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Research Paper

ON α-ALMOST ARTINIAN TYPE MODULES

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Abstract. In this article we introduce and study the concepts of α-almost Artinian type and α- Krull type modules. Using these concepts we extend some of the basic results of α-almost Artinian and α-Krull modules to α-almost Artinian type and α-Krull type modules. We observe that if M is an α-Krull type module then the uncountably generated Krull dimension of M is either α or $\alpha + 1$.

1. Introduction

The concept of Noetherian dimension of a module M, (the dual of Krull dimension of M, in the sense of Rentschler and Gabriel, see [19, 28]) introduced in Lemonnier [29], and Karamzadeh [22], is almost as old as Krull dimension of M, and their existence are equivalent. Later, Chambless [4] studied dual Krull dimension and called it $N-$dimension. Roberts [30] calls this dual dimension again Krull dimension. The latter dimension is also called dual Krull

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dimension in some other articles, see for example, [1, 2]. The former dimension has received some attention; see [1, 2, 23, 25, 26, 18]. In this article, all rings are associative with $1 \neq 0$, and all modules are unital right modules. If M is an R-module, then n-$\dim M$ and k-$\dim M$ will denote the Noetherian dimension and the Krull dimension of M, respectively.

Davoudian, Karamzadeh and Shirali in [16] introduce and study the concepts of α-short modules and α-almost Noetherian modules. We recall that an R-module M is called an α-short module, if for each submodule N of M, either n-$\dim N \leq \alpha$ or n-$\dim \frac{M}{N} \leq \alpha$ and α is the least ordinal number with this property. We also recall that an R-module M is called α-almost Noetherian, if for each proper submodule N of M, n-$\dim N < \alpha$ and α is the least ordinal number with this property, see [16]. Later Davoudian, Halali and Shirali undertook a systematic study of the concepts of α-almost Artinian and α-Krull modules, which are the dual of the concepts of α-almost Noetherian and α-short modules, respectivly, see [14]. We introduce and extensively investigate uncountably generated Krull dimension and uncountably generated Noetherian dimension of an R-module M, see [11]. The uncountably generated Noetherian dimension (resp., uncountably generated Krull dimension), which is denoted by ucn-$\dim M$ (resp., uck-$\dim M$) is defined to be the codeviation (resp., deviation) of the poset of all uncountably generated submodules of M. We recall that an R-module M is called α-critical type, where α is an ordinal, if uck-$\dim M = \alpha$ and uck-$\dim \frac{M}{N} < \alpha$ for any uncountably generated submodule N of M. M is said to be critica type if it is α-critical type for some α. We also extensively investigate the concepts of α-almost Noetherian type and α-short type modules, see [12]. Recall that an R-module M is called α-almost Noetherian type if for each uncountably generated submodule N of M, ucn-$\dim N < \alpha$ and α is the least ordinal with this property. We also recall that an R-module M is called α-short type if for each uncountably generated submodule N of M, either ucn-$\dim N < \alpha$ or ucn-$\dim \frac{M}{N} < \alpha$ and α is the least ordinal number with this property. It is convenient, when we are dealing with the latter dimensions, to begin our list of ordinales with -1. In this article we introduce and study the concepts of α-almost Artinian type and α-Krull type modules. These concepts are the dual of the concepts of α-almost Noetherian type and α-short type modules, respectively; and at the same time are the extension of the concepts of α-almost Artinian and α-Krull modules, respectively. Let us give a brief outline of this paper. Section one is the introduction. In section 2, we introduce and study the concept of α-almost Artinian type and α-Krull type modules. Hein [21] introduced almost Artinian modules and studied some of the properties of these modules. Later Davoudian, Halali and Shirali undertook a systematic study of the concept of α-almost Artinian modules. We recall that an R-module M is called α-almost Artinian, if for each non-zero submodule N of M, k-$\dim \frac{M}{N} < \alpha$ and α is the least ordinal number with this property, see [16]. We shall call an R-module M to be α-almost Artinian type if for each uncountably generated submodule N of M, $uck\text{-}\dim \frac{M}{N} < \alpha$ and α is the least ordinal number with this property. Using this concept we extend some of the basic results of α-almost Artinian modules to α-almost Artinian type modules. In particular, we observe that each α-almost Artinian type module M has uncountably generated Krull dimension and $uck\text{-}\dim M \leq \alpha$. We also introduce and study the concept of α-Krull type modules, which is the dual of α- short type modules, see [7]. We recall that an R-module M is called an α-Krull module, if for each submodule N of M, either $k\text{-}\dim N \leq \alpha$ or $k\text{-}\dim \frac{M}{N} \leq \alpha$ and α is the least ordinal number with this property. We shall call an R-module M to be α-Krull type if for each uncountably generated submodule N of M, either $uck\text{-}\dim N \leq \alpha$ or $uck\text{-}\dim \frac{M}{N} \leq \alpha$ and α is the least ordinal number with this property. In the last section we also investigate some properties of α-almost Artinian type and α-Krull type modules. Finally, we should emphasize here that the results in sections 2 and 3 are new and are the dual of the corresponding results in [12] and at the same time are the extensions of the results in [14]. For all concepts and basic properties of rings and modules which are not defined in this paper, we refer the reader to [3, 8, 5, 6, 13, 15, 9, 10, 19, 11].

