

Research Paper

DOUBLE DERIVATIONS OF *n*-HOM-LIE COLOR ALGEBRAS

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ABSTRACT. This paper aims to study the double derivation algebra $\mathcal{D}(\mathcal{L})$ of *n*-Hom-Lie color algebra \mathcal{L} and describe the relation between $\mathcal{D}(\mathcal{L})$ and the usual derivation Hom-Lie color algebra $Der(\mathcal{L})$. We prove that the inner derivation algebra $Inn(\mathcal{L})$ is an ideal of the double derivation algebra $\mathcal{D}(\mathcal{L})$. We also show that if \mathcal{L} is a perfect *n*-Hom Lie color algebra with certain constraints on the base field, then the centralizer of $Inn(\mathcal{L})$ in $\mathcal{D}(\mathcal{L})$ is trivial. In addition, we obtain that for every centerless perfect *n*-Hom-Lie color algebra \mathcal{L} , the triple derivations of the derivation algebra $Der(\mathcal{L})$ are exactly the derivations of $Der(\mathcal{L})$.

1. INTRODUCTION

The first motivation to study *n*-ary algebras appeared in physics when Nambu suggested in 1973 a generalization of Hamiltonian mechanics with more than one hamiltonian [29]. The mathematical algebraic foundations of Nambu mechanics have been developed by Takhtajan in [31, 32]. The abstract definition of *n*-Lie algebra is due to Filippov in 1985 [12], then Kasymov

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[18] more deeply investigated their properties. This approach uses the interpretation of Jacobi identity expressing the fact that the adjoint map is a derivation. Nambu mechanics, n -Lie algebras revealed have many applications in physics.

The theory of Hom-Lie algebras is originated from Hom-algebras introduced by J. T. Hartwig, D. Larsson and S. D. Silvestrov in [16] in the study of quasi-deformations of Lie algebras of vector fields, including q -deformations of Witt algebras and Virasoro algebras. Hom-type generalization of n -ary algebras, such as n -Hom-Lie algebras and other n -ary Hom-algebras of Lie type and associative type, were introduced in [7], by twisting the identities defining them using an algebra morphism. Further properties, construction methods, examples, cohomology and central extensions of n -ary Hom-algebras have been considered in [1, 2, 3, 33].

The concept of derivations appear in different mathematical fields with many different forms. In algebraic systems, derivations are linear maps that satisfy the Leibniz relation. There are several kinds of derivations in the theory of Lie algebras, such as Leibniz derivations [20], Jordan derivations [17] and δ -derivations [13, 14, 15, 19]. The notion of a generalized derivation of a Lie algebra and their subalgebras is a generalization of δ -derivation was due to Leger and Luks [24]. Their results were generalized by many authors. For example, in the case of color Lie algebras and Hom-Lie color algebras [11, 36], Hom and BiHom-Poisson color algebras [22, 23] Lie triple systems and Hom-Lie triple systems [39, 40], color n -ary Ω -algebras and multiplicative n -ary Hom- Ω -algebras [21, 10] and many other works. Another generalization of derivations of Lie algebras are triple derivations and generalized triple derivations. It was first introduced independently by Muller where it was called prederivation [28]. It can be easily checked that, for any Lie algebra, every derivation is a triple derivation, but the converse does not always hold [35]. The author proved that every triple derivation of a perfect Lie algebra with zero center is a derivation and every derivation of the derivation algebra is an inner derivation. In recent years, much progress has been obtained on the Lie triple derivations and n -Lie derivations of Lie algebras (see [4, 5, 6, 25, 26, 27, 35, 38]).

The double derivations provide a useful tool to study n -Lie algebras by making use of linear mappings. Double derivations are similar to the triple derivations of Lie algebras to some extent [35]. Recently, in [8] and [30] the authors studied double derivations of n -Lie algebras and n -Lie superalgebras. The main purpose of our work is to consider the double derivations of multiplicative n -Hom-Lie color algebras.

The paper is organized as follows. Section 2 contains some necessary basic definitions and notions. Moreover, we show that the double derivation algebra of a multiplicative n -Hom-Lie color algebra \mathcal{L} is a Hom-Lie color algebra and for a perfect n -Hom-Lie color algebra \mathcal{L} its inner derivation algebra is an ideal of the double derivation algebra. In Section 3, we construct

a new double derivation of a perfect centerless multiplicative n -Hom-Lie color algebra \mathcal{L} and obtain a Hom-Lie color algebra homomorphism. In section 4, we study triple derivations of the derivation algebra $Der(\mathcal{L})$ and the inner derivation algebra $Inn(\mathcal{L})$.

2. DOUBLE DERIVATIONS OF MULTIPLICATIVE n -HOM-LIE COLOR ALGEBRAS

Let us begin with some necessary important basic definitions and notations on graded spaces, graded algebras and n -Hom-Lie color algebras used in other sections. For a detailed discussion of this subject, we refer the reader to the literature [9]. We also prove that the inner derivation algebra $Inn(\mathcal{L})$ of \mathcal{L} is an ideal of the double derivation algebra of \mathcal{L} .

Let G be any additive abelian group, a vector space V is said to be G -graded, if there is a family $\{V_g\}_{g \in G}$ of vector subspaces such that $V = \bigoplus_{g \in G} V_g$. An element $v \in V$ is said to be *homogeneous of the degree g* if $v \in V_g$, $g \in G$, and in this case, g is called the *color* of v . We denote $\mathcal{H}(V)$ the set of all homogeneous elements in V .

Let $V = \bigoplus_{g \in G} V_g$ and $W = \bigoplus_{h \in G} W_h$ be two G -graded vector spaces. A linear mapping $f : V \rightarrow W$ is said to be *homogeneous of degree $\theta \in G$* if

$$f(V_g) \subset W_{g+\theta}, \quad \forall g \in G.$$

If in addition, f is homogeneous of degree zero, namely, $f(V_g) \subset W_g$ holds for any $g \in G$, then we call f is *even*.

An algebra A (with the juxtaposition product) is said to be G -graded if its underlying vector space is G -graded, i.e., $A = \bigoplus_{g \in G} A_g$, and if $A_g A_h \subset A_{g+h}$, for $g, h \in G$. A subalgebra of A is said to be graded if it is a graded as a subspace of A .

Let B be another G -graded algebra. A morphism $\varphi : A \rightarrow B$ of G -graded algebras is a homomorphism of the algebra A into the algebra B , which is an even mapping.

Definition 2.1. Let G be an additive abelian group. A map $\varepsilon : G \times G \rightarrow \mathbb{K} \setminus \{0\}$ is called a skew-symmetric *bi-character* on G if for all $g, h, k \in G$,

- (i) $\varepsilon(g, h)\varepsilon(h, g) = 1$,
- (ii) $\varepsilon(g + h, k) = \varepsilon(g, k)\varepsilon(h, k)$,
- (iii) $\varepsilon(g, h + k) = \varepsilon(g, h)\varepsilon(g, k)$.

The definition above implies that in particular, the following relations hold

$$\varepsilon(g, 0) = 1 = \varepsilon(0, g), \quad \varepsilon(g, g) = 1(\text{or } -1), \quad \forall g \in G.$$

Throughout this paper, if x and y are two homogeneous elements of a G -graded vector space and $|x|, |y|$ are their degrees respectively, then for convenience, we write $\varepsilon(x, y)$ instead of $\varepsilon(|x|, |y|)$. It is worth mentioning that, we unless otherwise stated, in the sequel the base

field of any vector space in this paper is \mathbb{K} , all the graded spaces are over the same abelian group G and the bi-character will be the same for all structures.

Definition 2.2. [12] An n -Lie algebra is a linear spaces \mathcal{L} equipped with n -ary operation satisfying the following identities:

- (i) $[x_1, \dots, x_i, x_{i+1}, \dots, x_n] = -[x_1, \dots, x_{i+1}, x_i, \dots, x_n]$, $i = 1, \dots, n - 1$,
- (ii) $[x_1, \dots, x_{n-1}, [y_1, \dots, y_n]] = \sum_{i=1}^n [y_1, \dots, y_{i-1}, [x_1, \dots, x_{n-1}, y_i], y_{i+1}, \dots, y_n]$,

for all $x_i, y_i \in \mathcal{L}$.

Definition 2.3. [9] An n -Hom-Lie color algebra is a graded linear space $\mathcal{L} = \bigoplus_{g \in G} \mathcal{L}_g$ with an n -linear map $[\cdot, \dots, \cdot] : \mathcal{L} \times \dots \times \mathcal{L} \rightarrow \mathcal{L}$, a bicharacter $\varepsilon : G \times G \rightarrow \mathbb{K} \setminus \{0\}$ and an even linear map $\alpha : \mathcal{L} \rightarrow \mathcal{L}$ such that

- (i) $[x_1, \dots, x_i, x_{i+1}, \dots, x_n] = -\varepsilon(x_i, x_{i+1})[x_1, \dots, x_{i+1}, x_i, \dots, x_n]$, $i = 1, \dots, n - 1$,
- (ii)

$$(1) \quad [\alpha(x_1), \dots, \alpha(x_{n-1}), [y_1, y_2, \dots, y_n]] = \sum_{i=1}^n \varepsilon(X, Y_i) [\alpha(y_1), \dots, \alpha(y_{i-1}), [x_1, \dots, x_{n-1}, y_i], \alpha(y_{i+1}), \dots, \alpha(y_n)],$$

where $x_i, y_i \in \mathcal{H}(\mathcal{L})$, $X = \sum_{i=1}^{n-1} x_i$, $Y_i = \sum_{j=1}^i y_{j-1}$ and $y_0 = e$.

