Algebraic Structures and Their Applications Vol. 12 No. 2 (2025) pp 141-153.

Research Paper

$\mathcal{N}-$IDEALS OF ALMOST DISTRIBUTIVE LATTICES

RAFI NOORBHASHA*, TESHALE AMARE NATNAEL AND MONIKARCHANA YATHALURU

Abstract. In this article, we present the concept of an $\mathcal{N}-$ideal within the structure of an Almost Distributive Lattice (ADL). We demonstrate that the collection of $\mathcal{N}-$ideals constitutes a distributive lattice, which is distinct from and not a sublattice of the lattice of ideals in an ADL. Additionally, we define $\mathcal{N}-$lets in ADLs. We derive the necessary and sufficient conditions for an ideal to be classified as an $\mathcal{N}-$ideal using $\mathcal{N}-$lets. Finally, we establish the conditions required for an ADL to achieve the property of being soft relatively complemented.

1. Introduction

In [10], Swamy and Rao developed an Almost Distributive Lattice(ADL) as a common abstraction of many existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other. In that paper they introduced the concept of an ideal in an ADL analogous to that in a distributive lattice and it was observed that the set $\mathcal{PI}(\mathcal{R})$ of all principal ideals of $\mathcal{R}$ forms a distributive lattice. Subsequent research by other

DOI: 10.22034/as.2024.21809.1725

MSC(2010): Primary: 06D99, 06D15.

Keywords: Almost distributive lattice (ADL), $\mathcal{N}-$ideal, Dual dense element, Soft relatively complemented ADL.

Received: 24 June 2024, Accepted: 02 December 2024.

∗Corresponding author

authors [3, 4, 5, 9] extended similar concepts from distributive lattices to ADLs. In this article, we present the notion of $\mathcal{N}-$ideals in ADLs and provide examples to illustrate these concepts. We derive several properties of $\mathcal{N}-$ideals and explore the class of $\mathcal{N}-$lets within an ADL, presenting some equivalent conditions for an ideal to be classified as an $\mathcal{N}-$let. Additionally, we describe the collection of prime $\mathcal{N}-$ideals and acquire certain outcomes related to them. Unless otherwise noted, an ADL with maximal elements is denoted by $\mathcal{R}$ in this work.

2. Preliminaries

The definitions and significant results from [7, 10] are gathered and given in this part; these will be needed during the entire document.

Definition 2.1. [10] An algebra $(\mathcal{R}, \vee, \wedge, 0)$ of type (2,2,0) satisfying the following specifications is an Almost Distributive Lattice (ADL) with zero:

(1) $(\varpi \vee \vartheta) \wedge \sigma = (\varpi \wedge \sigma) \vee (\vartheta \wedge \sigma)$,
(2) $\varpi \wedge (\vartheta \vee \sigma) = (\varpi \wedge \vartheta) \vee (\varpi \wedge \sigma)$,
(3) $(\varpi \vee \vartheta) \wedge \vartheta = \vartheta$,
(4) $(\varpi \vee \vartheta) \wedge \varpi = \varpi$,
(5) $\varpi \vee (\varpi \wedge \vartheta) = \varpi$,
(6) $0 \wedge \varpi = 0$, for any $\varpi, \vartheta, \sigma \in \mathcal{R}$.

When $\varpi = \varpi \wedge \vartheta$, or equivalently, $\varpi \vee \vartheta = \vartheta$, occurs for every $\varpi, \vartheta \in \mathcal{R}$, then $\varpi \leq \vartheta$. This defines a partial $\leq$ on $\mathcal{R}$ in an ADL $(\mathcal{R}, \vee, \wedge, 0)$. As a partial ordering on $\mathcal{R}$, this definition establishes $\leq$. When m in $\mathcal{R}$ holds maximum with respect to the partial ordering $\leq$ on $\mathcal{R}$, it is referred to as *maximal*, that is, for any $\varpi \in \mathcal{R}$, $m \leq \varpi \Rightarrow m = \varpi$. The set $\mathcal{M}_{Max.elts}$ is the collection of all such maximal elements within $\mathcal{R}$.

In Swamy's work [10], it is noted that an ADL denoted as $\mathcal{R}$ exhibits nearly all features of a distributive lattice [1, 2], apart from the non-commutativity between $\vee$ and $\wedge$ and the right distributivity of $\vee$ over $\wedge$. Either of these properties, if present, would classify $\mathcal{R}$ as a distributive lattice. A non-empty subset $\mathcal{I}$ of an ADL $\mathcal{R}$ is called an ideal of $\mathcal{R}$ if $\varpi \vee \vartheta \in \mathcal{I}$ and $\varpi \wedge \mu \in \mathcal{I}$ for any $\varpi, \vartheta \in \mathcal{I}$ and $\mu \in \mathcal{I}$. Also, a non-empty subset $\mathcal{F}$ of $\mathcal{R}$ is said to be a filter of $\mathcal{R}$ if $\varpi \wedge \vartheta \in \mathcal{F}$ and $\mu \vee \varpi \in \mathcal{F}$ for $\varpi, \vartheta \in \mathcal{F}$ and $\mu \in \mathcal{R}$. A maximum ideal (filter) contains every appropriate ideal (filter) of $\mathcal{R}$. For any subset $\mathcal{A}$ of $\mathcal{R}$ the smallest ideal containing $\mathcal{A}$ is given by $(\mathcal{A}] := \{(\bigvee_{i=1}^{n} \varpi_i) \wedge \mu \mid \varpi_i \in \mathcal{A}, \mu \in \mathcal{R}$ and $n \in \mathbb{N}\}$. An ideal like $\mathcal{A} = \{\varpi\}$ is written as $(\varpi]$ rather than $(\mathcal{A}]$; this is known as the principal ideal of $\mathcal{R}$. The same way, for each $\mathcal{A} \subseteq \mathcal{R}$, $[\mathcal{A}) := \{\mu \vee (\bigwedge_{i=1}^{n} \varpi_i) \mid \varpi_i \in \mathcal{A}, \mu \in \mathcal{R}$ and $n \in \mathbb{N}\}$. A filter like $\mathcal{A} = \{\varpi\}$ is written as $[\varpi)$ rather than $[\mathcal{A})$; this is known as the principal filter of $\mathcal{R}$. It can be confirmed that $(\varpi] \vee (\vartheta] = (\varpi \vee \vartheta]$, $(\varpi] \cap (\vartheta] = (\varpi \wedge \vartheta]$, $[\varpi) \vee [b) = [\varpi \wedge \vartheta)$ and $[\varpi) \cap [\vartheta) = [\varpi \vee \vartheta)$, hold for any $\varpi, \vartheta \in \mathcal{R}$. The set $(\mathcal{PI}(\mathcal{R}), \vee, \cap)$ of all principal ideals of $\mathcal{R}$ is therefore a sublattice of the distributive lattice $(\mathcal{I}(\mathcal{R}), \vee, \cap)$ of all ideals of $\mathcal{R}$. Furthermore, the set $(\mathcal{F}(\mathcal{R}), \vee, \cap)$ containing all filters for $\mathcal{R}$ is a bounded distributive lattice. In an ADL [8], observed that the prime ideal $\mathcal{P}$ of $\mathcal{R}$ can only exist if $\mathcal{R} \setminus \mathcal{P}$ is a prime filter of $\mathcal{R}$. For any non-empty subset $\mathcal{A}$ of $\mathcal{R}$, the set $\mathcal{A}^+ = \{\mu \in \mathcal{R} \mid \varpi \vee \mu \in \mathcal{M}_{Max.elts},$ for all $\varpi \in \mathcal{A}\}$ is a filter of $\mathcal{R}$. Usually, for every $\varpi \in \mathcal{R}$, $\{\varpi\}^+ = (\varpi)^+$, where $(\varpi) = [\varpi)$.