2. α-almost Artinian type and α-Krull type modules

In this section we introduce and study α-almost Artinian type and α-Krull type modules. We extend some of the basic results of α-almost Artinian modules to α-almost Artinian type modules.

Let us recall that the deviation of an arbitrary partially ordered set $E = (E, \leq)$, (shortly poset), denoted by $dev(E)$ is defined as follows: $dev(E) = -1$ if and only if E is a trivial poset, i.e., E has no two distinct comparable elements. If E is nontrivial but satisfies the descending chain condition on its elements, then $dev(E) = 0$. For a general ordinal α, we define $dev(E) = \alpha$, provided

(i) $dev(E) \neq \beta < \alpha$;

(ii) for any descending chain $x_1 \geq x_2 \geq ... \geq x_n \geq ...$ of elements of E there is some $n_0 \in N$ such that for all $n \geq n_0$ the deviation of the poset

$$\frac{x_n}{x_{n+1}} := \{x \in E | x_{n+1} \leq x \leq x_n\},$$

already defined and satisfies

$$dev(\frac{x_n}{x_{n+1}}) < \alpha.$$

If no ordinal α exists such that $dev(E) = \alpha$, we say E does not have deviation. For any R-module M we shall denote by UC(M) the poset of all uncountably generated submodules of M. The quasi-Krull dimension of the right R-module M, denoted by $uck\text{-}\dim M$, is defined to be the deviation of the poset $(\mathrm{UC(M)}, \subseteq)$, see [11, Definition 2.1].

We continue with our definition of α-almost Artinian type modules.

Definition 2.1. An R-module M is called α-almost Artinian type, if for each uncountably generated submodule N of M, $uck\text{-}\dim \frac{M}{N} < \alpha$ and α is the least ordinal number with this property.

We should remind the reader that the above concept is in fact the dual of α-almost Noetherian type modules, see [12, Definition 2.1].

Remark 2.2. If M is an α-almost Artinian type module, then each submodule and each factor module of M is β-almost Artinian type for some $\beta \leq \alpha$.

We recall that an R-module M is called α-critical type, if $uck\text{-}\dim M = \alpha$ and for each uncountably generated submodule N of M we have $uck\text{-}\dim \frac{M}{N} < \alpha$. M is called critical type if it is α-critical type for some ordinal number α, see [11, Definition 2.1]. The next three trivial, but useful facts, which are the dual of the corresponding facts in [12, Lemmas 1, 2, 3] are needed.

Lemma 2.3. *If M is an α-almost Artinian type module, then M has uncountably generated Krull dimension and $uck\text{-}\dim M \leq \alpha$. In particular, $uck\text{-}\dim M = \alpha$ if and only if M is α-critical type.*

Proof. For each proper uncountably generated submodule N of M, we have $uck\text{-}\dim \frac{M}{N} < \alpha$. In view of [11, Lemma 4], we get $uck\text{-}\dim M \leq \alpha$. The final part is now evident. □

Lemma 2.4. *If M is a module with $uck\text{-}\dim M = \alpha$, then either M is α-critical type, in which case it is α-almost Artinian type, or it is $\alpha + 1$-almost Artinian type.*