An n -Hom-Lie color algebra $(\mathcal{L}, [\cdot, \dots, \cdot], \alpha, \varepsilon)$ is said to be multiplicative if α is an endomorphism, i.e. a linear map on \mathcal{L} which is also a homomorphism with respect to multiplication $[\cdot, \dots, \cdot]$. We also call it a regular Hom-Lie color algebra if α is an automorphism. We recover n -Lie algebra when we have $\alpha = id$ and $G = \{e\}$. If $\alpha = id_{\mathcal{L}}$ and $G = \mathbb{Z}_2$ with $\varepsilon(x, y) = (-1)^{xy}$ for any $x, y \in \mathcal{H}(\mathcal{L})$, we get n -Lie super algebra. For some standard exampls, we refer the reader can be find in [9].

Definition 2.4. (1) A graded subspace S of an n -Hom-Lie color algebra \mathcal{L} is a n -Hom-Lie color subalgebra of \mathcal{L} if $\alpha(S) \subseteq S$ and $[S, \dots, S] \subseteq S$.

- (2) An n -Hom-Lie color ideal I of an n -Hom-Lie color algebra \mathcal{L} is a graded subspace of \mathcal{L} such that $\alpha(I) \subseteq I$ and $[I, \mathcal{L}, \dots, \mathcal{L}] \subseteq I$.

Let S_1, S_2, \dots, S_n be n -Hom-Lie subalgebras of an n -Hom-Lie color algebra \mathcal{L} . Denote by $[S_1, S_2, \dots, S_n]$ the n -Hom-Lie subalgebra of \mathcal{L} generated by all elements $[x_1, x_2, \dots, x_n]$, where $x_i \in S_i, i = 1, 2, \dots, n$. The algebra $[\mathcal{L}, \dots, \mathcal{L}]$ is called the derived algebra of \mathcal{L} , and is denoted by \mathcal{L}^1 . If $\mathcal{L}^1 = \mathcal{L}$, then \mathcal{L} is called a perfect n -Hom-Lie color algebra. The center of an n -Hom-Lie color algebra \mathcal{L} is denoted by $Z(\mathcal{L}) = \{x \in \mathcal{L} : [x, \mathcal{L}, \dots, \mathcal{L}] = 0\}$ which is a n -Hom-Lie color ideal of \mathcal{L} (see Theorem 2.16 in [9]). For a subset S of \mathcal{L} , the centralizer of S in \mathcal{L} is defined by $C_{\mathcal{L}}(S) = \{x \in \mathcal{L} : [x, S, \mathcal{L}, \dots, \mathcal{L}] = 0\}$. Note that $C_{\mathcal{L}}(\mathcal{L}) = Z(\mathcal{L})$.

From now on, we let $(\mathcal{L}, [\cdot, \dots, \cdot], \alpha, \varepsilon)$ be a multiplicative n -Hom-Lie color algebra, which we will denote by \mathcal{L} , for short. For any non-negative integer k , denote by α^k the k -times composition of α . In particular $\alpha^0 = id$, $\alpha^1 = \alpha$.

In the following we give a more general definition of derivations and related objects.

Definition 2.5. [9] For any $k \geq 1$, we call $D \in \text{End}(\mathcal{L})$ an α^k -derivation of degree d of the multiplicative n -Hom-Lie color algebra \mathcal{L} if

- (1) $D \circ \alpha = \alpha \circ D$.
- (2) For all $x_1, x_2, \dots, x_n \in \mathcal{H}(\mathcal{L})$,

$$D([x_1, \dots, x_n]) = \sum_{i=1}^n \varepsilon(d, X_i) [\alpha^k(x_1), \dots, \alpha^k(x_{i-1}), D(x_i), \alpha^k(x_{i+1}), \dots, \alpha^k(x_n)],$$

where $X_i = \sum_{j=1}^i x_{j-1}$ and $x_0 = e$.

We denote the set of all α^k -derivations of the multiplicative n -Hom-Lie color algebra \mathcal{L} by $\text{Der}_{\alpha^k}(\mathcal{L})$. For any $D \in \text{Der}_{\alpha^k}(\mathcal{L})$ and $D' \in \text{Der}_{\alpha^s}(\mathcal{L})$, let us define their color commutator $[D, D'] = D \circ D' - \varepsilon(d, d')D' \circ D$. Then $[D, D'] \in \text{Der}_{\alpha^{k+s}}$ (see Lemma 5.6 in [9]). Denote by $\text{Der}(\mathcal{L}) = \bigoplus_{k \geq 0} \text{Der}_{\alpha^k}(\mathcal{L})$. We also have that $(\text{Der}(\mathcal{L}), [\cdot, \cdot], \tilde{\alpha} = D \circ \alpha, \varepsilon)$ is a Hom-Lie color algebra (see Proposition 5.7 in [9]).

For $x_1, \dots, x_{n-1} \in \mathcal{H}(\mathcal{L})$ satisfying $\alpha(x_i) = x_i, i = 1, 2, \dots, n - 1$, we define the map $\text{ad}_k(x_1, \dots, x_{n-1}) : \mathcal{L} \rightarrow \mathcal{L}$ by

$$\text{ad}_k(x_1, \dots, x_{n-1})(y) := [x_1, \dots, x_{n-1}, \alpha^k(y)], \text{ and } k \geq 1,$$

for all $y \in \mathcal{H}(\mathcal{L})$. Then $\text{ad}_k(x_1, \dots, x_{n-1})$, is an α^{k+1} -derivation of \mathcal{L} (see Lemma 2.2 in [2]). We call $\text{ad}_k(x_1, \dots, x_{n-1})$ an *inner α^{k+1} -derivation*. Denote by $\text{Inn}_{\alpha^k}(\mathcal{L})$ the space generate by all the inner α^{k+1} -derivations. Denote by $\text{Inn}(\mathcal{L}) = \bigoplus_{k \geq 0} \text{Inn}_{\alpha^k}(\mathcal{L})$. It is not difficult to show that $(\text{Inn}(\mathcal{L}), [\cdot, \cdot], \tilde{\alpha} = D \circ \alpha, \varepsilon)$ is a Hom-Lie color algebra. In addition, $\text{Inn}(\mathcal{L})$ is a Hom-Lie color ideal of $\text{Der}(\mathcal{L})$ (see Proposition 2.4 in [2]).

Definition 2.6. Let $(\mathcal{L}, [\cdot, \dots, \cdot], \alpha, \varepsilon)$ be a multiplicative n -Hom-Lie color algebra with $n \geq 3$. A linear map $D : \mathcal{L} \rightarrow \mathcal{L}$ is called a *double α^k -derivation* of degree d of \mathcal{L} if

- (1) $D \circ \alpha = \alpha \circ D$,
- (2) for all $x_i, y_j \in \mathcal{H}(\mathcal{L})$, $i = 1, 2, \dots, n - 1$, $j = 1, 2, \dots, n$,

$$\begin{aligned} & D([x_1, \dots, x_{n-1}, [y_1, \dots, y_n]]) \\ &= \sum_{i=1}^{n-1} \varepsilon(d, X_i) [\alpha^k(x_1), \dots, D(x_i), \dots, \alpha^k(x_{n-1}), [\alpha^k(y_1), \dots, \alpha^k(y_n)]] \\ &+ \sum_{j=1}^n \varepsilon(d, X + Y_j) [\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), [\alpha^k(y_1), \dots, D(y_j), \dots, \alpha^k(y_n)]], \end{aligned}$$

where $X = \sum_{i=1}^{n-1} x_i$, $X_i = \sum_{j=1}^i x_{j-1}$, $Y_j = \sum_{k=1}^i y_{k-1}$ and $x_0 = e = y_0$.

Let us denote the set of all double α^k -derivations of the multiplicative n -Hom-Lie color algebra \mathcal{L} by $\mathcal{D}_{\alpha^k}(\mathcal{L})$ and denote by $\mathcal{D}(\mathcal{L}) = \bigoplus_{k \geq 0} \mathcal{D}_{\alpha^k}(\mathcal{L})$, the vector space spanned by the double derivations of \mathcal{L} . We call it *double derivation algebra* of \mathcal{L} (see Theorem 2.7 below).

It is obvious that the derivations of an n -Hom-Lie color algebra \mathcal{L} are double derivations, and hence we have $Der(\mathcal{L}) \subset \mathcal{D}(\mathcal{L})$. Note that there exists an n -Lie superalgebra \mathcal{L} such that $Der(\mathcal{L}) \neq \mathcal{D}(\mathcal{L})$ (see Remark 2.5 in [30]).