Lemma 2.2. [5] *If every two elements ϖ, ϑ of $\mathcal{R}$, then*

(1) $\varpi \leq \vartheta \Rightarrow (\varpi)^+ \subseteq (\vartheta)^+$,
(2) $(\varpi)^{+++} = (\varpi)^+$,
(3) $(\varpi \wedge \vartheta)^+ = (\varpi)^+ \cap (\vartheta)^+$,
(4) $(\varpi \vee \vartheta)^{++} = (\varpi)^{++} \cap (\vartheta)^{++}$,
(5) $(\varpi)^+ \subseteq (\vartheta)^+ \Rightarrow (\varpi)^{++} \subseteq (\vartheta)^{++}$,
(6) $\varpi \in (\varpi)^{++}$.

If $(e)^+ = \mathcal{M}_{Max.elts}$, then an element $e \in \mathcal{R}$ is considered dual dense. Within $\mathcal{R}$, the set E is the set of dual dense elements.

Definition 2.3. [6] An element ϖ of an ADL $\mathcal{R}$ is stated to be $E-$complemented, if $\varpi \wedge \vartheta \in E$ and $\varpi \vee \vartheta \in \mathcal{M}_{Max.elts}$, for some $\vartheta \in \mathcal{R}$. If every element in an ADL $\mathcal{R}$ is $E-$complemented, then the ADL $\mathcal{R}$ is known as an $E-$complemented ADL $\mathcal{R}$.

Theorem 2.4. [8] *If and only if there exists $\pi \notin \mathcal{P}$ such that $\mu \wedge \pi = 0$ for every $\mu \in \mathcal{P}$, a prime ideal $\mathcal{P}$ of $\mathcal{R}$ is minimal.*

3. $\mathcal{N}-$ideal of an ADL

Here we present the notion of an $\mathcal{N}-$ideal in an ADL with specific examples. We notice that the collection of $\mathcal{N}-$ideals constitutes a distributive lattice, which does not form a sub-distributive lattice within the collection of ideals of an ADL. Additionally, we define $\mathcal{N}-$lets in the context of an ADL. In this case of $\mathcal{N}-$lets, we present sufficient and necessary requirements for an ideal to be considered a $\mathcal{N}-$ideal. Finally, We extract the sufficient and required conditions so that an ADL can be established as soft relative complemented.

We now start with the definition that follows.

Definition 3.1. For any ideal $\mathcal{S}$ of $\mathcal{R}$, let us consider

$$\mathcal{S}^{\blacklozenge} = \{\mu \in \mathcal{R} \mid (\mu)^+ \subseteq (\varpi)^+, \quad \text{for some } \varpi \in \mathcal{S}\}.$$

In particular, for every $\varpi \in \mathcal{R}, (\varpi]^{\blacklozenge} = \{\mu \in \mathcal{R} \mid (\mu)^+ \subseteq (\varpi)^+\}$.

Lemma 3.2. *The following assertions are valid within an ADL*

(i) $\mathcal{R}^{\blacklozenge} = \mathcal{R} = (\mathcal{M}_{Max.elts}]^{\blacklozenge}$,

(ii) $E^{\blacklozenge} = E = (0]^{\blacklozenge}$,

(iii) *For any* $e \in E, (e]^{\blacklozenge} = E$,

(iv) *For every ideal* $\mathcal{S}$ *of* $\mathcal{R}, \mathcal{S} \subseteq \mathcal{S}^{\blacklozenge}$ *and* $E \subseteq \mathcal{S}^{\blacklozenge}$.

Lemma 3.3. *We have, for every pair of ideals* $\mathcal{S}, \mathcal{T}$ *of* $\mathcal{R}$

(i) $\mathcal{S} \subseteq \mathcal{T}$ *implies* $\mathcal{S}^{\blacklozenge} \subseteq \mathcal{T}^{\blacklozenge}$,

(ii) $\mathcal{S}^{\blacklozenge\blacklozenge} = \mathcal{S}^{\blacklozenge}$,

(iii) $(\mathcal{S} \cap \mathcal{T})^{\blacklozenge} = \mathcal{S}^{\blacklozenge} \cap \mathcal{T}^{\blacklozenge}$,

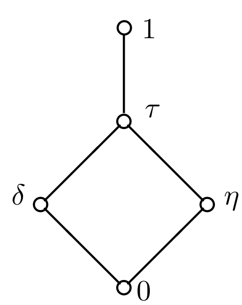
(iv) $(\mathcal{S} \vee \mathcal{T})^{\blacklozenge} = (\mathcal{S}^{\blacklozenge} \vee \mathcal{T}^{\blacklozenge})^{\blacklozenge}$.

Proof. (i) Let $\mu \in \mathcal{S}^{\blacklozenge}$. Then $(\mu)^+ \subseteq (\varpi)^+$ for some $\varpi \in \mathcal{S} \subseteq \mathcal{T}$. Therefore $\mu \in \mathcal{T}^{\blacklozenge}$ and hence $\mathcal{S}^{\blacklozenge} \subseteq \mathcal{T}^{\blacklozenge}$.
(ii) It is clear that $\mathcal{S}^{\blacklozenge} \subseteq \mathcal{S}^{\blacklozenge\blacklozenge}$. Let $\mu \in \mathcal{S}^{\blacklozenge\blacklozenge}$. Then $(\mu)^+ \subseteq (\varpi)^+$ for some $\varpi \in \mathcal{S}^{\blacklozenge}$. For this $\varpi \in \mathcal{S}^{\blacklozenge}, (\varpi)^+ \subseteq (\vartheta)^+$ for some $\vartheta \in \mathcal{S}$. Therefore $(\mu)^+ \subseteq (\vartheta)^+$ for some $\vartheta \in \mathcal{S}$. Hence $\mu \in \mathcal{S}^{\blacklozenge}$. Thus $\mathcal{S}^{\blacklozenge\blacklozenge} = \mathcal{S}^{\blacklozenge}$.
(iii) Clearly, $(\mathcal{S} \cap \mathcal{T})^{\blacklozenge} \subseteq \mathcal{T}^{\blacklozenge} \cap \mathcal{S}^{\blacklozenge}$. Let $\mu \in \mathcal{S}^{\blacklozenge} \cap \mathcal{T}^{\blacklozenge}$. Then $(\mu)^+ \subseteq (\varpi)^+$ and $(\mu)^+ \subseteq (\vartheta)^+$ for some $\varpi \in \mathcal{S}$ and $\vartheta \in \mathcal{T}$. Therefore $(\mu)^+ \subseteq (\varpi)^+ \cap (\vartheta)^+ = (\varpi \wedge \vartheta)^+$, where $\varpi \wedge \vartheta \in \mathcal{S} \cap \mathcal{T}$. Hence $\mu \in (\mathcal{S} \cap \mathcal{T})^{\blacklozenge}$. Thus $\mathcal{S}^{\blacklozenge} \cap \mathcal{T}^{\blacklozenge} = (\mathcal{S} \cap \mathcal{T})^{\blacklozenge}$.
(iv) Clearly, $\mathcal{S} \vee \mathcal{T} \subseteq \mathcal{S}^{\blacklozenge} \vee \mathcal{T}^{\blacklozenge}$ and $(\mathcal{S} \vee \mathcal{T})^{\blacklozenge} \subseteq (\mathcal{S}^{\blacklozenge} \vee \mathcal{T}^{\blacklozenge})^{\blacklozenge}$. Let $\mu \in (\mathcal{S}^{\blacklozenge} \vee \mathcal{T}^{\blacklozenge})^{\blacklozenge}$, then there are $\varpi \vee \vartheta \in \mathcal{S}^{\blacklozenge} \vee \mathcal{T}^{\blacklozenge}$ such that $(\mu)^+ \subseteq (\varpi \vee \vartheta)^+$, where $\varpi \in \mathcal{S}^{\blacklozenge}$, $\vartheta \in \mathcal{T}^{\blacklozenge}$. For $\varpi \in \mathcal{S}^{\blacklozenge}$ and $\vartheta \in \mathcal{T}^{\blacklozenge}$, there exist $\rho \in \mathcal{S}$ and $\kappa \in \mathcal{T}$ such that $(\varpi)^+ \subseteq (\rho)^+$ and $(\vartheta)^+ \subseteq (\kappa)^+$. Therefore $(\varpi)^{++} \subseteq (\rho)^{++}, (\vartheta)^{++} \subseteq (\kappa)^{++}$ and $(\varpi)^{++} \cap (\vartheta)^{++} \subseteq (\rho)^{++} \cap (\kappa)^{++}$. We get $(\varpi \vee \vartheta)^{++} \subseteq (\rho \vee \kappa)^{++}$. So that $(\mu)^+ \subseteq (\varpi \vee \vartheta)^+ \subseteq (\rho \vee \kappa)^+$, where $\rho \vee \kappa \in \mathcal{S} \vee \mathcal{T}$. Hence $\mu \in (\mathcal{S} \vee \mathcal{T})^{\blacklozenge}$. Thus $(\mathcal{S}^{\blacklozenge} \vee \mathcal{T}^{\blacklozenge})^{\blacklozenge} = (\mathcal{S} \vee \mathcal{T})^{\blacklozenge}$. □