Proof. Let M be an α-critical type module, then for each uncountably generated submodule N of M, we have $uck\text{-}\dim \frac{M}{N} < \alpha$. Hence M is β-almost Artinian type, for some ordinal number $\beta \leq \alpha$. If $\beta < \alpha$, then by Lemma 2.3 we have $uck\text{-}\dim M \leq \beta$ which is a contradiction. If M is not critical type, then there exists a uncountably generated submodule N of M such that $uck\text{-}\dim \frac{M}{N} = \alpha$. This implies that M is γ-almost Artinian type for some $\gamma \geq \alpha + 1$. But for each uncountably generated submodule N of M, we have $uck\text{-}\dim \frac{M}{N} \leq \alpha < \alpha + 1$, see [11, Theorem 1]. Therefore M is $\alpha + 1$-almost Artinian type. □

Lemma 2.5. *If M is an α-almost Artinian type module, then either M is α-critical type or $\alpha = uck\text{-}\dim M + 1$. In particular, if M is an α-almost Artinian type module, where α is a limit ordinal, then M is α-critical type.*

Proof. We infer that M has uncountably generated Krull dimension and uck-$\dim M \leq \alpha$, by lemma 2.3. If uck-$\dim M = \alpha$, then in view of Lemma 2.3, M is α-critical type. Now let uck-$\dim M < \alpha$, then by Lemma 2.4, we get $\alpha = uck$-$\dim M + 1$ and we are done. The final part is now evident. □

The following results are now immediate.

Corollary 2.6. *Let M be a $\beta + 1$-almost Artinian type module, then either uck-$\dim M = \beta$ or uck-$\dim M = \beta + 1$.*

Proposition 2.7. *An R-module M has uncountably generated Krull dimension if and only if M is α-almost Artinian type for some ordinal α.*

We continue with the following definition, which is in fact the dual of α-short type modules, see [12, Definition 3], and in the subsequent results we try to present counterparts of the appropriate results in [12].

Definition 2.8. An R-Module M is called α-Krull type, if for each uncountably generated submodule N of M, either uck-$\dim N \leq \alpha$ or uck-$\dim \frac{M}{N} \leq \alpha$, and α is the least ordinal number with this property.

Now, we cite the following example.

Example 2.9. If $M_1 = M_2 = Z_{p^\infty}$, then M_1 and M_2 are -1-Krull type (resp. 0-almost Artinian type) Z-modules such that $M_1 \oplus M_2$ is 0-Krull type (resp. 1-almost Artinian type). Now let $M_1 = M_2 = Z$. In this case the Z-module Z is -1-Krull type (resp. -1-almost Artinian type), the Z-module $Z \oplus Z$ is also -1-Krull type (resp. -1-almost quasi Artinian). We should also note that $Z_{p^\infty} \oplus Z$ is a 0-Krull type Z-module which is 1-almost quasi Artinian.

Remark 2.10. If M is an R-module with uck-$\dim M = \alpha$, then M is β-Krull type for some $\beta \leq \alpha$.

In view of [11, Lemma 2.3 and Theorem 2.4], we have the following result.

Remark 2.11. If M is an α-Krull type module, then each submodule and each factor module of M is β-Krull type for some $\beta \leq \alpha$.

We need the following result.

Lemma 2.12. *If M is an R-module and for each uncountably generated submodule N of M, either N or $\frac{M}{N}$ has uncountably generated Krull dimension, then so does M.*

Proof. Let $M_1 \supseteq M_2 \supseteq \ldots$ be any descending chain of uncountably generated submodules of M. If there exists some i such that M_i has uncountably generated Krull dimension, then each $\frac{M_k}{M_{k+1}}$ has Krull type dimension for each $k \geq i$, see [11, Lemma 2.3]. Otherwise $\frac{M}{M_i}$ has Krull type dimension for each i. Thus in either case there exists some integer k such that each $\frac{M_i}{M_{i+1}}$ has uncountably generated Krull dimension for each $i \geq k$, see [11, Lemma 2.3]. Consequently M has uncountably generated Krull dimension. □

The previous result and Remark 2.10, immediately yield the next result.

