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Research Paper

AN APPROXIMATE NOTION IN THE HOMOLOGY OF THE ENVELOPING DUAL BANACH ALGEBRAS

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Abstract. In this paper, we introduce a notion of approximate WAP-biprojectivity for the enveloping dual Banach algebras. We also find some relations between approximate Connes amenability, approximate biprojectivity, left φ-contractibility and approximate WAP-biprojectivity. Moreover, we propose a criterion to show that the enveloping dual Banach algebras associated to triangular Banach algebras are not approximately WAP-biprojective. Finally, we present some examples of the enveloping dual Banach algebras associated to $l^1(S)$ and $\ell^1(\mathbb{N})$ (equipped with a new multiplication) and also study their approximate WAP-biprojectivity, where S is a unital weakly cancellative semigroup.

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1. Introduction and Preliminaries

Biprojectivity for Banach algebras was proved to be an important and fertile notion in the homological theory [7]. In fact, a Banach algebra $\mathcal{A}$ is biprojective if there exists a bounded $\mathcal{A}$-bimodule morphism $\rho : \mathcal{A} \to \mathcal{A}\widehat{\otimes}\mathcal{A}$ such that $\pi_{\mathcal{A}} \circ \rho(a) = a$, where $\pi_{\mathcal{A}}$ is the multiplication operator given by $\pi_{\mathcal{A}}(a \otimes b) = ab$ for every $a, b \in \mathcal{A}$. Indeed, the measure algebra $M(G)$ on a locally compact group G is biprojective if and only if G is finite. The approximate homological notion is similar to the approximate biprojectivity of Banach algebras has been introduced by Zhang [18]. In fact, a Banach algebra $\mathcal{A}$ is called approximately biprojective if there exists a net (ρ_α) of continuous $\mathcal{A}$-bimodule morphisms from $\mathcal{A}$ into $\mathcal{A}\widehat{\otimes}\mathcal{A}$ such that $\pi_{\mathcal{A}} \circ \rho_\alpha(a) \to a$ for every $a \in \mathcal{A}$.

Here, we recall some definitions, notations and terminologies. Let $\mathcal{A}$ be a Banach algebra. An $\mathcal{A}$-bimodule E is called *dual* if there is a closed submodule E_* of E^* such that $E = (E_*)^*$. A Banach algebra $\mathcal{A}$ is called dual if it is dual as a Banach $\mathcal{A}$-bimodule. Examples of dual Banach algebras include all von Neumann algebras, the algebra $\mathfrak{B}(E) = (E\widehat{\otimes}E^*)^*$ of all bounded operators on a reflexive Banach space E where $\widehat{\otimes}$ stands for the projective tensor product. Also the measure algebra $M(G) = C_0(G)^*$, the Fourier-Stieltjes algebra $B(G) = C^*(G)^*$, and the second dual B^{**} of an Arens regular Banach algebra B.

The groundwork of dual Banach algebras was laid by Runde [13]. It is obvious that every Banach algebra is not always dual Banach algebra but recently Choi et al. showed that there exists a dual Banach algebra associated to an arbitrary Banach algebra which is called the enveloping dual Banach algebra [1].

An element $x \in E$ is called *weakly almost periodic* if the module maps $\mathcal{A} \to E$; $a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ are weakly compact. The set of all weakly almost periodic elements of E is denoted by $WAP(E)$ which is a norm closed subbimodule of E [14, Definition 4.1]. For a Banach algebra $\mathcal{A}$, we write $F(\mathcal{A})_*$ for the $\mathcal{A}$-bimodule $WAP(\mathcal{A}^*)$ which is the left introverted subspace of $\mathcal{A}^*$ in the sense of [9, §1]. Runde observed that $F(\mathcal{A}) = WAP(\mathcal{A}^*)^*$ is a dual Banach algebra with the first Arens product $\Box$ inherited from $\mathcal{A}^{**}$. He also showed that $F(\mathcal{A})$ is a canonical dual Banach algebra associated to $\mathcal{A}$ [14, Theorem 4.10]. Choi et al. in [1] called $F(\mathcal{A})$ the enveloping dual Banach algebra associated to $\mathcal{A}$. They showed that if $\mathcal{A}$ is a Banach algebra and X is a Banach $\mathcal{A}$-bimodule, then $F_{\mathcal{A}}(X) = WAP(X^*)^*$ is a normal dual $F(\mathcal{A})$-bimodule [1, Theorem 4.3].