Definition 3.4. If $\mathcal{S}^{\blacklozenge} = \mathcal{S}$, then an ideal $\mathcal{S}$ of $\mathcal{R}$ is known to as an $\mathcal{N}-$ideal.

It is noted that all ideals of $\mathcal{R}$ do not necessarily have to be $\mathcal{N}-$ideals.

Example 3.5. Consider a discrete ADL $\mathcal{D} = \{0, \varpi\}$ and a distributive lattice $\mathcal{R}' = \{0, \delta, \eta, \tau, 1\}$ whose Hasse-diagram is given below



Clearly, $\mathcal{R} = \mathcal{D} \times \mathcal{R}' = \{(0,0),(0,\delta),(0,\eta),(0,\tau),(0,1),(\varpi,0),(\varpi,\delta),(\varpi,\eta),(\varpi,\tau),(\varpi,1)\}$ is an ADL with zero element $(0,0)$. Take $\mathcal{R} = \{o,\upsilon,\phi,\varpi,\chi,\psi,\omega,\pi,e,\xi\}$, where $o=(0,0), \upsilon = (0,\delta), \phi = (0,\eta), \varpi = (0,\tau), \chi = (0,1), \psi = (\varpi,0), \omega = (\varpi,\delta), \pi = (\varpi,\eta), e = (\varpi,\tau), \xi = (\varpi,1)$. Define $\wedge$, $\vee$ of $\mathcal{R}$ as

$\wedge$	o	υ	ϕ	ϖ	χ	ψ	ω	π	e	ξ
o	o	o	o	o	o	o	o	o	o	o
υ	o	υ	o	υ	υ	o	υ	o	υ	υ
ϕ	o	o	ϕ	ϕ	ϕ	o	o	ϕ	ϕ	ϕ
ϖ	o	υ	ϕ	ϖ	ϖ	o	υ	ϕ	ϖ	ϖ
χ	o	υ	ϕ	ϖ	χ	o	υ	ϕ	ϖ	χ
ψ	o	o	o	o	o	ψ	ψ	ψ	ψ	ψ
ω	o	υ	o	υ	υ	ψ	ω	ψ	ω	ω
π	o	o	ϕ	ϕ	ϕ	ψ	ψ	π	π	π
e	o	υ	ϕ	ϖ	ϖ	ψ	ω	π	e	e
ξ	o	υ	ϕ	ϖ	χ	ψ	ω	π	e	ξ

$\vee$	o	υ	ϕ	ϖ	χ	ψ	ω	π	e	ξ
o	o	υ	ϕ	ϖ	χ	ψ	ω	π	e	ξ
υ	υ	υ	ϖ	ϖ	χ	ω	ω	e	e	ξ
ϕ	ϕ	ϖ	ϕ	ϖ	χ	π	e	π	e	ξ
ϖ	ϖ	ϖ	ϖ	ϖ	χ	e	e	e	e	ξ
χ	χ	χ	χ	χ	χ	ξ	ξ	ξ	ξ	ξ
ψ	ψ	ω	π	e	ξ	ψ	ω	π	e	ξ
ω	ω	ω	e	e	ξ	ω	ω	e	e	ξ
π	π	e	π	e	ξ	π	e	π	e	ξ
e	e	e	e	e	ξ	e	e	e	e	ξ
ξ	ξ	ξ	ξ	ξ	ξ	ξ	ξ	ξ	ξ	ξ

Here $(o)^+ = (\upsilon)^+ = (\phi)^+ = (\varpi)^+ = \{\xi\}$; $(\chi)^+ = \{\psi,\omega,\pi,e,\xi\}$; $(\psi)^+ = (\omega)^+ = (\pi)^+ = (e)^+ = \{\chi,\xi\}$ and $(\xi)^+ = \mathcal{R}$. Clearly, $\mathcal{S} = \{o,\psi\}$ is an ideal and $\mathcal{S} \subseteq \mathcal{S}^{\blacklozenge}$ but $\mathcal{S}^{\blacklozenge} \nsubseteq \mathcal{S}$. Hence $\mathcal{S}$ is not $\mathcal{N}-$ideal of $\mathcal{R}$. Clearly, $\mathcal{S}_1 = \{o,\upsilon,\phi,\varpi\}$ is an $\mathcal{N}-$ideal of $\mathcal{R}$.

Lemma 3.6. *For any ideal $\mathcal{S}$, $\mathcal{S}^{\blacklozenge}$ is the smallest $\mathcal{N}-$ideal containing $\mathcal{S}$.*

Proof. Clearly, we get that $\mathcal{S} \subseteq \mathcal{S}^{\blacklozenge}$. Let $\varpi,\vartheta \in \mathcal{S}^{\blacklozenge}$. Then there exist $\mu,\pi \in \mathcal{S}$ such that $(\varpi)^+ \subseteq (\mu)^+$ and $(\vartheta)^+ \subseteq (\pi)^+$. This implies $(\varpi)^{++} \cap (\vartheta)^{++} \subseteq (\mu)^{++} \cap (\pi)^{++}$ and hence $(\varpi \vee \vartheta)^{++} \subseteq (\mu \vee \pi)^{++}$. Therefore $(\varpi \vee \vartheta)^+ \subseteq (\mu \vee \pi)^+$. Since $\mathcal{S}$ is an ideal of $\mathcal{R}$ and $\mu,\pi \in \mathcal{S}$, we get $\mu \vee \pi \in \mathcal{S}$. Which implies $\varpi \vee \vartheta \in \mathcal{S}^{\blacklozenge}$. Let $\varpi \in \mathcal{S}^{\blacklozenge}$. Then there exist $\mu \in \mathcal{S}$ such that $(\varpi)^+ \subseteq (\mu)^+$. Therefore $(\varpi \wedge \tau)^+ \subseteq (\mu \wedge \tau)^+$, for any $\tau \in \mathcal{R}$. Since $\mathcal{S}$ is an ideal of $\mathcal{R}$, we get $\mu \wedge \tau \in \mathcal{S}$ and hence $\varpi \wedge \tau \in \mathcal{S}^{\blacklozenge}$. Therefore $\mathcal{S}^{\blacklozenge}$ is an ideal of $\mathcal{R}$ containing $\mathcal{S}$. Suppose $\mathcal{T}$ is an ideal of $\mathcal{R}$ containing $\mathcal{S}$. Let $\varpi \in \mathcal{S}^{\blacklozenge}$. Then there exist $\mu \in \mathcal{S}$ such that $(\varpi)^+ \subseteq (\mu)^+$. Since $\mathcal{S} \subseteq \mathcal{T}$, we get that $\varpi \in \mathcal{T}^{\blacklozenge}$. Hence $\mathcal{S}^{\blacklozenge}$ is the smallest $\mathcal{N}-$ideal containing $\mathcal{S}$. □

For every ideal $\mathcal{S}$ of $\mathcal{R}$, it could be noted that $\mathcal{S}^{\blacklozenge}$ is the smallest $\mathcal{N}-$ideal containing $\mathcal{S}$. In this particular case, the following is true.