Corollary 2.13. *Let M be an α-Krull type module. Then M has uncountably generated Krull dimension and $uck\text{-}\dim M \geq \alpha$.*

Proposition 2.14. *An R-module M has uncountably generated Krull dimension if and only if M is α-Krull type for some ordinal α.*

Proposition 2.15. *If M is an α-Krull type R-module, then either $uck\text{-}\dim M = \alpha$ or $uck\text{-}\dim M = \alpha + 1$.*

Proof. Clearly in view of Remark 2.10 and Corollary 2.13, we have $uck\text{-}\dim M \geq \alpha$. If $uck\text{-}\dim M \neq \alpha$, then $uck\text{-}\dim M \geq \alpha + 1$. Now let $M_1 \supseteq M_2 \supseteq \ldots$ be any descending chain of uncountably generated submodules of M. If there exists some k such that $uck\text{-}\dim M_k \leq \alpha$, then $uck\text{-}\dim \frac{M_i}{M_{i+1}} \leq uck\text{-}\dim M_i \leq uck\text{-}\dim M_k \leq \alpha$ for each $i \geq k$, [11, Lemma 2.3]. Otherwise $uck\text{-}\dim \frac{M}{M_i} \leq \alpha$ (note, M is α-Krull type) for each i, hence $uck\text{-}\dim \frac{M_i}{M_{i+1}} \leq \alpha$ for each i. Thus in any case there exists an integer k such that for each $i \geq k$, $uck\text{-}\dim \frac{M_i}{M_{i+1}} \leq \alpha$. This shows that $uck\text{-}\dim M \leq \alpha + 1$, i.e., $uck\text{-}\dim M = \alpha + 1$. □

Remark 2.16. An R-module M is -1-Krull type if and only if it is Noetherian or 1-atomic, (note, an R-module M is called α-atomic, if $n\text{-}\dim M = \alpha$ and $n\text{-}\dim N < \alpha$ for each proper submodules N of M).

Proposition 2.17. *Let M be an R-module, with $uck\text{-}\dim M = \alpha$, where α is a limit ordinal. Then M is α-Krull type.*

Proof. We know that M is β-Krull type for some $\beta \leq \alpha$. If $\beta < \alpha$, then by Proposition 2.15, $uck\text{-}\dim M \leq \beta + 1 < \alpha$, which is a contradiction. Thus M is α-Krull type. □

Proposition 2.18. *Let M be an R-module and $uck\text{-}\dim M = \alpha = \beta + 1$. Then M is either α-Krull type or it is β-Krull type.*

Proof. We know that M is γ-Krull type for some $\gamma \leq \alpha$. If $\gamma < \beta$ then by Proposition 2.15, we have $uck\text{-}\dim M \leq \gamma + 1 < \beta + 1$, which is impossible. Hence we are done. □

For the critical type modules we have the following proposition.

Proposition 2.19. *Let M be an $\beta + 1$-critical type R-module, where $\alpha = \beta + 1$. Then M is a β-Krull type module.*

Proof. Let N be a uncountably generated submodule of M, then $uck\text{-}\dim \frac{M}{N} < \alpha$. Thus $uck\text{-}\dim \frac{M}{N} \leq \beta$. This shows that for some $\beta' \leq \beta$, M is β'-Krull type. If $\beta' < \beta$, then $\beta' + 1 \leq \beta < \alpha$. But $uck\text{-}\dim M \leq \beta' + 1 \leq \beta < \alpha$, by Proposition 2.15, which is a contradiction. Thus $\beta' = \beta$ and we are done. □

The following remark, which is a trivial consequence of the previous fact, shows that the converse of Proposition 2.17, is not true in general.

Remark 2.20. Let M be an $\alpha + 1$-critical type R-module, where α is a limit ordinal. Then M is an α-Krull type module.

In view of Proposition 2.15 and Lemma 2.4, the following remark is now evident.

Remark 2.21. If M is a β-Krull type R-module, then it is an α-almost Artinian type module such that $\beta \leq \alpha \leq \beta + 2$, see Proposition 2.15 and Lemma 2.4. We note that every 1-critical type module is 0-Krull type which is also 1-almost Artinian type and every α-critical type module, where α is a limit ordinal, is an α-Krull type module which is also α-almost Artinian type, see Lemma 2.5 and Proposition 2.17.