A dual Banach $\mathcal{A}$-bimodule E is normal if for each $x \in E$ the module maps $\mathcal{A} \longrightarrow E$; $a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ are weak*-weak* (briefly, wk^*-wk^*) continuous. Let $\mathcal{A}$ be a Banach algebra and E be a Banach $\mathcal{A}$-bimodule. A bounded linear map $D : \mathcal{A} \longrightarrow E$ is called a bounded derivation if for every $a, b \in \mathcal{A}$, $D(ab) = a \cdot D(b) + D(a) \cdot b$. A derivation $D : \mathcal{A} \longrightarrow E$ is called inner if there exists an element x in E such that $D(a) = a \cdot x - x \cdot a$ $(a \in \mathcal{A})$. A dual

Banach algebra $\mathcal{A}$ is said to be *Connes amenable* if for every normal dual Banach $\mathcal{A}$-bimodule E, every wk^*-continuous derivation $D : \mathcal{A} \longrightarrow E$ is inner. For a given dual Banach algebra $\mathcal{A}$ and a Banach $\mathcal{A}$-bimodule E, $\sigma wc(E)$ denotes the set of all elements $x \in E$ such that the module maps $\mathcal{A} \longrightarrow E$; $a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ are wk^*-wk-continuous. One can see that $\sigma wc(E)$ is a closed submodule of E and E^* is normal if and only if $E = \sigma wc(E)$ (see [13] and [14] for more details). Moreover, for a dual Banach algebra $\mathcal{A}$, consider the product morphism $\pi_{\mathcal{A}} : \mathcal{A}\widehat{\otimes}\mathcal{A} \longrightarrow \mathcal{A}$ given by $\pi_{\mathcal{A}}(a \otimes b) = ab$ for every $a, b \in \mathcal{A}$. Since $\sigma wc(\mathcal{A}_*) = \mathcal{A}_*$, the adjoint of $\pi_{\mathcal{A}}$ maps $\mathcal{A}_*$ into $\sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^*$. Therefore $\pi_{\mathcal{A}}^{**}$ drops to an $\mathcal{A}$-bimodule morphism $\pi_{\sigma wc} : (\sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^*)^* \longrightarrow \mathcal{A}$. An element $M \in (\sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^*)^*$ satisfying

$$a \cdot M = M \cdot a, \quad \text{and} \quad a\pi_{\sigma wc}M = a \quad (a \in \mathcal{A}),$$

is called a σwc-virtual diagonal for $\mathcal{A}$. Runde showed that a dual Banach algebra $\mathcal{A}$ is Connes amenable if and only if there is a σwc-virtual diagonal for $\mathcal{A}$ [14, Theorem 4.8].

In [1], the authors investigated the Connes amenability of the enveloping dual Banach algebra. Indeed, they characterized Connes amenability for this specific dual Banach algebra in terms of diagonal type elements so-called *WAP-virtual diagonal* [1, Theorem 6.12]. There is a natural variant of biprojectivity, as introduced and studied in [16] that is better adapted to categories of enveloping dual Banach algebras. This notion has become known as WAP-biprojectivity. Shariati et al. showed that for an infinite commutative compact group G, the convolution Banach algebra $F(L^2(G))$ is not WAP-biprojective [16].

The concept of approximate Connes amenability which is a generalization of Connes amenability was introduced in [4]. A unital dual Banach algebra $\mathcal{A}$ is approximately Connes amenable if and only if there exists a net (M_α) in $(\sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^*)^*$ such that $a \cdot M_\alpha - M_\alpha \cdot a \to 0$ and $\pi_{\sigma wc}(M_\alpha)a \to a$ for every $a \in \mathcal{A}$ [4, Theorem 3.3]. This concept has been extended for the category of enveloping dual Banach algebras in [17]. For a given Banach algebra $\mathcal{A}$, $F(\mathcal{A})$ is approximately Connes amenable if and only if $\mathcal{A}$ has an approximately WAP-virtual diagonal [17, Theorem 2.6].