Theorem 2.7. *Let $(\mathcal{L}, [., \dots, .], \alpha, \varepsilon)$ be a multiplicative n -Hom-Lie color algebra. Then $\mathcal{D}(\mathcal{L})$ is a Hom-Lie color subalgebra of the general linear Hom-Lie color algebra $(gl(\mathcal{L}), [., .], \tilde{\alpha}, \varepsilon)$ with $\tilde{\alpha}(D) = D \circ \alpha$ for all $D \in \mathcal{D}(\mathcal{L})$.*

Proof. First, let us prove that $\tilde{\alpha}(\mathcal{D}(\mathcal{L})) \subseteq \mathcal{D}(\mathcal{L})$. Suppose that $D \in \mathcal{D}_{\alpha^k}(\mathcal{L})$, for any $x_1, x_2, \dots, x_{n-1}, y_1, \dots, y_n \in \mathcal{H}(\mathcal{L})$, we have

$$\begin{aligned}
& \tilde{\alpha}(D) \left(\left[x_1, \dots, x_{n-1}, [y_1, \dots, y_n] \right] \right) \\
&= (D \circ \alpha) \left(\left[x_1, \dots, x_{n-1}, [y_1, \dots, y_n] \right] \right) \\
&= D \left(\left[\alpha(x_1), \dots, \alpha(x_{n-1}), [\alpha(y_1), \dots, \alpha(y_n)] \right] \right) \\
&= \sum_{i=1}^{n-1} \varepsilon(d, X_i) [\alpha^{k+1}(x_1), \dots, D(\alpha(x_i)), \dots, \alpha^{k+1}(x_{n-1}), [\alpha^{k+1}(y_1), \dots, \alpha^{k+1}(y_n)]] \\
&\quad + \sum_{j=1}^n \varepsilon(d, X + Y_j) [\alpha^{k+1}(x_1), \dots, \alpha^{k+1}(x_{n-1}), [\alpha^{k+1}(y_1), \dots, D(\alpha(y_j)), \dots, \alpha^{k+1}(y_n)]] \\
&= \sum_{i=1}^{n-1} \varepsilon(d, X_i) [\alpha^{k+1}(x_1), \dots, (D \circ \alpha)(x_i), \dots, \alpha^{k+1}(x_{n-1}), [\alpha^{k+1}(y_1), \dots, \alpha^{k+1}(y_n)]] \\
&\quad + \sum_{j=1}^n \varepsilon(d, X + Y_j) [\alpha^{k+1}(x_1), \dots, \alpha^{k+1}(x_{n-1}), [\alpha^{k+1}(y_1), \dots, (D \circ \alpha)(y_j), \dots, \alpha^{k+1}(y_n)]] \\
&= \sum_{i=1}^{n-1} \varepsilon(d, X_i) [\alpha^{k+1}(x_1), \dots, (\tilde{\alpha}(D))(x_i), \dots, \alpha^{k+1}(x_{n-1}), [\alpha^{k+1}(y_1), \dots, \alpha^{k+1}(y_n)]] \\
&\quad + \sum_{j=1}^n \varepsilon(d, X + Y_j) [\alpha^{k+1}(x_1), \dots, \alpha^{k+1}(x_{n-1}), [\alpha^{k+1}(y_1), \dots, (\tilde{\alpha}(D))(y_j), \dots, \alpha^{k+1}(y_n)]] .
\end{aligned}$$

This means that $\tilde{\alpha}(D)$ is a double α^{k+1} -derivation i.e. $\tilde{\alpha}(D) \in \mathcal{D}_{\alpha^{k+1}}(\mathcal{L})$.

Next, assum that $D_1 \in \mathcal{D}_{\alpha^k}(\mathcal{L})$ of degree d_1 and $D_2 \in \mathcal{D}_{\alpha^s}(\mathcal{L})$ of degree d_2 . It is clear that $[D_1, D_2] \circ \alpha = \alpha \circ [D_1, D_2]$. For any $x_1, x_2, \dots, x_{n-1}, y_1, \dots, y_n \in \mathcal{H}(\mathcal{L})$, we also have

$$\begin{aligned}
 & D_1 D_2 \left([x_1, \dots, x_{n-1}, [y_1, \dots, y_n]] \right) \\
 &= D_1 \left(\sum_{i=1}^{n-1} \varepsilon(d_2, X_i) [\alpha^s(x_1), \dots, D_2(x_i), \dots, \alpha^s(x_{n-1}), [\alpha^s(y_1), \dots, \alpha^s(y_n)]] \right. \\
 &\quad \left. + \sum_{j=1}^n \varepsilon(d_2, X + Y_j) [\alpha^s(x_1), \dots, \alpha^s(x_{n-1}), [\alpha^s(y_1), \dots, D_2(y_j), \dots, \alpha^s(y_n)]] \right) \\
 &= \sum_{i=1}^{n-1} \varepsilon(d_1 + d_2, X_i) [\alpha^{k+s}(x_1), \dots, D_1 D_2(x_i), \dots, \alpha^{k+s}(x_{n-1}), [\alpha^{k+s}(y_1), \dots, \alpha^{k+s}(y_n)]] \\
 &\quad + \sum_{i=1}^{n-1} \sum_{r < i} \varepsilon(d_1, X_r) \varepsilon(d_2, X_i) \left[\alpha^{k+s}(x_1), \dots, D_1(\alpha^s(x_r)), \dots, D_2(\alpha^k(x_i)), \dots, \alpha^{k+s}(x_{n-1}), \right. \\
 &\quad \left. [\alpha^{k+s}(y_1), \dots, \alpha^{k+s}(y_n)] \right] \\
 &\quad + \sum_{i=1}^{n-1} \sum_{i < r} \varepsilon(d_1, X_r) \varepsilon(d_2, X_i) \varepsilon(d_1, d_2) \left[\alpha^{k+s}(x_1), \dots, D_2(\alpha^k(x_i)), \dots, D_1(\alpha^s(x_r)), \dots, \right. \\
 &\quad \left. \alpha^{k+s}(x_{n-1}), [\alpha^{k+s}(y_1), \dots, \alpha^{k+s}(y_n)] \right] \\
 &\quad + \sum_{i=1}^{n-1} \sum_{t=1}^{n-1} \varepsilon(d_2, X_i) \varepsilon(d_1, X) \varepsilon(d_1, d_2) \left[\alpha^{k+s}(x_1), \dots, D_2(\alpha^k(x_i)), \dots, \alpha^{k+s}(x_{n-1}), \right. \\
 &\quad \left. [\alpha^{k+s}(y_1), \dots, D_1(\alpha^s(y_t)), \dots, \alpha^{k+s}(y_n)] \right] \\
 &\quad + \sum_{j=1}^n \sum_{m=1}^{n-1} \varepsilon(d_1, X_m) \varepsilon(d_2, X + Y_j) \left[\alpha^{k+s}(x_1), \dots, D_1(\alpha^s(x_j)), \dots, \alpha^{k+s}(x_{n-1}), \right. \\
 &\quad \left. [\alpha^{k+s}(y_1), \dots, D_2(\alpha^k(y_m)), \dots, \alpha^{k+s}(y_n)] \right] \\
 &\quad + \sum_{j=1}^n \sum_{l < j} \varepsilon(d_1, X + Y_l) \varepsilon(d_2, X + Y_j) \left[\alpha^{k+s}(x_1), \dots, \alpha^{k+s}(x_{n-1}), \right. \\
 &\quad \left. [\alpha^{k+s}(y_1), \dots, D_1(\alpha^s(y_l)), \dots, D_2(\alpha^k(y_j)), \dots, \alpha^{k+s}(y_n)] \right] \\
 &\quad + \sum_{j=1}^n \sum_{j < l} \varepsilon(d_1, X + Y_l) \varepsilon(d_2, X + Y_j) \varepsilon(d_1, d_2) \left[\alpha^{k+s}(x_1), \dots, \alpha^{k+s}(x_{n-1}), \right. \\
 &\quad \left. [\alpha^{k+s}(y_1), \dots, D_2(\alpha^k(y_j)), \dots, D_1(\alpha^s(y_l)), \dots, \alpha^{k+s}(y_n)] \right] \\
 &\quad + \sum_{j=1}^n \varepsilon(d_1 + d_2, X + Y_j) \left[\alpha^{k+s}(x_1), \dots, \alpha^{k+s}(x_{n-1}), [\alpha^{k+s}(y_1), \dots, D_1 D_2(x_j), \dots, \alpha^{k+s}(y_n)] \right].
 \end{aligned}$$