Proposition 2.22. *Let M be an R-module such that $uck\text{-}\dim M = \alpha + 1$. Then M is either an α-Krull type R-module or there exists a uncountably generated submodule N of M such that $uck\text{-}\dim \frac{M}{N} = uck\text{-}\dim N = \alpha + 1$.*

Proof. We know that M is α-Krull type or an $\alpha + 1$-Krull type R-module, by Proposition 2.18. Let us assume that M is not an α-Krull type R-module, hence there exists a uncountably generated submodule N of M such that $uck\text{-}\dim N \geq \alpha + 1$ and $uck\text{-}\dim \frac{M}{N} \geq \alpha + 1$. This shows that $uck\text{-}\dim N = \alpha + 1$ and $uck\text{-}\dim \frac{M}{N} = \alpha + 1$ and we are through. □

Proposition 2.23. *Let M be an α-Krull type R-module. Then either M is β-almost Artinian type for some ordinal $\beta \leq \alpha + 1$ or there exists a uncountably generated submodule N of M with $uck\text{-}\dim N \leq \alpha$.*

Proof. Suppose that M is not β-almost Artinian type for any $\beta \leq \alpha+1$. This means that there must exist a uncountably generated submodule N of M such that $uck\text{-}\dim \frac{M}{N} \nleq \alpha$. Inasmuch as M is α-Krull type, we infer that $uck\text{-}\dim N \leq \alpha$ and we are done. □

3. Properties of α-Krull type modules and α-almost Artinian type modules

In this section some properties of α-Krull type and α-almost Artinian type modules over an arbitrary ring R are investigated.

First, in view of Proposition 2.15, we have the following two results.

Proposition 3.1. *Let R be a ring and M be an α-Krull type module, which is not a critical type module, then M contains a uncountably generated submodule L such that $uck\text{-}\dim L \leq \alpha$.*

Proof. Since M is not critical type, we infer that there exists a proper uncountably generated submodule $L \subset M$, such that $uck\text{-}\dim \frac{M}{L} = uck\text{-}\dim M$. We know that $uck\text{-}\dim M = \alpha$ or $uck\text{-}\dim M = \alpha + 1$, by Proposition 2.15. If $uck\text{-}\dim M = \alpha$ it is clear that $uck\text{-}\dim L \leq \alpha$. Hence we may suppose that $uck\text{-}\dim \frac{M}{L} = uck\text{-}\dim M = \alpha + 1$. Consequently, $uck\text{-}\dim L \leq \alpha$ and we are done. □

Theorem 3.2. *Let M be an R-module and α be an ordinal number. Let for any uncountably generated submodule N of M, $\frac{M}{N}$ be γ-Krull type for some ordinal number $\gamma \leq \alpha$. Then $uck\text{-}\dim M \leq \alpha + 2$. In particular M is μ-Krull type for some ordinal number $\mu \leq \alpha + 1$.*

Proof. Let $N \subset M$ be a uncountably generated submodule of M. Since $\frac{M}{N}$ is γ-Krull type for some ordinal number $\gamma \leq \alpha$, we infer that $uck\text{-}\dim \frac{M}{N} \leq \gamma + 1 \leq \alpha + 1$, by Proposition 2.15. This immediately implies that $uck\text{-}\dim M \leq \alpha + 2$, see [11, Lemma 2.7]. Now the last part of theorem is immediate. □

The next result is the dual of Theorem 3.2.

Theorem 3.3. *Let α be an ordinal number and M be an R-module such that every proper uncountably generated submodule of M is γ-Krull type for some ordinal number $\gamma \leq \alpha$. Then $uck\text{-}\dim M \leq \alpha + 1$. In particular M is μ-Krull type for some $\mu \leq \alpha + 1$.*

Proof. Let $N \subset M$ be any proper uncountably generated submodule of M, such that N is γ-Krull type for some ordinal number γ with $\gamma \leq \alpha$. We infer that $uck\text{-}\dim N \leq \gamma + 1 \leq \alpha + 1$, by Proposition 2.15. But we know that $uck\text{-}\dim M = \sup\{uck\text{-}\dim N : N \subset M, N \in \mathrm{UC(M)}\}$, see [11, Lemma 2.6]. This shows that $uck\text{-}\dim M \leq \alpha + 1$. Now the last part of theorem is immediate. □

The next immediate result is the counterparts of Theorems 3.2, 3.3, for α-almost Artinian type modules.