The organization of the paper is as follows. First, we introduce the notion of approximately WAP-biprojective for the enveloping dual Banach algebra $F(\mathcal{A})$ associated to a Banach algebra $\mathcal{A}$. It is natural to compare this new notion with other approximate homological concepts. In this regard, we prove that the approximate biprojectivity of a Banach algebra $\mathcal{A}$ implies that $F(\mathcal{A})$ is approximate WAP-biprojective and also for a dual Banach algebra $\mathcal{A}$, we obtain the relationship between approximate WAP-biprojectivity of $F(\mathcal{A})$ and approximate Connes biprojectivity of $\mathcal{A}$. More precisely, we prove that for a dual Banach algebra $\mathcal{A}$ with a left approximate identity if $F(\mathcal{A})$ is approximately WAP-biprojective, then $\mathcal{A}$ is left φ-contractible, where φ is a wk^*-continuous character on $\mathcal{A}$. Using this tool we show that for a totally ordered set I which has a smallest element and a dual Banach algebra $\mathcal{A}$ with a right identity and the non-empty wk^*-continuous character space, the enveloping dual Banach algebra of a class of $I\times I$-upper triangular matrix $F(UP(I,\mathcal{A}))$ is approximately WAP-biprojective if and only if $F(\mathcal{A})$ is approximately WAP-biprojective and I is singleton. We present an example of the enveloping dual Banach algebra, $F(L^2(G))$ which distinguishes our new notion from another notion WAP-biprojectivity, where G is an infinite commutative compact group. For a unital weakly cancellative semigroup S with additional assumption wk^*-continuity of φ_S, $F(l^1(S))$ is approximately WAP-biprojective if and only if S is a finite group.

2. Approximate WAP-biprojectivity

In this section, we define a new concept on $F(\mathcal{A})$ and study some its properties and compare with the previous concepts.

Let $\Delta_{WAP}: F_{\mathcal{A}}(\mathcal{A}\widehat{\otimes}\mathcal{A})\longrightarrow F(\mathcal{A})$ be the wk^*-wk^* continuous $\mathcal{A}$-bimodule map induced by $\pi_{\mathcal{A}}:\mathcal{A}\widehat{\otimes}\mathcal{A}\longrightarrow\mathcal{A}$. Note that Δ_{WAP} is also an $F(\mathcal{A})$-bimodule map (see [1, Corollary 5.2] for more details). Composing the canonical inclusion map $\mathcal{A}\hookrightarrow\mathcal{A}^{**}$ with the adjoint of the inclusion map $F(\mathcal{A})_*\hookrightarrow\mathcal{A}^*$, we obtain a continuous homomorphism of Banach algebras $\eta_{\mathcal{A}}:\mathcal{A}\longrightarrow F(\mathcal{A})$ which has a wk^*-dense range. We write $\bar{a}$ instead of $\eta_{\mathcal{A}}(a)$ [1, Definition 6.4]. Let $\mathcal{A}$ be a Banach algebra. An element $M\in F_{\mathcal{A}}(\mathcal{A}\widehat{\otimes}\mathcal{A})$ is called a WAP-virtual diagonal for $\mathcal{A}$ if for every $a\in\mathcal{A}$

$$a\cdot M=M\cdot a\quad\text{and}\quad\Delta_{WAP}(M)\cdot a=\bar{a}.$$

Let $\mathcal{A}$ be a Banach algebra. Then $F(\mathcal{A})$ is called WAP-biprojective if there exists a wk^*-wk^* continuous $\mathcal{A}$-bimodule morphism $\rho:F(\mathcal{A})\longrightarrow F_{\mathcal{A}}(\mathcal{A}\widehat{\otimes}\mathcal{A})$ such that

$$\Delta_{WAP}\circ\rho=id_{F(\mathcal{A})},$$

see [16]. Motivated by the above notations, we now ready to give the definition of our new notion as follows.

Definition 2.1. Let $\mathcal{A}$ be a Banach algebra. Then, $F(\mathcal{A})$ is called *approximately* WAP-*biprojective* if there exists a net (ρ_α) of wk^*-wk^* continuous $\mathcal{A}$-bimodule morphism ρ_α : $F(\mathcal{A})\longrightarrow F_{\mathcal{A}}(\mathcal{A}\widehat{\otimes}\mathcal{A})$ such that $\Delta_{WAP}\circ\rho_\alpha(\psi)\rightarrow\psi$ for all $\psi\in F(\mathcal{A})$.

Definition 2.1 indicates that every WAP-biprojective enveloping dual Banach algebra is approximately WAP-biprojective.

For two Banach spaces X and Y, we denote by $B(X,Y)$, the space of all bounded linear operators from X to Y. Recall that the *weak* operator topology* (W^*OT) on $B(X,Y^*)$ is the locally convex topology determined by the seminorms $\{p_{x,y}: x\in X, y\in Y\}$, where $p_{x,y}(T)=|\langle y,Tx\rangle|$. Indeed, the net $(T_\alpha)\subset B(X,Y^*)$ converges to T in the weak* operator topology of $B(X,Y^*)$ if $T_\alpha(x)$ converges to $T(x)$ in the weak* topology of Y^* for every $x\in X$ [2].