Lemma 3.7. *Let $\mathcal{S}$ be any proper ideal of $\mathcal{R}$. Then $\mathcal{S}^{\blacklozenge}$ is a proper $\mathcal{N}-$ideal.*

Proof. Clearly, $\mathcal{S}^{\blacklozenge}$ is an $\mathcal{N}-$ideal. Suppose $\mathcal{R} = \mathcal{S}^{\blacklozenge}$. Since a maximal element $\xi \in \mathcal{R} = \mathcal{S}^{\blacklozenge}$, we have $\mathcal{R} = (\xi)^+ \subseteq (\varpi)^+$, $\varpi \in \mathcal{S}$. Thus $\varpi = \xi$ and $\xi \in \mathcal{S}$. Hence $\mathcal{S} = \mathcal{R}$. Which leads contradiction. Therefore $\mathcal{S}^{\blacklozenge}$ is proper. □

Theorem 3.8. *Every maximal ideal is an $\mathcal{N}-$ideal.*

This may not be the reverse of the above outcome.

Example 3.9. From the above example 3.5, we get $\mathcal{S} = \{o, \upsilon, \phi, \varpi\}$ is an $\mathcal{N}-$ideal of $\mathcal{R}$ but not a maximal ideal, as $\mathcal{T} = \{o, \upsilon, \phi, \varpi, \chi\}$ is a proper ideal containing $\mathcal{S}$.

Remark 3.10. Every $\mathcal{N}-$ideal need not be prime.

As we can see from the example 3.5, $\mathcal{S} = \{o, \upsilon, \phi, \varpi\}$ is an $\mathcal{N}-$ideal of $\mathcal{R}$ but not prime, as $\chi \wedge \omega = \upsilon \in \mathcal{S}$, and $\chi \notin \mathcal{S},\ \omega \notin \mathcal{S}$.

Remark 3.11. Not all prime ideals must be $\mathcal{N}-$ideals.

As we can see from the example 3.5, a prime ideal $\mathcal{S} = \{o, \upsilon, \phi, \varpi, \chi\}$ of $\mathcal{R}$ but not an $\mathcal{N}-$ideal.

Remark 3.12. Not all minimal prime ideals must be $\mathcal{N}-$ideals.

As we can see from the example 3.5, a prime ideal $\mathcal{S} = \{o, \upsilon, \psi, \omega\}$ is minimal but not an $\mathcal{N}-$ideal.

The set of $\mathcal{N}-$ideals of $\mathcal{R}$ can be represented as $\mathcal{N}_{\mathcal{I}}(\mathcal{R})$. It is evident that the lattice $\mathcal{N}_{\mathcal{I}}(\mathcal{R})$ is distributive.

Theorem 3.13. *$\mathcal{N}_{\mathcal{I}}(\mathcal{R})$ can be a distributive lattice with the operations $\mathcal{S}^{\blacklozenge} \cap \mathcal{T}^{\blacklozenge} = (\mathcal{S} \cap \mathcal{T})^{\blacklozenge}$ and $\mathcal{S} \sqcup \mathcal{T} = (\mathcal{S} \vee \mathcal{T})^{\blacklozenge}$, for any $\mathcal{S}, \mathcal{T} \in \mathcal{N}_{\mathcal{I}}(\mathcal{R})$.*

Proof. Let $\mathcal{S},\mathcal{T} \in \mathcal{N}_{\mathcal{I}}(\mathcal{R})$. Clearly, $(\mathcal{S}\cap\mathcal{T})^{\blacklozenge}$ is the infimum of $\mathcal{S}$ and $\mathcal{T}$ in $\mathcal{N}_{\mathcal{I}}(\mathcal{R})$. Also, we have $(\mathcal{S}\vee\mathcal{T})^{\blacklozenge}$ is an upper bound of $\mathcal{S}$ and $\mathcal{T}$. Let $\mathcal{U} \in \mathcal{N}_{\mathcal{I}}(\mathcal{R})$ such that $\mathcal{S}^{\blacklozenge} \subseteq \mathcal{U}, \mathcal{T}^{\blacklozenge} \subseteq \mathcal{U}$ and $\mu \in (\mathcal{S}\vee\mathcal{T})^{\blacklozenge}$. Then $(\mu)^{+} \subseteq (\varpi)^{+}$ for some $\varpi \in \mathcal{S}\vee\mathcal{T} \subseteq \mathcal{U}$. Therefore $\mu \in \mathcal{U}^{\blacklozenge} = \mathcal{U}$ (since $\mathcal{U} \in \mathcal{N}_{\mathcal{I}}(\mathcal{R})$). Thus $(\mathcal{S}\vee\mathcal{T})^{\blacklozenge} = \mathcal{S}\sqcup\mathcal{T}$ is the supremum of $\mathcal{S}$ and $\mathcal{T}$ in $\mathcal{N}_{\mathcal{I}}(\mathcal{R})$. Let $\mathcal{S},\mathcal{T},\mathcal{U} \in \mathcal{N}_{\mathcal{I}}(\mathcal{R})$. Then $\mathcal{S}\cap(\mathcal{T}\sqcup\mathcal{U}) = \mathcal{S}^{\blacklozenge}\cap(\mathcal{T}\vee\mathcal{U})^{\blacklozenge} = (\mathcal{S}\cap(\mathcal{T}\vee\mathcal{U}))^{\blacklozenge} = \{(\mathcal{S}\cap\mathcal{T})\vee(\mathcal{S}\cap\mathcal{U})\}^{\blacklozenge} = (\mathcal{S}\cap\mathcal{T})\sqcup(\mathcal{S}\cap\mathcal{U})$. Therefore $(\mathcal{N}_{\mathcal{I}}(\mathcal{R}),\wedge,\sqcup)$ is a distributive lattice with the greatest element $\mathcal{R}^{\blacklozenge} = \mathcal{R} = (\mathcal{M}_{Max.elts}]^{\blacklozenge}$. □

Theorem 3.14. *There is an epimorphism from $\mathcal{I}(\mathcal{R})$ onto $\mathcal{N}_{\mathcal{I}}(\mathcal{R})$.*

Proof. Let $\mathcal{S},\mathcal{T} \in \mathcal{I}(\mathcal{R})$. Define a map $\Phi : \mathcal{I}(\mathcal{R}) \to \mathcal{N}_{\mathcal{I}}(\mathcal{R})$ by $\Phi(\mathcal{S}) = \mathcal{S}^{\blacklozenge}$. Then $\Phi(\mathcal{S})\cap\Phi(\mathcal{T}) = \mathcal{S}^{\blacklozenge}\cap\mathcal{T}^{\blacklozenge} = (\mathcal{S}\cap\mathcal{T})^{\blacklozenge} = \Phi(\mathcal{S}\cap\mathcal{T})$ and $\Phi(\mathcal{S})\sqcup\Phi(\mathcal{T}) = \mathcal{S}^{\blacklozenge}\sqcup\mathcal{T}^{\blacklozenge} = (\mathcal{S}^{\blacklozenge}\vee\mathcal{T}^{\blacklozenge})^{\blacklozenge} = (\mathcal{S}\vee\mathcal{T})^{\blacklozenge} = \Phi(\mathcal{S}\vee\mathcal{T})$. Therefore Φ is a homomorphism. Since $\mathcal{N}_{\mathcal{I}}(\mathcal{R}) \subseteq \mathcal{I}(\mathcal{R}), \Phi$ is an onto homomorphism. □

Definition 3.15. An ideal $\mathcal{S}$ of $\mathcal{R}$ is known an $\mathcal{N}-$let, if $\mathcal{S} = (\varpi]^{\blacklozenge}$, for some $\varpi \in \mathcal{R}$.