Proposition 3.4. *Let M be an R-module and α be an ordinal number. If each proper uncountably generated submodule N of M (resp. for each proper uncountably generated submodule N of M, $\frac{M}{N}$) is γ- almost Artinian type with $\gamma \leq \alpha$, then M is a μ-almost Artinian type module with $\mu \leq \alpha + 1$, $uck\text{-}\dim M \leq \alpha$ (resp. with $\mu \leq \alpha + 1$, $uck\text{-}\dim M \leq \alpha + 1$).*

Clearly every α-almost Artinian type (resp. α-Krull type) module has uncountably generated Krull dimension (i.e., it has uncountably generated Noetherian dimension too, for by a nice result due to Lemonnier, every module has uncountably generated Noetherian dimension if and only if it has uncountably generated Krull dimension, see the comment which follows [11, Theorem 3.11]). Consequently, we have the following immediate result.

Proposition 3.5. *The following statements are equivalent for a ring R.*

(a) *Every R-module with uncountably generated Krull dimension is Noetherian.*
(b) *Every α-Krull type R-module is Noetherian for all α.*
(c) *Every α-almost Artinian type R-module is Noetherian for all α.*

Moreover, if R is a right perfect ring (i.e., every R-module is a Loewy module) then every α-Krull type (resp. α-almost Artinian type) R-module is both Artinian and Noetherian, see [26, Proposition 2.1].

Before concluding this section with our last observation, let us cite the next result which is in [26, Theorem 2.9], see also [20, Theorem 3.2].

Theorem 3.6. *For a commutative ring R the following statements are equivalent.*

(a) *Every R-module with finite Noetherian dimension is Noetherian.*
(b) *Every Artinian R-module is Noetherian.*
(c) *Every R-module with Noetherian dimension is both Artinian and Noetherian.*

Now in view of the above theorem, [16, Proposition 2.21], [14, Proposition 4.18], [7, Proposition 2.24], [12, Proposition 10], and also [27, Corollary 2.15], we observe the following result.

Proposition 3.7. *The following statements are equivalent for a commutative ring R.*

(a) *Every Artinian R-module is Noetherian.*
(b) *Every quotient finite dimensional m-Krull module is both Artinian and Noetherian for all integers $m \geq -1$.*
(c) *Every quotient finite dimensional α-Krull module is both Artinian and Noetherian for all ordinals α.*

(d) *Every quotient finite dimensional m-Krull type module is both Artinian and Noetherian for all integers $m \geq -1$.*

(e) *Every quotient finite dimensional α-Krull type module is both Artinian and Noetherian for all ordinals α.*

(f) *Every quotient finite dimensional m-almost Artinian R-module is both Artinian and Noetherian for all non-negative integers m.*

(g) *Every quotient finite dimensional α-almost Artinian R-module is both Artinian and Noetherian for all ordinals α.*

(h) *Every quotient finite dimensional m-almost Artinian type R-module is both Artinian and Noetherian for all non-negative integers m.*

(i) *Every quotient finite dimensional α-almost Artinian type R-module is both Artinian and Noetherian for all ordinals α.*

(j) *Every quotient finite dimensional m-quasi short module is both Artinian and Noetherian for all integers $m \geq -1$.*

(k) *Every quotient finite dimensional α-quasi short module is both Artinian and Noetherian for all ordinals α.*

(l) *Every quotient finite dimensional m-almost Noetherian type R-module is both Artinian and Noetherian for all non-negative integers m.*

(m) *Every quotient finite dimensional α-almost Noetherian type R-module is both Artinian and Noetherian for all ordinals α.*

(n) *Every quotient finite dimensional m-short module is both Artinian and Noetherian for all integers $m \geq -1$.*

(o) *Every quotient finite dimensional α-short module is both Artinian and Noetherian for all ordinals α.*

(p) *Every quotient finite dimensional m-almost Noetherian R-module is both Artinian and Noetherian for all non-negative integers m.*

(q) *Every quotient finite dimensional α-almost Noetherian R-module is both Artinian and Noetherian for all ordinals α.*

(r) *No homomorphic image of R can be isomorphic to a dense subring of a complete local domain of uncountably generated Krull dimension* 1.

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