Remark 2.2. If the net (ρ_α) in definition 2.1 is bounded, then the notions approximate WAP-biprojectivity and WAP-biprojectivity are the same. To see this, we should remind that on bounded sets, the W^*OT coincides with the wk^*-topology of $B(F(\mathcal{A}), F_{\mathcal{A}}(\mathcal{A}\widehat{\otimes}\mathcal{A}))$, where identified with $(F(\mathcal{A})\widehat{\otimes}WAP(\mathcal{A}\widehat{\otimes}\mathcal{A})^*)^*$. Since the unit ball of $B(F(\mathcal{A}), F_{\mathcal{A}}(\mathcal{A}\widehat{\otimes}\mathcal{A}))$ is W^*OT-compact, the net (ρ_α) has W^*OT-limit point. Set $\rho := W^*OT\text{-}\lim_\alpha \rho_\alpha$. It follows that

$$\rho(xy) = wk^*\text{-}\lim_\alpha \rho_\alpha(xy) = wk^*\text{-}\lim_\alpha x\cdot\rho_\alpha(y) = x\cdot wk^*\text{-}\lim_\alpha \rho_\alpha(y) = x\cdot\rho(y),$$

and by similarity for the right action, $\rho(xy) = \rho(x)\cdot y$. In addition

$$\Delta_{WAP}\circ\rho(\psi) = \Delta_{WAP}(wk^*\text{-}\lim_\alpha \rho_\alpha(\psi)) = wk^*\text{-}\lim_\alpha \Delta_{WAP}\circ\rho_\alpha(\psi) = \psi.$$

Let $\mathcal{A}$ be a Banach algebra and $F(\mathcal{A})$ has an identity element e. An approximately WAP-virtual diagonal for $\mathcal{A}$ is a net $(M_\alpha)\subseteq F_{\mathcal{A}}(\mathcal{A}\widehat{\otimes}\mathcal{A})$ such that for each $a\in\mathcal{A}$

$$a\cdot M - M\cdot a \to 0 \quad \text{and} \quad \Delta_{WAP}(M)\to e.$$

Theorem 2.3. *Let $\mathcal{A}$ be a Banach algebra. If $F(\mathcal{A})$ is approximately WAP-biprojective with an identity, then $F(\mathcal{A})$ is approximately Connes amenable.*

Proof. Suppose that $F(\mathcal{A})$ is approximately WAP-biprojective with an identity e. Then, there exists a wk^*-wk^* continuous $\mathcal{A}$-bimodule morphism $\rho_\alpha : F(\mathcal{A}) \longrightarrow F_{\mathcal{A}}(\mathcal{A}\widehat{\otimes}\mathcal{A})$ such that $\Delta_{WAP}\circ\rho_\alpha$ tends to $id_{F(\mathcal{A})}$. Put $M_\alpha := \rho_\alpha(e)$. Hence, (M_α) is a net in $F_{\mathcal{A}}(\mathcal{A}\widehat{\otimes}\mathcal{A})$ and by [16, Lemma 2.2 (i)] for every $a\in\mathcal{A}$ and $\eta\in F(\mathcal{A})$, we have

$$a\cdot M_\alpha = a\cdot\rho_\alpha(e) = \rho_\alpha(a\cdot e) = \rho_\alpha(\bar{a}\Box e) = \rho_\alpha(e\Box\bar{a}) = \rho_\alpha(e\cdot a) = \rho_\alpha(e)\cdot a = M_\alpha\cdot a,$$

and

$$\lim_\alpha \Delta_{WAP}(M_\alpha)\Box\eta = \lim_\alpha(\Delta_{WAP}\circ\rho_\alpha(e))\Box\eta = e\Box\eta = \eta.$$

The last relations show that M_α is an approximately WAP-virtual diagonal for $\mathcal{A}$. It follows that $F(\mathcal{A})$ is approximately Connes amenable [17, Theorem 2.6]. □

Recall from [15] that a dual Banach algebra $\mathcal{A}$ is called *approximately Connes biprojective* if there exists a (not necessarily bounded) net $(\rho_\alpha)_\alpha$ of continuous $\mathcal{A}$-bimodule morphisms from $\mathcal{A}$ into $(\sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^*)^*$ such that

$$\pi_{\sigma wc}\circ\rho_\alpha(a)\to a, \quad (a\in\mathcal{A}).$$

Theorem 2.4. *Let $\mathcal{A}$ be a dual Banach algebra. If $F(\mathcal{A})$ is approximately WAP-biprojective, then $\mathcal{A}$ is approximately Connes biprojective.*