Theorem 3.16. *Every $\mathcal{N}-$let is an $\mathcal{N}-$ideal.*

Proof. Let $\mu \in \mathcal{R}$ and $\nu \in (\mu]^{\blacklozenge\blacklozenge}$. Then $(\nu)^{+} \subseteq (\varpi)^{+}$, for some $\varpi \in (\mu]^{\blacklozenge}$ and $(\varpi)^{+} \subseteq (\mu)^{+}$. Therefore $(\nu)^{+} \subseteq (\mu)^{+}$ and hence $\nu \in (\mu]^{\blacklozenge}$. Thus $(\mu]^{\blacklozenge}$ is an $\mathcal{N}-$ideal. □

Lemma 3.17. *For any $\varpi, \vartheta \in \mathcal{R}$, we have*

(i) $\varpi \leq \vartheta \Rightarrow (\varpi]^{\blacklozenge} \subseteq (\vartheta]^{\blacklozenge}$,

(ii) $\varpi \in (\vartheta]^{\blacklozenge} \Rightarrow (\varpi]^{\blacklozenge} \subseteq (\vartheta]^{\blacklozenge}$,

(iii) $(\varpi]^{\blacklozenge} = E \Leftrightarrow \varpi \in E$,

(iv) $(\varpi]^{\blacklozenge} = \mathcal{R} \Leftrightarrow \varpi \in \mathcal{M}_{Max.elts}$,

(v) *For any* $m \in \mathcal{M}_{Max.elts}$, $(m]^{\blacklozenge} = \mathcal{R}$,

(vi) $(\varpi]^{\blacklozenge} \cap (\vartheta]^{\blacklozenge} = (\varpi\wedge\vartheta]^{\blacklozenge}$.

Proof. (i) Suppose $\varpi \leq \vartheta$. Then $(\varpi] \subseteq (\vartheta]$. Therefore $(\varpi]^{\blacklozenge} \subseteq (\vartheta]^{\blacklozenge}$ (by lemma 3.3(i)).
(ii) Suppose $\varpi \in (\vartheta]^{\blacklozenge}$. Then $(\varpi] \subseteq (\vartheta]^{\blacklozenge}$. Therefore $(\varpi]^{\blacklozenge} \subseteq (\vartheta]^{\blacklozenge\blacklozenge} = (\vartheta]^{\blacklozenge}$ (since $(\vartheta]^{\blacklozenge}$ is $\mathcal{N}-$let).
(iii) Suppose $(\varpi]^{\blacklozenge} = E$. Then $\varpi \in (\varpi]^{\blacklozenge} = E$. Let $e \in E$, then $(e]^{\blacklozenge} = \{\mu \in \mathcal{R} \mid (\mu)^{+} \subseteq (e)^{+} = \mathcal{M}_{Max.elts}\} = E$.
(iv) Suppose $(\varpi]^{\blacklozenge} = \mathcal{R}$. Then $m \in \mathcal{R} = (\varpi]^{\blacklozenge}$, where $m \in \mathcal{M}_{Max.elts}$. Therefore $\mathcal{R} = (m)^{+} \subseteq (\varpi)^{+}$. Hence $\varpi \in \mathcal{M}_{Max.elts}$. It's trivial to see that the converse holds.
(v) It clear from (iii).
(vi) $(\varpi]^{\blacklozenge} \cap (\vartheta]^{\blacklozenge} = ((\varpi] \cap (\vartheta])^{\blacklozenge} = (\varpi\wedge\vartheta]^{\blacklozenge}$ (by lemma 3.3). □

Lemma 3.18. *For every $\varpi, \vartheta \in \mathcal{R}$, the following holds*

(i) $\varpi \vee \vartheta \in \mathcal{M}_{Max.elts}$ *implies* $(\varpi]^{\blacklozenge} \vee (\vartheta]^{\blacklozenge} = \mathcal{R}$,

(ii) $\varpi \wedge \vartheta \in E \Leftrightarrow (\varpi]^{\blacklozenge} \cap (\vartheta]^{\blacklozenge} = E$,

(iii) $\varpi \notin \mathcal{M}_{Max.elts} \Rightarrow (\varpi)^{+} \cap (\varpi]^{\blacklozenge} = \phi$,

(iv) $(\varpi)^{+} = (\vartheta)^{+} \Leftrightarrow (\varpi]^{\blacklozenge} = (\vartheta]^{\blacklozenge}$,

(v) $(\varpi]^{\blacklozenge} = (\vartheta]^{\blacklozenge} \Rightarrow (\varpi \wedge \sigma]^{\blacklozenge} = (\vartheta \wedge \sigma]^{\blacklozenge}$ *for all* $\sigma \in \mathcal{R}$,

(vi) $(\varpi]^{\blacklozenge} = (\vartheta]^{\blacklozenge} \Rightarrow (\varpi \vee \sigma]^{\blacklozenge} = (\vartheta \vee \sigma]^{\blacklozenge}$ *for all* $\sigma \in \mathcal{R}$.

Proof. (i) Suppose that $\varpi \vee \vartheta$ is maximal element say m. Then $\mathcal{R} = (m] = (\varpi \vee \vartheta] = (\varpi] \vee (\vartheta] \subseteq (\varpi]^{\blacklozenge} \vee (\vartheta]^{\blacklozenge} \subseteq \mathcal{R}$. Therefore $(\varpi]^{\blacklozenge} \vee (\vartheta]^{\blacklozenge} = \mathcal{R}$.

(ii) As Lemma 3.17(iii) shows, it is obvious.

(iii) Suppose $\varpi \notin \mathcal{M}_{Max.elts}$. Let $\mu \in (\varpi)^{+} \cap (\varpi]^{\blacklozenge}$. Then $(\mu)^{+} \subseteq (\varpi)^{+}$ and $\mu \vee \varpi \in \mathcal{M}_{Max.elts}$. Therefore $\varpi \in (\mu)^{+} \subseteq (\varpi)^{+}$. Hence $\varpi \vee \varpi \in \mathcal{M}_{Max.elts}$ which leads contradiction. Therefore $(\varpi)^{+} \cap (\varpi]^{\blacklozenge} = \phi$.

(iv) Assume $(\varpi)^{+} = (\vartheta)^{+}$. Then $\varpi \in (\vartheta]^{\blacklozenge}$ and $\vartheta \in (\varpi]^{\blacklozenge}$. Therefore $(\varpi]^{\blacklozenge} \subseteq (\vartheta]^{\blacklozenge}$ and $(\vartheta]^{\blacklozenge} \subseteq (\varpi]^{\blacklozenge}$. Hence $(\varpi]^{\blacklozenge} = (\vartheta]^{\blacklozenge}$. Conversely assume $(\varpi]^{\blacklozenge} = (\vartheta]^{\blacklozenge}$. Then $\varpi \in (\vartheta]^{\blacklozenge}$ and $\vartheta \in (\varpi]^{\blacklozenge}$. Therefore $(\varpi)^{+} \subseteq (\vartheta)^{+}$ and $(\vartheta)^{+} \subseteq (\varpi)^{+}$ and hence $(\varpi)^{+} = (\vartheta)^{+}$.

(v) Assume $(\varpi]^{\blacklozenge} = (\vartheta]^{\blacklozenge}$. For any $\nu \in \mathcal{R}, \nu \in (\varpi \wedge \sigma)^{+} \Leftrightarrow \nu \vee (\varpi \wedge \sigma) \in \mathcal{M}_{Max.elts} \Leftrightarrow (\nu \vee \varpi) \wedge (\nu \vee \sigma) \in \mathcal{M}_{Max.elts} \Leftrightarrow (\nu \vee \varpi), (\nu \vee \sigma) \in \mathcal{M}_{Max.elts} \Leftrightarrow \nu \in (\varpi)^{+} = (\vartheta)^{+}$ and $(\nu \vee \sigma) \in \mathcal{M}_{Max.elts} \Leftrightarrow (\nu \vee \vartheta), (\nu \vee \sigma) \in \mathcal{M}_{Max.elts} \Leftrightarrow (\nu \vee \vartheta) \wedge (\nu \vee \sigma) \in \mathcal{M}_{Max.elts} \Leftrightarrow \nu \vee (\vartheta \wedge c) \in \mathcal{M}_{Max.elts} \Leftrightarrow \nu \in (\vartheta \wedge \sigma)^{+}$. Therefore $(\varpi \wedge \sigma)^{+} = (\vartheta \wedge \sigma)^{+}$. By (iv), we get $(\varpi \wedge \sigma]^{\blacklozenge} = (\vartheta \wedge \sigma]^{\blacklozenge}$.