Similarly,

$$\begin{aligned}
& \varepsilon(d_1, d_2) D_2 D_1 \left([x_1, \dots, x_{n-1}, [y_1, \dots, y_n]] \right) \\
&= \sum_{i=1}^{n-1} \varepsilon(d_1 + d_2, X_i) \left[\alpha^{k+s}(x_1), \dots, D_2 D_1(x_i), \dots, \alpha^{k+s}(x_{n-1}), [\alpha^{k+s}(y_1), \dots, \alpha^{k+s}(y_n)] \right] \\
&+ \sum_{i=1}^{n-1} \sum_{r < i} \varepsilon(d_2, X_r) \varepsilon(d_1, X_i) \left[\alpha^{k+s}(x_1), \dots, D_2(\alpha^k(x_r)), \dots, D_1(\alpha^s(x_i)), \dots, \alpha^{k+s}(x_{n-1}), \right. \\
&\quad \left. [\alpha^{k+s}(y_1), \dots, \alpha^{k+s}(y_n)] \right] \\
&+ \sum_{i=1}^{n-1} \sum_{i < r} \varepsilon(d_2, X_r) \varepsilon(d_1, X_i) \varepsilon(d_2, d_1) \left[\alpha^{k+s}(x_1), \dots, D_1(\alpha^s(x_i)), \dots, D_2(\alpha^k(x_r)), \dots, \right. \\
&\quad \left. \alpha^{k+s}(x_{n-1}), [\alpha^{k+s}(y_1), \dots, \alpha^{k+s}(y_n)] \right] \\
&+ \sum_{i=1}^{n-1} \sum_{t=1}^{n-1} \varepsilon(d_1, X_i) \varepsilon(d_2, X) \varepsilon(d_2, d_1) \left[\alpha^{k+s}(x_1), \dots, D_1(\alpha^s(x_i)), \dots, \alpha^{k+s}(x_{n-1}), \right. \\
&\quad \left. [\alpha^{k+s}(y_1), \dots, D_2(\alpha^k(y_t)), \dots, \alpha^{k+s}(y_n)] \right] \\
&+ \sum_{j=1}^n \sum_{m=1}^{n-1} \varepsilon(d_2, X_m) \varepsilon(d_1, X + Y_j) \left[\alpha^{k+s}(x_1), \dots, D_2(\alpha^k(x_j)), \dots, \alpha^{k+s}(x_{n-1}), \right. \\
&\quad \left. [\alpha^{k+s}(y_1), \dots, D_1(\alpha^k(y_m)), \dots, \alpha^{k+s}(y_n)] \right] \\
&+ \sum_{j=1}^n \sum_{l < j} \varepsilon(d_2, X + Y_l) \varepsilon(d_1, X + Y_j) \left[\alpha^{k+s}(x_1), \dots, \alpha^{k+s}(x_{n-1}), \right. \\
&\quad \left. [\alpha^{k+s}(y_1), \dots, D_2(\alpha^k(y_l)), \dots, D_1(\alpha^s(y_j)), \dots, \alpha^{k+s}(y_n)] \right] \\
&+ \sum_{j=1}^n \sum_{j < l} \varepsilon(d_2, X + Y_l) \varepsilon(d_1, X + Y_j) \varepsilon(d_2, d_1) \left[\alpha^{k+s}(x_1), \dots, \alpha^{k+s}(x_{n-1}), \right. \\
&\quad \left. [\alpha^{k+s}(y_1), \dots, D_1(\alpha^k(y_j)), \dots, D_2(\alpha^k(y_l)), \dots, \alpha^{k+s}(y_n)] \right] \\
&+ \sum_{j=1}^n \varepsilon(d_1 + d_2, X + Y_j) \left[\alpha^{k+s}(x_1), \dots, \alpha^{k+s}(x_{n-1}), [\alpha^{k+s}(y_1), \dots, D_2 D_1(x_j), \dots, \alpha^{k+s}(y_n)] \right].
\end{aligned}$$

Then from easy computation we have

$$[D_1, D_2] \left([x_1, \dots, x_{n-1}, [y_1, \dots, y_n]] \right) = (D_1 D_2 - \varepsilon(d_1, d_2) D_2 D_1) \left([x_1, \dots, x_{n-1}, [y_1, \dots, y_n]] \right)$$

$$\begin{aligned}
 &= \sum_{i=1}^{n-1} \left[\alpha^s(x_1), \dots, (D_1D_2 - \varepsilon(d_1, d_2)D_2D_1)(x_i), \dots, \alpha^s(x_{n-1}), [\alpha^s(y_1), \dots, \alpha^s(y_n)] \right] \\
 &+ \sum_{j=1}^n \varepsilon(d_1 + d_2, X + Y_j) \left[\alpha^{k+s}(x_1), \dots, \alpha^{k+s}(x_{n-1}), \right. \\
 &\quad \left. [\alpha^{k+s}(y_1), \dots, (D_1D_2 - \varepsilon(d_1, d_2)D_2D_1)(y_j), \dots, \alpha^{k+s}(y_n)] \right] \\
 &= \sum_{i=1}^{n-1} \left[\alpha^s(x_1), \dots, [D_1, D_2](x_i), \dots, \alpha^s(x_{n-1}), [\alpha^s(y_1), \dots, \alpha^s(y_n)] \right] \\
 &+ \sum_{j=1}^n \varepsilon(d_1 + d_2, X + Y_j) \\
 &\quad \cdot \left[\alpha^{k+s}(x_1), \dots, \alpha^{k+s}(x_{n-1}), [\alpha^{k+s}(y_1), \dots, [D_1, D_2](x_j), \dots, \alpha^{k+s}(y_n)] \right].
 \end{aligned}$$

This implies that, $[D_1, D_2]$ is a double α^{k+s} -derivation of degree $d_1 + d_2$ and so $[D_1, D_2] \in \mathcal{D}(\mathcal{L})$ as required. \square

Theorem 2.8. *If $(\mathcal{L}, [., \dots, .], \alpha, \varepsilon)$ is a perfect multiplicative n -Hom-Lie color algebra, then the inner derivation algebra of \mathcal{L} is a Hom-Lie color ideal of the Hom-Lie color algebra $(\mathcal{D}(\mathcal{L}), [., .], \tilde{\alpha}, \varepsilon)$.*

Proof. First, we are going to check that $\tilde{\alpha}(Inn(\mathcal{L})) \subseteq Inn(\mathcal{L})$. For $x_1, x_2, \dots, x_{n-1} \in \mathcal{H}(\mathcal{L})$ and any $y \in \mathcal{H}(\mathcal{L})$, we have

$$\begin{aligned}
 \tilde{\alpha} \left(\text{ad}_k(x_1, \dots, x_{n-1}) \right) (y) &= \text{ad}_k(x_1, \dots, x_{n-1})(\alpha(y)) \\
 &= [x_1, \dots, x_{n-1}, \alpha^{k+1}(y)] \\
 &= \text{ad}_{k+1}(x_1, \dots, x_{n-1})(y).
 \end{aligned}$$

This means that $\tilde{\alpha}(\text{ad}_k(x_1, \dots, x_{n-1}))$ is an inner α^{k+2} -derivation i.e. $\tilde{\alpha}(Inn(\mathcal{L})) \subseteq Inn(\mathcal{L})$.

Next, let $D \in \mathcal{D}_{\alpha^s}(\mathcal{L})$ and $\text{ad}_k(x_1, x_2, \dots, x_{n-1})$ be an inner α^{k+1} -derivation. Since \mathcal{L} is perfect, there exists a finite index set I and $x_{i_j} \in \mathcal{L}, i \in I, j = 1, \dots, n$, such that

$$x_1 = \sum_{i \in I} [x_{i_1}, x_{i_2}, \dots, x_{i_n}].$$

For any $y \in \mathcal{L}$, we have

$$\begin{aligned}
 &[D, \text{ad}_k(x_1, \dots, x_{n-1})](y) \\
 &= D \circ \text{ad}_k(x_1, \dots, x_{n-1})(y) - \varepsilon(d, X) \text{ad}_k(x_1, \dots, x_{n-1}) \circ D(y) \\
 &= D([\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), \alpha^k(y)]) - \varepsilon(d, X) [\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), \alpha^k(D(y))]
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{i \in I} D \left(\left[[\alpha^k(x_{i_1}), \alpha^k(x_{i_2}), \dots, \alpha^k(x_{i_n})], \alpha^k(x_2), \dots, \alpha^k(x_{n-1}), \alpha^k(y)] \right) \right. \\
&\quad \left. - \varepsilon(d, X) [\alpha^{k+s}(x_1), \dots, \alpha^{k+s}(x_{n-1}), D(\alpha^k(y))] \right) \\
&= \sum_{i \in I} \left(\sum_{j=2}^{n-2} \varepsilon(d, x_2 + \dots + x_{j-1}) \right. \\
&\quad \cdot [\alpha^{k+s}(x_2), \dots, D(x_j), \dots, \alpha^{k+s}(x_{n-1}), \alpha^{k+s}(y), [\alpha^{k+s}(x_{i_1}), \dots, \alpha^{k+s}(x_{i_n})]] \\
&\quad \left. + \varepsilon(d, x_2 + \dots + x_{n-1}) \right. \\
&\quad \cdot [\alpha^{k+s}(x_2), \dots, \alpha^{k+s}(x_{n-1}), D(\alpha^k(y)), [\alpha^{k+s}(x_{i_1}), \dots, \alpha^{k+s}(x_{i_n})]] \\
&\quad \left. + \sum_{t=1}^n \varepsilon(d, x_2 + \dots + x_{n-1} + y + X_{i_t}) \right. \\
&\quad \cdot [\alpha^{k+s}(x_2), \dots, \alpha^{k+s}(x_{n-1}), \alpha^{k+s}(y), [\alpha^{k+s}(x_{i_1}), \dots, D(x_{i_t}), \dots, \alpha^{k+s}(x_{i_n})]] \left. \right) \\
&\quad \left. - \varepsilon(d, X) [\alpha^{k+s}(x_1), \dots, \alpha^{k+s}(x_{n-1}), D(\alpha^k(y))] \right) \\
&= \sum_{i \in I} \left(\sum_{j=2}^{n-1} \varepsilon(d, X_j) \right. \\
&\quad \cdot [[\alpha^{k+s}(x_{i_1}), \dots, \alpha^{k+s}(x_{i_n})], \alpha^{k+s}(x_2), \dots, D(x_j), \dots, \alpha^{k+s}(x_{n-1}), \alpha^{k+s}(y)] \\
&\quad \left. + \varepsilon(d, X) [[\alpha^{k+s}(x_{i_1}), \dots, \alpha^{k+s}(x_{i_n})], \alpha^{k+s}(x_2), \dots, \alpha^{k+s}(x_{n-1}), D(\alpha^k(y))] \right) \\
&\quad \left. + \sum_{i \in I} \left(\sum_{t=1}^n \varepsilon(d, X_{i_t}) \right. \right. \\
&\quad \cdot [[\alpha^{k+s}(x_{i_1}), \dots, \alpha^{k+s}(x_{i_n})], \alpha^{k+s}(x_2), \dots, \alpha^{k+s}(x_{n-1}), \alpha^{k+s}(y)] \left. \right) \\
&\quad \left. - \varepsilon(d, X) [\alpha^{k+s}(x_1), \dots, \alpha^{k+s}(x_{n-1}), D(\alpha^k(y))] \right) \\
&= \sum_{j=2}^{n-1} \varepsilon(d, X_j) [\alpha^{k+s}(x_1), \dots, D(x_j), \dots, \alpha^{k+s}(x_{n-1}), \alpha^{k+s}(y)] \\
&\quad + \sum_{i \in I} \left(\sum_{t=1}^n \varepsilon(d, X_{i_t}) \right. \\
&\quad \cdot [[\alpha^{k+s}(x_{i_1}), \dots, D(x_{i_t}), \dots, \alpha^{k+s}(x_{i_n})], \alpha^{k+s}(x_2), \dots, \alpha^{k+s}(x_{n-1}), \alpha^{k+s}(y)] \left. \right) \\
&= \sum_{j=2}^{n-1} \varepsilon(d, X_j) ad_{k+s}(x_1, \dots, D(x_j), \dots, x_{n-1})(y) \\
&\quad + \sum_{i \in I} \sum_{t=1}^n \varepsilon(d, X_{i_t}) ad_{k+s}([x_{i_1}, \dots, D(x_{i_t}), \dots, x_{i_n}], x_2, \dots, x_{n-1})(y).
\end{aligned}$$