Proof. By assumptions, there is a net of bounded $\mathcal{A}$-bimodule morphism $\rho_\alpha : F(\mathcal{A}) \longrightarrow F_{\mathcal{A}}(\mathcal{A}\widehat{\otimes}\mathcal{A})$ satisfies $\lim_\alpha \Delta_{WAP} \circ \rho_\alpha = id_{F(\mathcal{A})}$. According to [14], $\sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^* \subseteq WAP(\mathcal{A}\widehat{\otimes}\mathcal{A})^*$ induces a quotient map $\theta : F_{\mathcal{A}}(\mathcal{A}\widehat{\otimes}\mathcal{A}) \longrightarrow (\sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^*)^*$ defined by $\theta(u) = u|_{\sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^*}$, where $u \in F_{\mathcal{A}}(\mathcal{A}\widehat{\otimes}\mathcal{A})$. Since $\mathcal{A}$ is normal as Banach $\mathcal{A}$-bimodule, again by [14], $\mathcal{A}_* \subseteq WAP(\mathcal{A}^*)$ and so there is a quotient map $\xi : F(\mathcal{A}) \longrightarrow \mathcal{A}$ such that $\xi \circ \eta_{\mathcal{A}} = id_{\mathcal{A}}$. We claim that $\xi \circ \Delta_{WAP} = \pi_{\sigma wc} \circ \theta$. For doing it, we have

$$\begin{aligned}\langle f, \xi \circ \Delta_{WAP}(v)\rangle &= \langle f, (\Delta_{WAP}(v))|_{\mathcal{A}_*}\rangle = \langle f, \Delta_{WAP}(v)\rangle = \langle \pi^*|_{WAP(\mathcal{A}^*)}(f), v\rangle \\ &= \langle \pi^*_{\mathcal{A}}(f), v\rangle = \langle \pi^*|_{\mathcal{A}_*}(f), v\rangle = \langle \pi^*|_{\mathcal{A}_*}(f), v|_{\sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^*}\rangle \\ &= \langle \pi^*|_{\mathcal{A}_*}(f), \theta(v)\rangle = \langle f, \pi_{\sigma wc} \circ \theta(v)\rangle,\end{aligned}$$

for all $v \in F_{\mathcal{A}}(\mathcal{A}\widehat{\otimes}\mathcal{A})$ and $f \in \mathcal{A}_*$. Let $\psi_\alpha = \theta \circ \rho_\alpha \circ \eta_{\mathcal{A}}$. We obtain a net of bounded $\mathcal{A}$-bimodule morphism $\psi_\alpha : \mathcal{A} \longrightarrow (\sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^*)^*$ such that

$$\pi_{\sigma wc} \circ \psi_\alpha = \pi_{\sigma wc} \circ \theta \circ \rho_\alpha \circ \eta_{\mathcal{A}} = \xi \circ \Delta_{WAP} \circ \rho_\alpha \circ \eta_{\mathcal{A}}.$$

Since $\Delta_{WAP} \circ \rho_\alpha$ tends to $id_{F(\mathcal{A})}$, we get $\lim_\alpha \pi_{\sigma wc} \circ \psi_\alpha(a) = \xi \circ id_{F(\mathcal{A})} \circ \eta_{\mathcal{A}}(a) = a$, as required. $\square$

Corollary 2.5. *Let $\mathcal{A}$ be a reflexive Banach algebra. If $F(\mathcal{A})$ is approximately WAP-biprojective, then $F(\mathcal{A})$ is approximately Connes biprojective.*

Recall from [6] that a Banach algebra $\mathcal{A}$ is called *pseudo-contractible* if there exists a not necessarily bounded net (u_α) in $\mathcal{A}\widehat{\otimes}\mathcal{A}$ such that $a \cdot u_\alpha = u_\alpha \cdot a$ and $\pi_{\mathcal{A}}(u_\alpha)a \to a$ for every $a \in \mathcal{A}$.

Theorem 2.6. *Let $\mathcal{A}$ be a Banach algebra. If $\mathcal{A}$ is pseudo-contractible, then $F(\mathcal{A})$ is approximately WAP-biprojective.*