(vi) Assume $(\varpi]^{\blacklozenge} = (\vartheta]^{\blacklozenge}$. For any $\nu \in \mathcal{R}, \nu \in (\varpi \vee \sigma)^{+} \Leftrightarrow \nu \vee \varpi \vee \sigma \in \mathcal{M}_{Max.elts} \Leftrightarrow \nu \vee \sigma \vee \varpi \in \mathcal{M}_{Max.elts} \Leftrightarrow \nu \vee \sigma \in (\varpi)^{+} = (\vartheta)^{+} \Leftrightarrow \nu \vee \sigma \vee \vartheta \in \mathcal{M}_{Max.elts} \Leftrightarrow \nu \vee \vartheta \vee \sigma \in \mathcal{M}_{Max.elts} \Leftrightarrow \nu \in (\vartheta \vee \sigma)^{+} \Leftrightarrow (\varpi \vee \sigma)^{+} = (\vartheta \vee \sigma)^{+}$. By (iv), we get $(\varpi \vee \sigma]^{\blacklozenge} = (\vartheta \vee \sigma]^{\blacklozenge}$. □

Theorem 3.19. *The following constraints are identical for every ideal $\mathcal{S}$ in $\mathcal{R}$:*

(i) $\mathcal{S}$ *is an* $\mathcal{N}-$*ideal,*

(ii) *For* $\mu \in \mathcal{R}, \mu \in \mathcal{S}$ *implies* $(\mu]^{\blacklozenge} \subseteq \mathcal{S}$,

(iii) *For* $\mu, \pi \in \mathcal{R}, (\mu)^{+} = (\pi)^{+}$ *and* $\mu \in \mathcal{S}$ *implies* $\pi \in \mathcal{S}$,

(iv) *For* $\mu, \pi \in \mathcal{R}, (\mu]^{\blacklozenge} = (\pi]^{\blacklozenge}$ *and* $\mu \in \mathcal{S}$ *implies* $\pi \in \mathcal{S}$,

(v) $\mathcal{S} = \bigcup\limits_{\mu \in \mathcal{S}} (\mu]^{\blacklozenge}$.

Proof. (i) $\Rightarrow$ (ii): Suppose (i). Let $\mu \in \mathcal{S}$. Therefore $(\mu] \subseteq \mathcal{S}$ and hence $(\mu]^{\blacklozenge} \subseteq \mathcal{S}^{\blacklozenge} = \mathcal{S}$. Thus $(\mu]^{\blacklozenge} \subseteq \mathcal{S}$.

(ii) $\Rightarrow$ (iii): Suppose (ii). Let $\mu, \pi \in \mathcal{R}$ with $(\mu)^{+} = (\pi)^{+}$ and $\mu \in \mathcal{S}$. Then, by Lemma 3.18, $(\pi]^{\blacklozenge} = (\mu]^{\blacklozenge} \subseteq \mathcal{S}$. Hence $\pi \in \mathcal{S}$.

(iii) $\Rightarrow$(iv): It is obvious by Lemma 3.18.

(iv) $\Rightarrow$ (v): Suppose (iv). Let $\mu \in \mathcal{S}$. Then $\mu \in (\mu]^{\blacklozenge}$. Hence $\mathcal{S} \subseteq \bigcup_{\mu\in\mathcal{S}} (\mu]^{\blacklozenge}$. Let $\mu \in \mathcal{S}$ and $\pi \in (\mu]^{\blacklozenge}$, then $(\pi]^{\blacklozenge} \subseteq (\mu]^{\blacklozenge}$. Therefore $(\pi]^{\blacklozenge} = (\pi]^{\blacklozenge} \cap (\mu]^{\blacklozenge} = (\pi \wedge \mu]^{\blacklozenge}$ and $\pi \wedge \mu \in \mathcal{S}$. By our assumption, $\pi \in \mathcal{S}$. Then $(\mu]^{\blacklozenge} \subseteq \mathcal{S}$ and which leads $\bigcup_{\mu\in\mathcal{S}} (\mu]^{\blacklozenge} \subseteq \mathcal{S}$.

(v) $\Rightarrow$ (i): Suppose (v). Let $\mu \in \mathcal{S}^{\blacklozenge}$. Then $(\mu)^+ \subseteq (\varpi)^+$, for some $\varpi \in \mathcal{S}$. So that $\mu \in (\varpi]^{\blacklozenge}$ and which gives $\mu \in \bigcup_{\pi\in\mathcal{S}} (\pi]^{\blacklozenge} = \mathcal{S}$. Thus $\mathcal{S}$ is an $\mathcal{N}-$ideal. □

We will write $\mathcal{N}_{\mathcal{I}}^{\blacklozenge}(\mathcal{R})$ to represent the set of $\mathcal{N}-$lets of $\mathcal{R}$.

Theorem 3.20. *$(\mathcal{N}_{\mathcal{I}}^{\blacklozenge}(\mathcal{R}), \cap, \sqcup)$ is a sublattice of $\mathcal{N}_{\mathcal{I}}(\mathcal{R})$, where the largest element in $\mathcal{N}_{\mathcal{I}}^{\blacklozenge}(\mathcal{R})$ is $(\mathcal{M}_{Max.elts})^{\blacklozenge}$. In addition, if and only if $\mathcal{R}$ has a dual dense element, $\mathcal{N}_{\mathcal{I}}^{\blacklozenge}(\mathcal{R})$ has the smallest element.*

Proof. By theorem 3.13, $(\mathcal{N}_{\mathcal{I}}^{\blacklozenge}(\mathcal{R}), \cap, \sqcup)$ is a sublattice of a distributive lattice $(\mathcal{N}_{\mathcal{I}}(\mathcal{R}), \cap, \sqcup)$ with the greatest element $(\mathcal{M}_{Max.elts}]^{\blacklozenge} = \mathcal{R}$. Assume that $e \in E$. Let $\mu \in (e]^{\blacklozenge}$. Then $(\mu)^+ \subseteq (e)^+ = \mathcal{M}_{Max.elts} \subseteq (\varpi)^+$ for every $\varpi \in \mathcal{R}$. Which gives $e \in (\varpi]^{\blacklozenge}$ for all $\varpi \in \mathcal{R}$. Hence $(e]^{\blacklozenge} \subseteq (\varpi]^{\blacklozenge}$ for every $\varpi \in \mathcal{R}$. Thus $(e]^{\blacklozenge}$ is the smallest element in $\mathcal{N}_{\mathcal{I}}^{\blacklozenge}(\mathcal{R})$. Conversely assume that $\mathcal{N}_{\mathcal{I}}^{\blacklozenge}(\mathcal{R})$ has the smallest element, say $(\varpi]^{\blacklozenge}$ for some $\varpi \in \mathcal{R}$. Let $\mu \in (\varpi)^+$. Then $\mu \vee \varpi \in \mathcal{M}_{Max.elts}$. Therefore $(\mu]^{\blacklozenge} \vee (\varpi]^{\blacklozenge} = (\mu]^{\blacklozenge} = \mathcal{R}$. Hence $\mu \in E$. □

Definition 3.21. $\mathcal{R}$ is called disjunctive, if for every $\mu, \pi \in \mathcal{R}, \mu \neq \pi$ implies $(\mu)^+ \neq (\pi)^+$.