Consequently, $[D, ad_k(x_1, \dots, x_{n-1})] \in Inn_{\alpha^{k+s+1}}(\mathcal{L})$, which implies that $Inn(\mathcal{L})$ is a Hom-Lie color ideal of Hom-Lie color algebra $(\mathcal{D}(\mathcal{L}), [\cdot, \cdot], \tilde{\alpha}, \varepsilon)$. \square

Note that, if \mathcal{L} is not a perfect multiplicative n -Hom-Lie color algebra, then $Inn(\mathcal{L})$ may not be an ideal of $\mathcal{D}(\mathcal{L})$ (see Remark 2.8 in [30]).

3. DOUBLE DERIVATIONS OF PERFECT MULTIPLICATIVE n -HOM-LIE COLOR ALGEBRAS

In this section, we focus on the perfectness of multiplicative n -Hom-Lie color algebras by centering our attention in those of centerless.

Let $(\mathcal{L}, [\cdot, \dots, \cdot], \alpha, \varepsilon)$ be an arbitrary centerless multiplicative n -Hom-Lie color algebra. For all $D \in \mathcal{D}_{\alpha^k}(\mathcal{L})$, we define a linear map $\delta_D : \mathcal{L} \rightarrow \mathcal{L}$, with $\delta_D \circ \alpha = \alpha \circ \delta_D$ by

$$(2) \quad \delta_D(x) = \begin{cases} D(x), & x \in \mathcal{L} \setminus \mathcal{L}^1, \\ \sum_{i \in I} \sum_{t=1}^n \varepsilon(d, X_{it}) [\alpha^k(x_{i_1}), \dots, D(x_{i_t}), \dots, \alpha^k(x_{i_n})], & x = \sum_{i \in I} [x_{i_1}, \dots, x_{i_n}] \in \mathcal{L}^1, \end{cases}$$

for all $x \in \mathcal{H}(\mathcal{L})$.

Lemma 3.1. *The linear map δ_D defined as Eq. (2) is well defined.*

Proof. Suppose that

$$x = \sum_{i \in I} [x_{i_1}, \dots, x_{i_n}] = \sum_{j \in J} [x_{j_1}, \dots, x_{j_n}] = y,$$

we set $\delta_D(x) = A$ and $\delta_D(y) = B$. For $D \in \mathcal{D}_{\alpha^k}(\mathcal{L})$ of degree d and $z_1, \dots, z_{n-1} \in \mathcal{H}(\mathcal{L})$, we have

$$\begin{aligned} & [\alpha^k(z_1), \dots, \alpha^k(z_{n-1}), A] \\ &= \left[\alpha^k(z_1), \dots, \alpha^k(z_{n-1}), \sum_{i \in I} \sum_{t=1}^n \varepsilon(d, X_{it}) [\alpha^k(x_{i_1}), \dots, D(x_{i_t}), \dots, \alpha^k(x_{i_n})] \right] \\ & \varepsilon(d, Z) \left(D([z_1, \dots, z_{n-1}, x]) - \sum_{s=1}^{n-1} \varepsilon(d, Z_s) [\alpha^k(z_1), \dots, D(z_s), \dots, \alpha^k(z_{n-1}), \alpha^k(x)] \right) \\ &= \varepsilon(d, Z) \left(D([z_1, \dots, z_{n-1}, y]) - \sum_{s=1}^{n-1} \varepsilon(d, Z_s) [\alpha^k(z_1), \dots, D(z_s), \dots, \alpha^k(z_{n-1}), \alpha^k(y)] \right) \\ &= \varepsilon(d, Z) \left(D \left([z_1, \dots, z_{n-1}, \sum_{j \in J} [x_{j_1}, \dots, x_{j_n}]] \right) - \right. \\ & \quad \left. \sum_{s=1}^{n-1} \varepsilon(d, Z_s) [\alpha^k(z_1), \dots, D(z_s), \dots, \alpha^k(z_{n-1}), \sum_{j \in J} [\alpha^k(x_{j_1}), \dots, \alpha^k(x_{j_n})]] \right) \end{aligned}$$

$$\begin{aligned}
&= \left[\alpha^k(z_1), \dots, \alpha^k(z_{n-1}), \sum_{j \in J} \sum_{s=1}^n \varepsilon(d, Y_{j_s}) [\alpha^k(y_{j_1}), \dots, D(y_{j_s}), \dots, \alpha^k(y_{j_n})] \right] \\
&= [\alpha^k(z_1), \dots, \alpha^k(z_{n-1}), B].
\end{aligned}$$

Therefore, we get that $[\alpha^k(z_1), \dots, \alpha^k(z_{n-1}), A - B] = 0$. This implies $A - B \in Z(\mathcal{L}) = 0$, we obtain $\delta_D(x) = \delta_D(y)$. Thus, δ_D is well-defined. \square

Consequently, we obtain a linear map $\delta : \mathcal{D}(\mathcal{L}) \longrightarrow \text{End}(\mathcal{L})$ defined by

$$\delta(D) = \delta_D, \quad \forall D \in \mathcal{D}(\mathcal{L}).$$

Proposition 3.2. *Let $(\mathcal{L}, [., \dots, .], \alpha, \varepsilon)$ be a centerless perfect multiplicative n -Hom-Lie color algebra. Suppose that $D \in \mathcal{D}(\mathcal{L})$, then the following assertions hold.*

- (1) δ_D is a double derivation of \mathcal{L} .
- (2) For all $x_1, \dots, x_n \in \mathcal{H}(\mathcal{L})$, we have

$$(D - \delta_D)([x_1, \dots, x_n]) = \varepsilon(d, X_i) [\alpha^k(x_1), \dots, (D - \delta_D)(x_i), \dots, \alpha^k(x_n)], \quad i = 1, \dots, n.$$

- (3) $\delta_{D-\delta_D} = -n(D - \delta_D)$.

Proof. (1) Suppose that $D \in \mathcal{D}_{\alpha^k}(\mathcal{L})$. Since \mathcal{L} is perfect, we can write $x_j = \sum_{i \in I_j} [x_{i_{j_1}}, \dots, x_{i_{j_n}}]$, $j = 1, \dots, n-1$ and $y_s = \sum_{p \in P_s} [y_{p_{s_1}}, \dots, y_{p_{s_n}}]$, $s = 1, \dots, n$. From Eq. (2), we have

$$\delta_D(x_j) = \sum_{l=1}^n \sum_{i \in I_j} \varepsilon(d, X_{I_{j_l}}) [\alpha^k(x_{i_{j_1}}), \dots, D(x_{i_{j_l}}), \dots, \alpha^k(x_{i_{j_n}})],$$

and similarly for $\delta_D(y_s)$. Next, we obtain

$$\begin{aligned}
&\sum_{j=1}^{n-1} \varepsilon(d, X_j) \left[\alpha^k(x_1), \dots, \delta_D(x_j), \dots, \alpha^k(x_{n-1}), [\alpha^k(y_1), \dots, \alpha^k(y_n)] \right] \\
&\quad + \sum_{s=1}^n \varepsilon(d, X + Y_s) \left[\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), [\alpha^k(y_1), \dots, \delta_D(y_s), \dots, \alpha^k(y_n)] \right] \\
&= \sum_{j=1}^{n-1} \varepsilon(d, X_j) \left[\alpha^k(x_1), \dots, \sum_{l=1}^n \sum_{i \in I_j} \varepsilon(d, X_{I_{j_l}}) \right. \\
&\quad \left. [\alpha^k(x_{i_{j_1}}), \dots, D(x_{i_{j_l}}), \dots, \alpha^k(x_{i_{j_n}})], \dots, \alpha^k(x_{n-1}), [\alpha^k(y_1), \dots, \alpha^k(y_n)] \right] \\
&\quad + \sum_{s=1}^n \varepsilon(d, X + Y_s) \left[\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), [\alpha^k(y_1), \dots, \sum_{l=1}^n \varepsilon(d, Y_{p_{s_l}}) \right. \\
&\quad \left. [\alpha^k(y_{p_{s_1}}), \dots, D(y_{p_{s_l}}), \dots, \alpha^k(y_n)], \dots, \alpha^k(y_n)] \right]
\end{aligned}$$