Proof. The pseudo-contractibility of $\mathcal{A}$ implies that there exists a net (u_α) in $\mathcal{A}\widehat{\otimes}\mathcal{A}$ such that $a \cdot u_\alpha = u_\alpha \cdot a$ and $\pi_{\mathcal{A}}(u_\alpha)a \to a$ for all $a \in \mathcal{A}$. Define $\rho_\alpha : F(\mathcal{A}) \longrightarrow F_{\mathcal{A}}(\mathcal{A}\widehat{\otimes}\mathcal{A})$ via $\rho_\alpha(\psi) = \psi \bullet \overline{u_\alpha}$, for all $\psi \in F(\mathcal{A})$, where $\bullet$ denotes the module action of $F(\mathcal{A})$ on $F_{\mathcal{A}}(\mathcal{A}\widehat{\otimes}\mathcal{A})$ and $\overline{u_\alpha} = \eta_{\mathcal{A}}(u_\alpha)$. Since $F_{\mathcal{A}}(\mathcal{A}\widehat{\otimes}\mathcal{A})$ is a normal dual $F(\mathcal{A})$-bimodule [1, Theorem 4.3], ρ_α is wk^*-wk^* continuous for every α. Using Lemma [16, Lemma 2.2], for each $a \in \mathcal{A}$ and $\psi \in F(\mathcal{A})$ we have

$$a \cdot \rho_\alpha(\psi) = a \cdot (\psi \bullet \overline{u_\alpha}) = \overline{a} \bullet (\psi \bullet \overline{u_\alpha}) = (\overline{a}\Box\psi) \bullet \overline{u_\alpha} = (a \cdot \psi) \bullet \overline{u_\alpha} = \rho_\alpha(a \cdot \psi),$$

and also

$$\rho_\alpha(\psi)\cdot a = (\psi \bullet \overline{u_\alpha})\cdot a = (\psi \bullet \overline{u_\alpha}) \bullet \overline{a} = \psi \bullet (\overline{u_\alpha} \bullet \overline{a})$$
$$= \psi \bullet (\overline{u_\alpha \cdot a}) = \psi \bullet (\overline{a \cdot u_\alpha}) = \psi \bullet (\overline{a} \bullet \overline{u_\alpha})$$
$$= (\psi \Box \overline{a}) \bullet \overline{u_\alpha} = (\psi \cdot a) \bullet \overline{u_\alpha} = \rho_\alpha(\psi \cdot a).$$

Hence, ρ_α is an $\mathcal{A}$-bimodule morphism for every α. Since Δ_{WAP} is an $F(\mathcal{A})$-bimodule morphism [1, Corollary 5.2], for every $\psi \in F(\mathcal{A})$ we obtain

$$\Delta_{WAP} \circ \rho_\alpha(\psi) = \Delta_{WAP}(\psi \bullet \overline{u_\alpha}) = \psi \Box \Delta_{WAP}(\overline{u_\alpha}) = \psi \Box \overline{\pi(u_\alpha)} = \psi \cdot \pi(u_\alpha). \tag{1}$$

On the other hand

$$|\langle f, \psi \cdot \pi(u_\alpha) - \psi \rangle| = |\langle \pi(u_\alpha)\cdot f, \psi\rangle - \psi(f)|$$
$$= |\langle \pi(u_\alpha)\cdot f - f, \psi\rangle|$$
$$\leq \|\psi\| \|\pi(u_\alpha)\cdot f - f\| \qquad ,(f \in WAP(\mathcal{A}^*)).$$

Thus, $\psi \cdot \pi(u_\alpha) \to \psi$ is an immediate consequence of $\pi(u_\alpha)\cdot f \to f$. It follows from (1), $\Delta_{WAP} \circ \rho_\alpha(\psi) \to \psi$. Therefore, $F(\mathcal{A})$ is approximately WAP-biprojective. □

Recall that a Banach algebra $\mathcal{A}$ is *left φ-contractible*, where φ is a linear multiplication functional on $\mathcal{A}$ if there exists $m \in \mathcal{A}$ such that $am = \varphi(a)m$ and $\varphi(m) = 1$, for every $a \in \mathcal{A}$ [8]. The set of all wk^*-continuous homomorphism from $\mathcal{A}$ into $\mathbb{C}$ is denoted by $\Delta_{wk^*}(\mathcal{A})$.

Corollary 2.7. *Let $\mathcal{A}$ be a dual Banach algebra with a left approximate identity and $\varphi \in \Delta_{wk^*}(\mathcal{A})$. If $F(\mathcal{A})$ is approximately WAP-biprojective, then $\mathcal{A}$ is left φ-contractible.*

Proof. Applying part (ii) of Theorem 2.4, we see that the approximate WAP-biprojectivity of $F(\mathcal{A})$ implies that $\mathcal{A}$ is approximately Connes biprojective. It now follows from the hypothesis and [15, Theorem 2.4] that $\mathcal{A}$ is left φ-contractible. □

Remark 2.8. Parallel to [15, Theorem 2.4], we state the following result. Let $\mathcal{A}$ be an approximately Connes biprojective dual Banach algebra and $\varphi \in \Delta_{wk^*}(\mathcal{A})$ such that $\mathcal{A}$ has right approximate identity. Then, $\mathcal{A}$ is right φ-contractible

