_i(x), B_j(y)\rangle \delta_{\ell+1}D_k(z)\Big),
\end{aligned}$$

for all $x, y, z \in \mathcal{M}$. If we put $i+\ell = r$ in the first summation, $j+\ell = r$ in the second summation and $k+\ell = r$ in the third summation, then we get

$$\begin{aligned}
&(n+1)\Phi_{n+1}(\langle x,y\rangle z)\\
&= \sum_{r+j+k=n}\sum_{\ell=0}^{r}\Big(\langle \alpha_{\ell+1}A_{r-\ell}(x), B_j(y)\rangle D_k(z)\Big) + \sum_{i+r+k=n}\sum_{\ell=0}^{r}\Big(\langle A_i(x), \beta_{\ell+1}B_{r-\ell}(y)\rangle D_k(z)\Big)\\
&\quad + \sum_{i+j+r=n}\sum_{\ell=0}^{r}\Big(\langle A_i(x), B_j(y)\rangle \delta_{\ell+1}D_{r-\ell}(z)\Big)\\
&= \sum_{i+j+k=n}\sum_{\ell=0}^{i}\Big(\langle \alpha_{\ell+1}A_{i-\ell}(x), B_j(y)\rangle D_k(z)\Big) + \sum_{i+j+k=n}\sum_{\ell=0}^{j}\Big(\langle A_i(x), \beta_{\ell+1}B_{j-\ell}(y)\rangle D_k(z)\Big)\\
&\quad + \sum_{i+j+k=n}\sum_{\ell=0}^{k}\Big(\langle A_i(x), B_j(y)\rangle \delta_{\ell+1}D_{k-\ell}(z)\Big)\\
&= \sum_{i+j+k=n}\Big(\langle (i+1)A_{i+1}(x), B_j(y)\rangle D_k(z) + \langle A_i(x), (j+1)B_{j+1}(y)\rangle D_k(z)\\
&\quad +\langle A_i(x), B_j(y)\rangle (k+1)D_{k+1}(z)\Big)\\
&= \sum_{i+j+k=n+1}\Big(\langle iA_i(x), B_j(y)\rangle D_k(z) + \langle A_i(x), jB_j(y)\rangle D_k(z) + \langle A_i(x), B_j(y)\rangle kD_k(z)\Big)\\
&= \sum_{i+j+k=n+1}(i+j+k)\langle A_i(x), B_j(y)\rangle D_k(z) = (n+1)\sum_{i+j+k=n+1}\langle A_i(x), B_j(y)\rangle D_k(z),
\end{aligned}$$

for all $x, y, z \in \mathcal{M}$. Thus $\{\Phi_n\}_{n=0}^{\infty}$ is a pexider Hilbert C*-module higher $\{A_n, B_n, D_n\}$-derivation. This completes the proof. □

Example 2.8. Using Theorem 2.7, the five terms of $\{\Phi_n\}_{n=0}^{\infty}$ are

$$\Phi_0 = I,$$

$$\Phi_1 = \varphi_1$$

$$\Phi_2 = \frac{1}{2}\varphi_1\varphi_1 + \frac{1}{2}\varphi_2,$$

$$\Phi_3 = \frac{1}{6}\varphi_1\varphi_1\varphi_1 + \frac{1}{6}\varphi_1\varphi_2 + \frac{1}{3}\varphi_2\varphi_1 + \frac{1}{3}\varphi_3,$$

$$\Phi_4 = \frac{1}{24}\varphi_1\varphi_1\varphi_1\varphi_1 + \frac{1}{24}\varphi_1\varphi_1\varphi_2 + \frac{1}{12}\varphi_1\varphi_2\varphi_1 + \frac{1}{12}\varphi_1\varphi_3 + \frac{1}{8}\varphi_2\varphi_1\varphi_1 + \frac{1}{8}\varphi_2\varphi_2 + \frac{1}{4}\varphi_3\varphi_1 + \frac{1}{4}\varphi_4.$$

Corollary 2.9. *If for each pexider Hilbert C*-module higher* $\{A_n, B_n, D_n\}$ *derivation* $\{\Phi_n\}_{n=0}^{\infty}$ *on* $\mathcal{M}$*, we define* $\varphi_n, \alpha_n, \beta_n, \delta_n : \mathcal{M} \to \mathcal{M}$ *by equations (2), then by Theorem 2.5,* $\{\varphi_n\}_{n=1}^{\infty}$ *is a sequence of pexider Hilbert C*-module* $\{\alpha_n, \beta_n, \delta_n\}$*-derivations on* $\mathcal{M}$*. Conversely, if for any sequence of pexider Hilbert C*-module* $\{\alpha_n, \beta_n, \delta_n\}$*-derivations* $\{\varphi_n\}_{n=1}^{\infty}$*, we define* $\Phi_n, A_n, B_n, D_n : \mathcal{M} \to \mathcal{M}$ *by* $\Phi_0 = A_0 = B_0 = D_0 = I$ *and equations (4), then by Theorm 2.7,* $\{\Phi_n\}_{n=0}^{\infty}$ *is a pexider Hilbert C*-module higher* $\{A_n, B_n, D_n\}$*-derivation on* $\mathcal{M}$*. This shows that there is a one to one correspondence between the set of all pexider Hilbert C*-module higher* $\{A_n, B_n, D_n\}$*-derivations on* $\mathcal{M}$ *and the set of all sequences of pexider Hilbert C*-module* $\{\alpha_n, \beta_n, \delta_n\}$*-derivations on* $\mathcal{M}$*.*

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Sayed Khalil Ekrami

Department of Mathematics,

Payame Noor University,

P.O. Box 19395-3697,

Tehran, Iran.

ekrami@pnu.ac.ir, khalil.ekrami@gmail.com