$$\begin{aligned}
 &= \sum_{i \in I_j} \left(\sum_{j=1}^{n-1} (-1)^{(n-1)(n-j)} \varepsilon(d, X_j) \varepsilon(Y, d + X) \varepsilon \left(\sum_{t=j+1}^{n-1} (x_t, \widehat{X}_t + Y + d) \right) \right. \\
 &\quad \cdot \left[\alpha^k(x_{j+1}), \dots, \alpha^k(x_{n-1}), [\alpha^k(y_1), \dots, \alpha^k(y_n)], \alpha^k(x_1), \dots, \alpha^k(x_{j-1}), \right. \\
 &\quad \left. \left. \sum_{l=1}^n \varepsilon(d, X_{i_{j_l}}) [\alpha^k(x_{i_{j_l}}), \dots, D(x_{i_{j_l}}), \dots, \alpha^k(x_{i_{j_n}})] \right] \right) \\
 &\quad + \sum_{p \in P_s} \left(\sum_{s=1}^n (-1)^{(n-1)(n-j)} \varepsilon(d, X + Y_s) \varepsilon \left(\sum_{t=s+1}^n (y_t, \widehat{Y}_t + d) \right) \right. \\
 &\quad \cdot \left[\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), [\alpha^k(y_{s+1}), \dots, \alpha^k(y_n)], \alpha^k(y_1), \dots, \alpha^k(y_{s_t}), \right. \\
 &\quad \left. \left. \sum_{l=1}^n \varepsilon(d, Y_{p_{s_l}}) [\alpha^k(y_{p_{s_l}}), \dots, D(y_{p_{s_l}}), \dots, \alpha^k(y_{p_{s_n}})] \right] \right) \\
 &= \sum_{i \in I_j} \left(\sum_{j=1}^n (-1)^{(n-1)(n-j)} \varepsilon(d, X_j) \varepsilon(Y, d + X) \varepsilon \left(\sum_{t=j+1}^{n-1} (x_t, \widehat{X}_t + Y + d) \right) \varepsilon(d, \sum_{t=j+1}^{n-1} (\widehat{X}_t + Y)) \right. \\
 &\quad \cdot \left(D([x_{j+1}, \dots, x_{n-1}, [y_1, \dots, y_n], x_1, \dots, x_{j-1}, [x_{i_{j_1}}, \dots, x_{i_{j_n}}]]) \right) \\
 &\quad - \sum_{m=j+1}^{n-1} \varepsilon(d, x_{j+1} + \dots + x_{m-1}) \left[\alpha^k(x_{j+1}), \dots, D(x_m), \dots, \alpha^k(x_{n-1}), \right. \\
 &\quad \left. [\alpha^k(y_1), \dots, \alpha^k(y_n)], \alpha^k(x_1), \dots, \alpha^k(x_{j-1}), [x_{i_{j_1}}, \dots, x_{i_{j_n}}] \right] \\
 &\quad - \sum_{m=1}^{j-1} \varepsilon(d, x_{j+1} + \dots + x_{n-1} + Y + X_m) \left[\alpha^k(x_{j+1}), \dots, \alpha^k(x_{n-1}), \right. \\
 &\quad \left. [\alpha^k(y_1), \dots, \alpha^k(y_n)], \alpha^k(x_1), \dots, D(x_m), \dots, \alpha^k(x_{j-1}), [x_{i_{j_1}}, \dots, x_{i_{j_n}}] \right] \\
 &\quad - \varepsilon(d, x_{j+1} + \dots + x_{n-1}) \left[\alpha^k(x_{j+1}), \dots, \alpha^k(x_{n-1}), D([y_1, \dots, y_n]), \right. \\
 &\quad \left. \alpha^k(x_1), \dots, \alpha^k(x_{j-1}), [x_{i_{j_1}}, \dots, x_{i_{j_n}}] \right) \\
 &\quad + \sum_{p \in P_s} \left(\sum_{s=1}^n (-1)^{(n-1)(n-j)} \varepsilon(d, X + Y_s) \varepsilon(y_s, \sum_{t=s+1}^n \widehat{Y}_t + d) \varepsilon(d, \sum_{t=s+1}^n \widehat{Y}_t) \right. \\
 &\quad \cdot \left([\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), D([y_{s+1}, \dots, y_n, y_1, \dots, y_{s-1}, [y_{p_{s_1}}, \dots, y_{p_{s_n}}]])] \right) \\
 &\quad - \sum_{m=s+1}^n \varepsilon(d, y_{s+1} + \dots + y_{m-1}) \left[\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), [\alpha^k(y_{s+1}), \dots, \right. \\
 &\quad \left. D(y_m), \dots, \alpha^k(y_n), \alpha^k(y_1), \dots, \alpha^k(y_{s-1}), [\alpha^k(y_{p_{s_1}}), \dots, \alpha^k(y_{p_{s_n}})]] \right]
 \end{aligned}$$

$$\begin{aligned}
& - \sum_{m=1}^{s-1} \varepsilon(d, y_{s+1} + \dots + y_n + Y_m) \left[\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), [\alpha^k(y_{s+1}), \dots, \right. \\
& \left. - \alpha^k(y_n), \alpha^k(y_1), \dots, D(y_m), \dots, \alpha^k(y_{s-1}), [\alpha^k(y_{p_{s_1}}), \dots, \alpha^k(y_{p_{s_n}})]] \right] \Big) \\
& = \sum_{j=1}^{n-1} (-1)^{(n-1)(n-j)} \varepsilon(d, X_j) \varepsilon(Y, d + X) \varepsilon \left(\sum_{t=j+1}^{n-1} (x_t, \widehat{X}_t + Y + d) \right) \\
& \cdot \left((-1)^{(n-1)(n-j)} \varepsilon \left(\sum_{t=j+1}^{n-1} (x_t, \widehat{X}_t + Y + d) \right) \varepsilon(Y, X) D([x_1, \dots, x_{n-1}, [y_1, \dots, y_n]]) \right) \\
& - \sum_{m=j+1}^{n-1} \varepsilon(d, x_{j+1} + \dots + x_{m-1}) \varepsilon(d + x_m, \widehat{X}_m + Y) \varepsilon \left(\sum_{t=j+1}^{n-1} (x_t, \widehat{X}_t + Y + d) \right) \varepsilon(Y, d + X) \\
& \cdot [\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), [\alpha^k(y_1), \dots, \alpha^k(y_n)]] \\
& - \sum_{m=1}^{j-1} (-1)^{(n-1)(n-j)} \varepsilon(d, x_{j+1} + \dots + x_{n-1} + Y + X_m) \varepsilon \left(\sum_{t=j+1}^{n-1} (x_t, \widehat{X}_t + Y + d) \right) \varepsilon(Y, X) \\
& \cdot [\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), [\alpha^k(y_1), \dots, \alpha^k(y_n)]] \\
& - (-1)^{(n-1)(n-j)} \varepsilon(d, x_{j-1} + \dots + x_{n-1}) \varepsilon \left(\sum_{t=j+1}^{n-1} (x_t, \widehat{X}_t + Y + d) \right) \\
& \cdot \varepsilon(d + Y, X) [\alpha^k(y_1), \dots, \alpha^k(y_n), D([y_1, \dots, y_n])] \Big) \\
& + \sum_{s=1}^n (-1)^{(n-1)(n-j)} \varepsilon(d, X + Y_s) \varepsilon \left(\sum_{t=j+1}^{n-1} (y_t, \widehat{Y}_t) \right) \left((-1)^{(n-1)(n-j)} \varepsilon \left(\sum_{t=j+1}^{n-1} (y_t, \widehat{Y}_t) \right) \right) \\
& \cdot [\alpha^k(y_1), \dots, \alpha^k(y_n), D([y_1, \dots, y_s, \dots, y_n])] \\
& - \sum_{m=s+1}^n \varepsilon(d, y_{s+1} + \dots + y_{m-1}) \varepsilon \left(\sum_{\substack{t=s+1 \\ t \neq m}}^n (y_t, d + \widehat{Y}_t) \right) \\
& \cdot \varepsilon(d + y_m, \widehat{Y}_m) [\alpha^k(y_1), \dots, \alpha^k(y_n), [\alpha^k(y_1), \dots, D(y_m), \dots, \alpha^k(y_n)]] \\
& - \sum_{m=1}^{s-1} (-1)^{(n-1)(s-1)} \varepsilon(d, y_{s+1} + \dots + y_n + Y_m) \varepsilon \left(\sum_{t=s+1}^n (y_t, d + \widehat{Y}_t) \right) \\
& \cdot [\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), [\alpha^k(y_1), \dots, D(y_m), \dots, \alpha^k(y_n)]] \Big) \\
& = \sum_{j=1}^{n-1} D \left([x_1, \dots, x_j, \dots, x_{n-1}, [y_1, \dots, y_n]] \right) - \sum_{j=1}^{n-1} \sum_{m=j+1}^{n-1} \varepsilon(d, X_m) \\
& + \varepsilon(d, X) [\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), D([y_1, \dots, y_n])] \\
& \cdot [\alpha^k(x_1), \dots, \alpha^k(x_j), \dots, D(x_m), \dots, \alpha^k(x_{n-1}), [\alpha^k(y_1), \dots, \alpha^k(y_n)]] \Big)
\end{aligned}$$