Theorem 3.22. *Every ideal is a $\mathcal{N}-$ideal when $\mathcal{R}$ is a disjunctive ADL.*

Proof. Suppose that an ideal $\mathcal{S}$ of $\mathcal{R}$ is not an $\mathcal{N}-$ideal. Then $(\mu]^{\blacklozenge} = (\pi]^{\blacklozenge}, \mu \in \mathcal{S}$ and $\pi \notin \mathcal{S}$, for some $\mu, \pi \in \mathcal{R}$. Therefore $(\mu)^+ = (\pi)^+$. Since $\mathcal{R}$ is disjunctive, $\mu = \pi$. Hence $\pi \in \mathcal{S}$, it gives a contradiction. Ths $\mathcal{S}$ is an $\mathcal{N}-$ideal. □

The above theorem's converse need not hold.

Example 3.23. Let $\mathcal{R}_1 = \{0, \varpi\}$ and $\mathcal{R}_2 = \{0, \vartheta_1, \vartheta_2\}$ be two discrete ADLs. Then $\mathcal{R}_1 \times \mathcal{R}_2 = \{(0,0), (0,\vartheta_1), (0,\vartheta_2), (\varpi,0), (\varpi,\vartheta_1), (\varpi,\vartheta_2)\}$. Take $L = \{0, \upsilon, \phi, \varpi, \xi, \zeta\}$, where $o = (0,0), \upsilon = (0,\vartheta_1), \phi = (0,\vartheta_2), \varpi = (\varpi,0), \xi = (\varpi,\vartheta_1), \zeta = (\varpi,\vartheta_2)$. Define $\wedge,\ \vee$ on $\mathcal{R}$ as follows:

$\wedge$	o	υ	ϕ	ϖ	ξ	ζ
o	o	o	o	o	o	o
υ	o	υ	ϕ	o	υ	ϕ
ϕ	o	υ	ϕ	o	υ	ϕ
ϖ	o	o	o	ϖ	ϖ	ϖ
ξ	o	υ	ϕ	ϖ	ξ	ζ
ζ	o	υ	ϕ	ϖ	ξ	ζ

$\vee$	o	υ	ϕ	ϖ	ξ	ζ
o	o	υ	ϕ	ϖ	ξ	ζ
υ	υ	υ	υ	ξ	ξ	ξ
ϕ	ϕ	ϕ	ϕ	ζ	ζ	ζ
ϖ	ϖ	ξ	ζ	ϖ	ξ	ζ
ξ	ξ	ξ	ξ	ξ	ξ	ξ
ζ	ζ	ζ	ζ	ζ	ζ	ζ

Then $(\mathcal{R}, \wedge, \vee, o)$ is an ADL. It is noticed that every ideal is an $\mathcal{N}-$ideal and $(\xi)^+ = (\zeta)^+$ but $\xi \neq \zeta$. Hence $\mathcal{R}$ is not a disjunctive ADL.

Definition 3.24. A relation ϖ on $\mathcal{R}$ define by $\varpi = \{(\mu, \pi) \in \mathcal{R} \times \mathcal{R} | (\mu]^{\blacklozenge} = (\pi]^{\blacklozenge}\}$.

ϖ is a congruence relation on $\mathcal{R}$, as may be easily observed.

Theorem 3.25. *The quotient ADL $\mathcal{R}/\varpi$ forms a distributive lattice with the operations $\mu/\varpi \wedge \pi/\varpi = (\mu \wedge \pi)/\varpi$ and $\mu/\varpi \vee \pi/\varpi = (\mu \vee \pi)/\varpi$. Moreover the least element is m/ϖ and the greatest element is e/ϖ.*

Definition 3.26. ADL $\mathcal{R}$ is referred to as soft relatively complemented if, for any $\varpi, \vartheta \in \mathcal{R}$, there is $\mu \in \mathcal{R}$ such that $\varpi \vee \mu \in \mathcal{M}_{Max.elts}$, $(\varpi \wedge \mu)^+ = (\varpi \wedge \vartheta)^+$.

Example 3.27. Let $\mathcal{D}_1 = \{0, \varpi\}$ and $\mathcal{D}_2 = \{0, \vartheta_1, \vartheta_2\}$ be two discrete ADLs. Then $\mathcal{D}_1 \times \mathcal{D}_2 \setminus \{(\varpi, 0)\} = \{(0,0), (0, \vartheta_1), (0, \vartheta_2), (\varpi, \vartheta_1), (\varpi, \vartheta_2)\}$. Take $\mathcal{R} = \{o, \upsilon, \phi, \xi, \zeta\}$, where $o = (0,0), \upsilon = (0, \vartheta_1), \phi = (0, \vartheta_2), \xi = (\varpi, \vartheta_1), \zeta = (\varpi, \vartheta_2)$. Define $\wedge$ and $\vee$ on $\mathcal{R}$ as follows:

$\wedge$	o	υ	ϕ	ξ	ζ
o	o	o	o	o	o
υ	o	υ	ϕ	υ	ϕ
ϕ	o	υ	ϕ	υ	ϕ
ξ	o	υ	ϕ	ξ	ζ
ζ	o	υ	ϕ	ξ	ζ

$\vee$	o	υ	ϕ	ξ	ζ
o	o	υ	ϕ	ξ	ζ
υ	υ	υ	υ	ξ	ξ
ϕ	ϕ	ϕ	ϕ	ζ	ζ
ξ	ξ	ξ	ξ	ξ	ξ
ζ	ζ	ζ	ζ	ζ	ζ

Then $(\mathcal{R}, \wedge, \vee)$ is an ADL, but not a lattice. Because, $\upsilon \wedge \zeta = \phi \neq \upsilon = \zeta \wedge \upsilon$. Take $\mathcal{M}_{Max.elts} = \{\xi, \zeta\}$. For any $\varpi, \vartheta \in \mathcal{R}$, define μ by

$$\mu = \begin{cases} m_1, & \text{if } \varpi \notin \mathcal{M}_{Max.elts}, \\ \vartheta, & \text{if } \varpi \in \mathcal{M}_{Max.elts}. \end{cases}$$

Then $\mu \in \mathcal{R},\ \varpi \vee \mu \in \mathcal{M}_{Max.elts}$ and $(\varpi \wedge \mu)^+ = (\varpi \wedge \vartheta)^+$. Therefore $\mathcal{R}$ is soft relatively complemented.

Definition 3.28. In $\mathcal{R}$, a filter $\mathcal{F}$ is called as dual dense complemented in $\mathcal{R}$ if there is a filter $\mathcal{G}$ in $\mathcal{R}$ such that $\mathcal{F} \cap \mathcal{G} = \mathcal{M}_{Max.elts}$ and $\mathcal{F} \vee \mathcal{G}$ is a filter generated by a dual dense element in $\mathcal{R}$.

Theorem 3.29. *The following conditions are equivalent in an ADL $\mathcal{R}$*

(i) *$\mathcal{R}$ is a soft relatively complemented,*

(ii) *$(\mathcal{N}_{\mathcal{I}}^{\blacklozenge}(\mathcal{R}), \cap, \sqcup, E, \mathcal{R})$ is a Boolean algebra,*

(iii) *For any $m \in \mathcal{M}_{Max.elts}$ and $e \in E$, $(\mathcal{R}/\varpi, \wedge, \vee, m/\varpi, e/\varpi)$ is a Boolean algebra,*