Let $\mathcal{A}$ be a Banach algebra and I be a totally ordered set. Then, the set of all $I \times I$-upper triangular matrices with the usual matrix operations and the norm $\| \, [a_{i,j}]_{i,j\in I} \, \| = \sum_{i,j\in I} \| \, a_{i,j} \, \| < \infty$, becomes a Banach algebra and it is denoted by

$$UP(I,\mathcal{A}) = \left\{ \left[\; a_{i,j} \; \right]_{i,j\in I} : a_{i,j} \in A \text{ and } \; a_{i,j} = 0 \; \text{ for every } i > j \right\}.$$

Theorem 2.9. *Let $\mathcal{A}$ be a dual Banach algebra with a right identity and $\varphi \in \Delta_{wk^*}(\mathcal{A})$. If I is a totally ordered set which has a smallest element, then $F(UP(I, \mathcal{A}))$ is approximately WAP-biprojective if and only if $|I| = 1$ and $F(\mathcal{A})$ is approximately WAP-biprojective.*

Proof. Suppose that $F(UP(I, \mathcal{A}))$ is approximately WAP-biprojective. By Theorem 2.4, $UP(I, \mathcal{A})$ is approximately Connes biprojective. Since $\mathcal{A}$ has a right identity 1, $UP(I, \mathcal{A})$ has a right approximate identity. Define a character $\psi_\varphi : UP(I, \mathcal{A}) \longrightarrow \mathbb{C}$ through $\psi_\varphi([a_{ij}]) = \varphi(a_{i_0 i_0})$. It is easy to checked that the wk^*-continuity of φ implies that the wk^*-continuity of ψ_φ. According by Remark 2.8, $UP(I, \mathcal{A})$ is right ψ_φ-contractible. Therefore, there exists X in $UP(I, \mathcal{A})$ satisfies $XY = \psi_\varphi(Y)X$ and $\psi_\varphi(X) = \varphi(X_{i_0 i_0}) = 1$, for all $Y \in UP(I, \mathcal{A})$. Assume contrary to our claim, that I has at least two elements. Choose $\varepsilon_{i_0 j}$ be an element in $UP(I, \mathcal{A})$ which (i_0, j)-th entry is 1 and others are zero, where $j \neq i_0$. Thus, $X\varepsilon_{i_0 j} = \psi_\varphi(\varepsilon_{i_0 j})X = 0$ and hence $X_{i_0 i_0} = 0$, which is a contradiction with $\psi_\varphi(X) = \varphi(X_{i_0 i_0}) = 1$.
The converse is clear. □

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and X be a Banach $\mathcal{A}$-$\mathcal{B}$-module. That is, X is a Banach left $\mathcal{A}$-module and a Banach right $\mathcal{B}$-module satisfying $(a\cdot x)\cdot b = a\cdot(x\cdot b)$ and $||a\cdot x\cdot b|| \leq ||a||||x||||b||$, for every $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $x \in X$. Consider

$$Tri(\mathcal{A}, \mathcal{B}, X) = \begin{pmatrix} \mathcal{A} & X \\ 0 & \mathcal{B} \end{pmatrix},$$

with the usual matrix operations and the norm

$$\left\| \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \right\| = ||a|| + ||x|| + ||b||, \quad (a \in \mathcal{A}, x \in X, b \in \mathcal{B}).$$

It can be checked that $Tri(\mathcal{A}, \mathcal{B}, X)$ becomes a Banach algebra which is called a triangular Banach algebra. Note that if $\mathcal{A}$ and $\mathcal{B}$ are dual Banach algebras and X be a normal dual Banach $\mathcal{A}$-$\mathcal{B}$-module, then $Tri(\mathcal{A}, \mathcal{B}, X)$ is a dual Banach algebra as well.

Recall from [2] that X is said to be essential provided that for every $x \in X$ there are $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $y, z \in X$ such that $x = a \cdot y = z \cdot b$. If $\mathcal{A}$ equipped with left approximate identity and X is essential, then $\mathcal{A}$ has left approximate identity for X. Further, X is called right faithful if $\{b \in \mathcal{B} : \ X \cdot b = 0\} = 0$.

The following proposition is a consequences of [15, Theorem 3.1] and Theorem 2.4 and therefore the proof can be omitted.

Proposition 2.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be a dual Banach algebra equipped with left approximate identity and $\varphi \in \Delta_{wk^*}(\mathcal{B})$. Suppose that X is an essential normal dual Banach $\mathcal{A}$-$\mathcal{B}$-module* *satisfying X is right faithful as a right $\mathcal{B}$-module. Then, $F(Tri(\mathcal{A},\mathcal{B},X))$ is not approximately WAP-biprojective.*

A direct consequence of Proposition 2.10 is as follows.