$$\begin{aligned}
 & - \sum_{j=1}^{n-1} \sum_{m=1}^{j-1} \varepsilon(d, X_m) \cdot \left[\alpha^k(x_1), \dots, D(x_m), \dots, \alpha^k(x_j), \dots, \alpha^k(x_{n-1}), [\alpha^k(y_1), \dots, \alpha^k(y_n)] \right] \\
 & - \sum_{j=1}^{n-1} \varepsilon(d, X) \left[\alpha^k(x_1), \dots, \alpha^k(x_j), \dots, \alpha^k(x_{n-1}), D([y_1, \dots, y_n]) \right] \\
 & + \sum_{s=1}^n \varepsilon(d, X) \left[\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), D([y_1, \dots, y_s, \dots, y_n]) \right] \\
 & - \sum_{s=1}^n \sum_{m=s+1}^n \varepsilon(d, X + Y_m) \left[\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), [\alpha^k(y_1), \dots, \alpha^k(y_s), \dots, D(y_m), \dots, \alpha^k(y_n)] \right] \\
 & - \sum_{s=1}^n \sum_{m=1}^{s-1} \varepsilon(d, X + Y_m) \left[\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), [\alpha^k(y_1), \dots, D(y_m), \dots, \alpha^k(y_s), \dots, \alpha^k(y_n)] \right] \\
 & = \sum_{j=1}^{n-1} D \left([x_1, \dots, x_j, \dots, x_{n-1}, [y_1, \dots, y_n]] \right) - \sum_{j=1}^{n-1} \sum_{\substack{m=1 \\ m \neq j}}^{n-1} \varepsilon(d, X_m) \\
 & \cdot \left[\alpha^k(x_1), \dots, D(x_m), \dots, \alpha^k(x_{n-1}), [\alpha^k(y_1), \dots, \alpha^k(y_n)] \right] \\
 & + \varepsilon(d, X) \left[\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), D([y_1, \dots, y_n]) \right] \\
 & - \sum_{s=1}^{n-1} \sum_{m=1}^n \varepsilon(d, X_m) \left[\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), [\alpha^k(y_1), \dots, D(y_m), \dots, \alpha^k(y_n)] \right] \\
 & = (n-1) D \left([x_1, \dots, x_j, \dots, x_{n-1}, [y_1, \dots, y_n]] \right) \\
 & - (n-2) \sum_{m=1}^n \varepsilon(d, X_m) \left[\alpha^k(x_1), \dots, D(x_m), \dots, \alpha^k(x_{n-1}), [\alpha^k(y_1), \dots, \alpha^k(y_n)] \right] \\
 & + \varepsilon(d, X) \left[\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), D([y_1, \dots, y_n]) \right] \\
 & - (n-1) \sum_{m=1}^{n-1} \varepsilon(d, X_m) \left[\alpha^k(x_1), \dots, D(x_m), \dots, \alpha^k(x_{n-1}), [\alpha^k(y_1), \dots, \alpha^k(y_n)] \right] \\
 & = (n-1) \left(\sum_{m=1}^{n-1} \varepsilon(d, X_m) \left[\alpha^k(x_1), \dots, D(x_m), \dots, \alpha^k(x_{n-1}), [\alpha^k(y_1), \dots, \alpha^k(y_n)] \right] \right. \\
 & \left. + \sum_{m=1}^n \varepsilon(d, X + Y_m) \left[\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), [\alpha^k(y_1), \dots, D(y_m), \dots, \alpha^k(y_n)] \right] \right) \\
 & - (n-2) \sum_{m=1}^{n-1} \varepsilon(d, X_m) \left[\alpha^k(x_1), \dots, D(x_m), \dots, \alpha^k(x_{n-1}), [\alpha^k(y_1), \dots, \alpha^k(y_n)] \right] \\
 & + \varepsilon(d, X) \left[\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), D([y_1, \dots, y_n]) \right]
 \end{aligned}$$

$$\begin{aligned}
& -(n-1) \sum_{m=1}^n \varepsilon(d, X + Y_m) \left[\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), [\alpha^k(y_1), \dots, D(y_m), \dots, \alpha^k(y_n)] \right] \\
&= \sum_{m=1}^{n-1} \varepsilon(d, X_m) [\alpha^k(x_1), \dots, D(x_m), \dots, \alpha^k(x_{n-1}), [\alpha^k(y_1), \dots, \alpha^k(y_n)]] \\
&\quad + \varepsilon(d, X) \left[\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), D([y_1, \dots, y_n]) \right] \\
&= \delta_D \left([x_1, \dots, x_{n-1}, [y_1, \dots, y_n]] \right),
\end{aligned}$$

where $\widehat{X}_t = \sum_{j=1}^{n-1} y_j$. Thus, $\delta_D \in \mathcal{D}_{\alpha^k}(\mathcal{L})$, and so is a double derivation of \mathcal{L} .

(2) For any $D \in \mathcal{D}_{\alpha^k}(\mathcal{L})$ and $x_i, y_j, z_i \in \mathcal{H}(\mathcal{L})$ with $1 \leq i \leq n-1, 1 \leq j \leq n$, it follows from Eq. (2) that

$$\begin{aligned}
& \left[\alpha^k(z_1), \dots, \alpha^k(z_{n-1}), D([x_1, \dots, x_{n-1}, [y_1, \dots, y_n]]) \right] \\
&= \sum_{i=1}^{n-1} \varepsilon(d, X_i) \left[\alpha^k(z_1), \dots, \alpha^k(z_{n-1}), [\alpha^k(x_1), \dots, D(x_i), \dots, \alpha^k(x_{n-1}), [\alpha^k(y_1), \dots, \alpha^k(y_n)]] \right] \\
&\quad + \sum_{j=1}^n \varepsilon(d, X + Y_j) \left[\alpha^k(z_1), \dots, \alpha^k(z_{n-1}), \right. \\
&\quad \left. [\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), [\alpha^k(y_1), \dots, D(y_j), \dots, \alpha^k(y_n)]] \right] \\
&= \left[\alpha^k(z_1), \dots, \alpha^k(z_{n-1}), \delta_D([x_1, \dots, x_{n-1}, [y_1, \dots, y_n]]) \right] \\
&\quad - \varepsilon(d, X) \left[\alpha^k(z_1), \dots, \alpha^k(z_{n-1}), [\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), D([y_1, \dots, y_n])] \right] \\
&\quad + \varepsilon(d, X) \left[\alpha^k(z_1), \dots, \alpha^k(z_{n-1}), [\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), \delta_D([y_1, \dots, y_n])] \right].
\end{aligned}$$

Hence,

$$\begin{aligned}
& \left[\alpha^k(z_1), \dots, \alpha^k(z_{n-1}), (D - \delta_D)([x_1, \dots, x_{n-1}, [y_1, \dots, y_n]]) \right] \\
&= - \left[\alpha^k(z_1), \dots, \alpha^k(z_{n-1}), [\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), \varepsilon(d, X)(D - \delta_D)([y_1, \dots, y_n])] \right].
\end{aligned}$$

Taking into account $[\mathcal{L}, \mathcal{L}, \dots, \mathcal{L}] = \mathcal{L}$ and $Z(\mathcal{L}) = 0$ we have that

$$(D - \delta_D)([x_1, \dots, x_{n-1}, [y_1, \dots, y_n]]) = -\varepsilon(d, X) [\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), (D - \delta_D)([y_1, \dots, y_n])],$$

and so

$$(D - \delta_D)([x_1, \dots, x_n]) = \varepsilon(d, X_i) [\alpha^k(x_1), \dots, (D - \delta_D)(x_i), \dots, \alpha^k(x_n)],$$

for all $x_1, \dots, x_n \in \mathcal{H}(\mathcal{L})$ and $i = 1, \dots, n$.

(3) By part (1), if D is a double derivation of \mathcal{L} , then δ_D is a double derivation of \mathcal{L} . Hence,

$D - \delta_D$ is a double derivation of \mathcal{L} . Now, by part (2) we find that

$$\begin{aligned} \delta_{D-\delta_D}([x_1, \dots, x_n]) &= \sum_{i=1}^n \varepsilon(d, X_i)[\alpha^k(x_1), \dots, (D - \delta_D)(x_i), \dots, \alpha^k(x_n)] \\ &= -n(D - \delta_D)([x_1, \dots, x_n]), \end{aligned}$$

which gives us the result. \square

Proposition 3.3. *If $(\mathcal{L}, [., \dots, .], \alpha, \varepsilon)$ is a centerless multiplicative perfect n -Hom-Lie color algebra. Then the following assertions hold.*

- (1) *If D is a double derivation of \mathcal{L} , then δ_D is a derivation of \mathcal{L} if and only if D is a derivation of \mathcal{L} . In particular, $\delta_D = D$ if D is a derivation of \mathcal{L} .*
- (2) *For all $x_1, \dots, x_n \in \mathcal{H}(\mathcal{L})$ and any α^k -double derivation D of \mathcal{L} ,*

$$[D, ad_s(x_1, \dots, x_n)] = ad_{k+s}(\delta_D(x_1), x_2, \dots, x_{n-1}) + \sum_{j=2}^{n-1} \varepsilon(d, X_j) ad_{k+s}(x_1, \dots, D(x_j), \dots, x_{n-1}).$$

Proof. (1) Let D be a derivation of \mathcal{L} . Then by Eq. (2),

$$\delta_D([x_1, \dots, x_n]) = D([x_1, \dots, x_n]).$$

Taking into account $[\mathcal{L}, \mathcal{L}, \dots, \mathcal{L}] = \mathcal{L}$, we get $\delta_D = D$. Next, suppose that $D \in \mathcal{D}_{\alpha^k}(\mathcal{L})$ but not contained in $Der(\mathcal{L})$ and so $\delta_D \neq D$. Then there exist $x_1, \dots, x_n \in \mathcal{H}(\mathcal{L})$ such that $(D - \delta_D)([x_1, \dots, x_n]) \neq 0$. Suppose that

$$x_j = \sum_{i \in I_j} [x_{i_{j_1}}, \dots, x_{i_{j_n}}], \quad j = 1, \dots, n.$$