(iv) *Every principal filter of $\mathcal{R}$ is dual dense complemented.*

Proof. (i) $\Rightarrow$ (ii): Assume (i). Let $\mu \in \mathcal{R}$ and $e \in E$. Then there exists $\pi \in \mathcal{R}$ such that $\mu \vee \pi \in \mathcal{M}_{Max.elts}$ and $(\mu \wedge \pi)^+ = (\mu \wedge e)^+ = \mathcal{M}_{Max.elts}$. Therefore $\mu \wedge \pi$ is dual dense. Now, $(\mu]^{\blacklozenge} \cap (\pi]^{\blacklozenge} = ((\mu] \cap (\pi])^{\blacklozenge} = (\mu \wedge \pi]^{\blacklozenge} = E$ and $(\mu]^{\blacklozenge} \sqcup (\pi]^{\blacklozenge} = ((\mu] \vee (\pi])^{\blacklozenge} = (\mu \vee \pi]^{\blacklozenge} = \mathcal{R}$. Therefore $\mathcal{N}_{\mathcal{I}}^{\blacklozenge}(\mathcal{R})$ is a Boolean algebra.
(ii) $\Rightarrow$ (iii): Assume (ii). Let $\mu \in \mathcal{R}$. Then there exists $\pi \in \mathcal{R}$ such that $(\mu]^{\blacklozenge} \cap (\pi]^{\blacklozenge} = E$ and $(\mu]^{\blacklozenge} \sqcup (\pi]^{\blacklozenge} = \mathcal{R}$. It leads $(\mu \wedge \pi]^{\blacklozenge} = E$ and $\mathcal{R} = (\mu \vee \pi]^{\blacklozenge}$. Thus $\mu \wedge \pi \in E$ and $\mu \vee \pi \in \mathcal{M}_{Max.elts}$ and hence $\mu/\varpi \vee \pi/\varpi = (\mu \vee \pi)/\varpi$ is a maximal element in $\mathcal{R}/\varpi$ and $\mu/\varpi \wedge \pi/\varpi = (\mu \wedge \pi)/\varpi$ is a dual dense element in $\mathcal{R}/\varpi$. Thus $\mathcal{R}/\varpi$ is a Boolean algebra.
(iii) $\Rightarrow$ (iv): Assume (iii). Let $\mu \in \mathcal{R}$. Then there exists $\pi \in \mathcal{R}$ such that $\mu/\varpi \vee \pi/\varpi = (\mu \vee \pi)/\varpi = m/\varpi$ and $\mu/\varpi \wedge \pi/\varpi = (\mu \wedge \pi)/\varpi = e/\varpi$. It leads $\mu \vee \pi \in \mathcal{M}_{Max.elts}$, $\mu \wedge \pi \in E$. Hence $[\mu) \cap [\pi) = [\mu \vee \pi) = \mathcal{M}_{Max.elts}$ and $[\mu) \vee [\pi) = [\mu \wedge \pi)$ is a filter generated by a dual dense element $\mu \wedge \pi$. Therefore $[\mu)$ is dual dense complemented.
(iv) $\Rightarrow$ (i) Let $\varpi, \vartheta \in \mathcal{R}$. Then there is $\sigma, e \in \mathcal{R}$ such that $[\varpi) \cap [\sigma) = \mathcal{M}_{Max.elts} = [\vartheta) \cap [e)$ and $[\varpi) \vee [\sigma)$ and $[\vartheta) \vee [e)$ are the principal filters generated by dual dense elements. Hence $\varpi \vee \sigma, \vartheta \vee e \in \mathcal{M}_{Max.elts}$ and $\varpi \wedge \sigma, \vartheta \wedge e \in E$. Take $\mu = \sigma \vee \vartheta$. Then $\varpi \vee \mu = \varpi \vee \sigma \vee \vartheta \in \mathcal{M}_{Max.elts}$ (since $\varpi \vee \sigma \in \mathcal{M}_{Max.elts}$) and $(\varpi \wedge \mu) \vee (\varpi \wedge \vartheta) = \varpi \wedge (\mu \vee \vartheta) = \varpi \wedge (\sigma \vee \vartheta \vee \vartheta) = \varpi \wedge \mu$. So that $(\varpi \wedge \vartheta)^+ \subseteq (\varpi \wedge \mu)^+$. Now, for $\nu \in \mathcal{R}, \nu \in (\varpi \wedge \mu)^+ \Rightarrow \nu \vee (\varpi \wedge \mu) \in \mathcal{M}_{Max.elts} \Rightarrow \nu \vee \varpi \in \mathcal{M}_{Max.elts}$ and $\nu \vee \sigma \vee \vartheta \in \mathcal{M}_{Max.elts} \Rightarrow \nu \vee \vartheta \vee (\varpi \wedge \sigma) \in \mathcal{M}_{Max.elts} \Rightarrow \nu \vee \vartheta \in \mathcal{M}_{Max.elts}$ (since $\varpi \wedge \sigma \in E$) $\Rightarrow \nu \vee (\varpi \wedge \vartheta) \in \mathcal{M}_{Max.elts} \Rightarrow \nu \in (\varpi \wedge \vartheta)^+$. Which gives $(\varpi \wedge \mu)^+ \subseteq (\varpi \wedge \vartheta)^+$ and hence $(\varpi \wedge \mu)^+ = (\varpi \wedge \vartheta)^+$. Therefore $\mathcal{R}$ is soft relatively complemented. □

Theorem 3.30. *The following are equivalent if $\mathcal{R}$ is an ADL where each dual dense element is zero*

(i) *$\mathcal{R}$ is $E-$complemented,*

(ii) *$\mathcal{R}$ is soft relatively complemented,*

(iii) *$(\mathcal{N}_{\mathcal{S}}^{\blacklozenge}(\mathcal{R}), \cap, \sqcup, \mathcal{M}_{Max.elts}, \mathcal{R})$ is a Boolean algebra,*

(iv) *$(\mathcal{R}/\varpi, \wedge, \vee, 0/\varpi, m/\varpi)$ is a Boolean algebra,*

(v) *Each principal filter of $\mathcal{R}$ is complemented.*

Theorem 3.31. *For any $\varpi, \vartheta \in \mathcal{R}$ there is $\mu \in \mathcal{R}$ such that $\varpi \vee \mu \in \mathcal{M}_{Max.elts}$ and $(\varpi \vee \mu]^{\blacklozenge} = (\varpi \vee \vartheta]^{\blacklozenge}$ if and only if $\mathcal{R}$ is soft relatively complemented.*

4. Acknowledgments

The authors sincerely thank the referee for their valuable suggestions, which have greatly improved this work.

References

[1] G. Birkhoff, *Lattice Theory*, 3rd ed., American Mathematical Society, colloquium publications, New York, 1967.

[2] G. Gratzer, *General Lattice Theory*, Academic Press, inc. New York, 1978.

[3] Y. S. Pawar, *The space of maximal ideals in an almost distributive lattice*, Int. Math. Forum., **6** No. 28 (2011) 1387-1396.

[4] Y. S. Pawar and I. A. Shaikh, *On prime, minimal prime and annihilator ideal in an almost distributive lattices*, Eur. J. Pure Appl. Math., **6** No. 1 (2013) 107-118.

[5] N. Rafi and R. K. Bandaru, *μ-filters of almost distributive lattices*, Chamchuri J. Math., **10** (2018) 53-65.

[6] N. Rafi, Y. Monikarchana and R. K. Bandaru, *ν-ideals of Almost Distributive Lattices*, Accepted for Publication in Palestine Journal of Mathematics.

[7] G. C. Rao, *Almost Distributive Lattices*, Doctoral Thesis, Department of Mathematics, Andhra University, Visakhapatnam, 1980.

[8] G. C. Rao and S. R. Kumar, *Minimal prime ideals in almost distributive lattices*, Int. J. Contemp. Sci., **4** No. 10 (2009) 475-484.

[9] G. C. Rao and M. S. Rao, *Annulets in almost distributive lattices*, Eur. J. Pure Appl. Math., **2** No. 1 (2009) 58-72.

[10] U. M. Swamy and G. C. Rao, *Almost distributive lattices*, Jour. Aust. Math. Soc., **31** No. 1 (1981) 77-91.

Rafi Noorbhasha
Department of Mathematics,
Bapatla Engineering College, Bapatla,
Andhra Pradesh, India-522 102.
rafimaths@gmail.com
Teshale Amare Natnael
Department of Mathematics,
University of Gondar,
Gondar, Ethiopia.
yenatnaelteshale@gmail.com

Monikarchana Yathaluru

Department of Mathematics,

Mohan Babu University,

A. Rangampet, Tirupati,

Andhra Pradesh, India-517 102.

285archana@gmail.com