Corollary 2.11. *Let $\mathcal{A}$ be a dual Banach algebra with a left approximate identity and $\varphi \in \Delta_{wk^*}(\mathcal{A})$. Then, $F(Tri(\mathcal{A},\mathcal{A},\mathcal{A}))$ is not approximately WAP-biprojective.*

The augmentation character on semigroup algebra $l^1(S)$ defined by

$$\sum_{s\in S}\alpha_s\delta_s \mapsto \sum_{s\in S}\alpha_s,$$

and it is denoted by φ_S. According by [3, Definition 3.14], the semigroup S is weakly left (respectively, right) cancellative if $s^{-1}F = \{x \in S : sx \in F\}$ (resp., $Fs^{-1} = \{x \in S : xs \in F\}$) is finite for all $s \in S$ and every finite subset F of S. Moreover, S is weakly cancellative if it is both weakly left cancellative and weakly right cancellative.

Proposition 2.12. *Let S be a unital weakly cancellative semigroup and $\varphi_S \in \Delta_{w^*}(l^1(S))$. Then, $F(l^1(S))$ be approximately WAP-biprojective if and only if S is finite.*

Proof. Suppose that $F(l^1(S))$ be approximately WAP-biprojective. Note that δ_e is an identity for $l^1(S)$, where e is an identity for S. Using Corollary 2.7, $l^1(S)$ is left φ_S-contractible. By [5, Proposition 4.4], φ_S-Connes amenability $l^1(S)$ is equivalent with left φ_S-contractibility. As shown in [5, Theorem 4.3], S must be a group. Hence, by Theorem 6.1 of [12], S is compact. This fact together with being discrete of S show that S is finite. However, it is clear that if S is finite, then $F(l^1(S))$ must be approximately WAP-biprojective. □

3. Examples

In this section, we present some examples pertinent to new concept and other notions.

Example 3.1. (i) For a locally compact group G, suppose that $F(M(G))$ is approximately WAP-biprojective. By Theorem 2.4, $M(G)$ is approximately Connes biprojective. Applying [15, Theorem 4.1], we deduce that G is amenable;

(ii) For an amenable, non-compact group G, it is known that the Banach algebra $L^1(G)$ is not approximately biprojective while $F(L^1(G))$ is Connes amenable [14]. Now, Corollary 2.5 from [16] implies that $F(L^1(G))$ is WAP-biprojective and so we conclude that $F(L^1(G))$ is approximately WAP-biprojective.

In the upcoming example we give a Banach algebra which is not WAP-biprojective but it is approximately WAP-biprojective.

Example 3.2. For an infinite commutative compact group G, it was shown in [18, §2] that the Banach algebra $L^2(G)$ with convolution multiplication is not biprojective. Moreover, It was proved in [16] that the Banach algebra $F(L^2(G))$ is not WAP-biprojective while it is pseudo-contractible. It now follows from Theorem 2.6 that $F(L^2(G)) = L^2(G)$ is approximately WAP-biprojective.

The following example emphasizes the fact that Theorem 2.3 does not hold for enveloping dual Banach algebras without identity. In fact, we give a Banach algebra which is approximately WAP-biprojective but not approximately Connes amenable.

Example 3.3. Consider the Banach algebra $\ell^1 = l^1(\mathbb{N})$ of all sequences $a = (a_n)$ of complex numbers with

$$\|a\| = \sum_{n=1}^{\infty} |a_n| < \infty.$$

It is well known that ℓ^1 with pointwise multiplication is pseudo-contractible. This latter and Theorem 2.6 necessitate that $F(\ell^1)$ is approximately WAP-biprojective. Furthermore, by [11], ℓ^1 is not approximately Connes amenable. Applying Corollary 3.4 from [10], we observe that $F(\ell^1)$ is not approximately Connes amenable.

Example 3.4. Consider the Banach algebra ℓ^1 as in Example 3.3 with the product

$$(a * b)(n) = \begin{cases} a(1)b(1), & \text{if} \quad n = 1, \\ a(1)b(n) + b(1)a(n) + a(n)b(n), & \text{if} \quad n > 1, \end{cases}$$

for all $a, b \in \ell^1$. According by [15, Example 2.5] $(\ell^1, *)$ is not an approximately Connes biprojective dual Banach algebra. Using Part (ii) of Theorem 2.4, we find that $F(\ell^1, *)$ is not approximately WAP-biprojective.

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