We have

$$\begin{aligned} &\sum_{j=1}^n \varepsilon(d, X_j)[\alpha^k(x_1), \dots, \delta_D(x_j), \dots, \alpha^k(x_n)] \\ &= \sum_{j=1}^n \varepsilon(d, X_j) \left[\alpha^k(x_1), \dots, \sum_{i \in I_j} \sum_{l=1}^n \varepsilon(d, X_{i_{j_l}})[\alpha^k(x_{i_{j_1}}), \dots, D(x_{i_{j_l}}), \dots, \alpha^k(x_{i_{j_n}})], \dots, \alpha^k(x_n) \right] \\ &= \sum_{j=1}^n \sum_{i \in I_j} (-1)^{(n-1)(n-j)} \varepsilon(d, X_j) \varepsilon \left(\sum_{m=j+1}^n (x_m, d + \widehat{X_m}) \right) \\ &\quad \cdot \left[\alpha^k(x_{j+1}), \dots, \alpha^k(x_n), \alpha^k(x_1), \dots, \sum_{l=1}^n \varepsilon(x, X_{i_{j_l}})[\alpha^k(x_{i_{j_1}}), \dots, D(x_{i_{j_l}}), \dots, \alpha^k(x_{i_{j_n}})] \right] \\ &= \sum_{j=1}^n \sum_{i \in I_j} (-1)^{(n-1)(n-j)} \varepsilon(d, X_j) \varepsilon \left(\sum_{m=j+1}^n (x_m, d + \widehat{X_m}) \right) \varepsilon(d, \widehat{X_j}) \end{aligned}$$

$$\begin{aligned}
& \cdot \left(D([x_{j+1}, \dots, x_n, x_1, \dots, x_{j-1}, [x_{i_{j_1}}, \dots, x_{i_{j_n}}]]) - \sum_{s=j+1}^n \varepsilon(d, x_{j+1} + \dots + x_{s-1}) \right. \\
& \cdot [\alpha^k(x_{j+1}), \dots, D(x_s), \dots, \alpha^k(x_n), \alpha^k(x_1), \dots, \alpha^k(x_{j-1}), [\alpha^k(x_{i_{j_1}}), \dots, \alpha^k(x_{i_{j_n}})]] \\
& - \sum_{s=1}^{j-1} \varepsilon(d, x_{j+1} + \dots + x_{s-1}) \\
& \cdot [\alpha^k(x_{j+1}), \dots, \alpha^k(x_n), \alpha^k(x_1), \dots, D(x_s), \dots, \alpha^k(x_{j-1}), [\alpha^k(x_{i_{j_1}}), \dots, \alpha^k(x_{i_{j_n}})]] \Big) \\
& = \sum_{j=1}^n (-1)^{(n-1)(n-j)} \varepsilon(d, X_j) \varepsilon\left(\sum_{m=j+1}^n (x_m, \widehat{X}_m)\right) \varepsilon(d, \widehat{X}_j) \\
& \cdot \left((-1)^{(n-1)(n-j)} \varepsilon\left(\sum_{m=j+1}^n (x_m, \widehat{X}_m)\right) D([x_1, \dots, x_n]) \right. \\
& - \sum_{s=j+1}^n \varepsilon(d, x_{j+1} + \dots + x_{s-1}) \varepsilon\left(\sum_{m=j+1}^n (x_m, d + \widehat{X}_m)\right) \varepsilon(d + x_s, \widehat{X}_s) \\
& \cdot [\alpha^k(x_1), \dots, D(x_s), \dots, \alpha^k(x_n)] \\
& - \sum_{s=1}^{j-1} (-1)^{(n-1)(n-j)} \varepsilon(d, x_{j+1} + \dots + x_{s-1}) \varepsilon\left(\sum_{m=j+1}^n (x_m, d + \widehat{X}_m)\right) \\
& \cdot [\alpha^k(x_1), \dots, D(x_s), \dots, \alpha^k(x_n)] \Big) \\
& = \sum_{j=1}^n D([x_1, \dots, x_n]) - \sum_{j=1}^n \sum_{s=j+1}^n \varepsilon(d, X_s) [\alpha^k(x_1), \dots, D(x_s), \dots, \alpha^k(x_n)] \\
& - \sum_{j=1}^n \sum_{s=1}^{j-1} \varepsilon(d, X_s) [\alpha^k(x_1), \dots, D(x_s), \dots, \alpha^k(x_n)] \\
& = nD([x_1, \dots, x_n]) - \sum_{j=1}^n \sum_{\substack{s=1 \\ s \neq j}}^n \varepsilon(d, X_s) [\alpha^k(x_1), \dots, D(x_s), \dots, \alpha^k(x_n)] \\
& = nD([x_1, \dots, x_n]) - (n-1) \sum_{s=1}^n \varepsilon(d, X_s) [\alpha^k(x_1), \dots, D(x_s), \dots, \alpha^k(x_n)] \\
& = nD([x_1, \dots, x_n]) - (n-1) \delta_D([x_1, \dots, x_n]) \\
& = n(D - \delta_D)([x_1, \dots, x_n]) + \delta_D([x_1, \dots, x_n]) \\
& \neq \delta_D([x_1, \dots, x_n]).
\end{aligned}$$

It follows that $\delta_D \notin \text{Der}_{\alpha^k}(\mathcal{L})$ and so δ_D is not a derivation of \mathcal{L} .

(2) For all $x_1, \dots, x_n \in \mathcal{H}(\mathcal{L})$ and any $D \in \mathcal{D}_{\alpha^k}(\mathcal{L})$, by Theorem 2.8 and Eq. (2), we have

$$\begin{aligned}
 [D, ad_s(x_1, \dots, x_n)] &= \sum_{j=2}^{n-1} \varepsilon(d, X_j) ad_{k+s}(x_1, \dots, D(x_j), \dots, x_{n-1}) \\
 &+ \sum_{i \in I} \sum_{t=1}^n \varepsilon(d, X_{i_t}) ad_{k+s}([x_{i_1}, \dots, D(x_{i_t}), \dots, x_{i_n}], x_2, \dots, x_{n-1}) \\
 &= \sum_{j=2}^{n-1} \varepsilon(d, X_j) ad_{k+s}(x_1, \dots, D(x_j), \dots, x_{n-1}) \\
 &+ \sum_{i \in I} \sum_{t=1}^n \varepsilon(d, X_{i_t}) ad_{k+s}([\alpha^k(x_{i_1}), \dots, D(x_{i_t}), \dots, \alpha^k(x_{i_n})], x_2, \dots, x_{n-1}) \\
 &= \sum_{j=2}^{n-1} \varepsilon(d, X_j) ad_{k+s}(x_1, \dots, D(x_j), \dots, x_{n-1}) + ad_{k+s}(\delta_D(x_1), x_2, \dots, x_{n-1}).
 \end{aligned}$$

Therefore, the result holds. \square

Proposition 3.4. *If $(\mathcal{L}, [\cdot, \dots, \cdot], \alpha, \varepsilon)$ is a centerless perfect n -Hom-Lie color algebra. Then the map $\delta : \mathcal{D}(\mathcal{L}) \rightarrow \text{End}(\mathcal{L})$ is a Hom-Lie color algebra homomorphism, that is, $\delta_{[D_1, D_2]} = [\delta_{D_1}, \delta_{D_2}]$.*

Proof. Suppose that $D_1 \in \mathcal{D}_{\alpha^k}(\mathcal{L})$ and $D_2 \in \mathcal{D}_{\alpha^s}(\mathcal{L})$. Since \mathcal{L} is perfect, for any $x \in \mathcal{H}(\mathcal{L})$ we can write $x = \sum_{i \in I} [x_{i_1}, \dots, x_{i_n}] \in \mathcal{L}$. By Eq. (2) we have

$$\begin{aligned}
 [\delta_{D_1}, \delta_{D_2}](x) &= (\delta_{D_1} \delta_{D_2} - \varepsilon(d_1, d_2) \delta_{D_2} \delta_{D_1})(x) \\
 &= \sum_{i \in I} \left(\delta_{D_1} (\delta_{D_2}([x_{i_1}, \dots, x_{i_n}])) - \varepsilon(d_1, d_2) \delta_{D_2} (\delta_{D_1}([x_{i_1}, \dots, x_{i_n}])) \right) \\
 &= \sum_{i \in I} \left(\delta_{D_1} \left(\sum_{j=1}^n \varepsilon(d_2, X_{i_j}) [\alpha^s(x_{i_1}), \dots, D_2(x_{i_j}), \dots, \alpha^s(x_{i_n})] \right) \right. \\
 &\quad \left. - \varepsilon(d_1, d_2) \delta_{D_2} \left(\sum_{j=1}^n \varepsilon(d_1, X_{i_j}) [\alpha^k(x_{i_1}), \dots, D_1(x_{i_j}), \dots, \alpha^k(x_{i_n})] \right) \right) \\
 &= \sum_{i \in I} \left(\sum_{j=1}^n \left(\sum_{l=1}^{j-1} \varepsilon(d_2, X_{i_j}) \varepsilon(d_1, X_{i_l}) \right) \right. \\
 &\quad \cdot [\alpha^{k+s}(x_{i_1}), \dots, D_1(\alpha^k(x_{i_l})), \dots, D_2(x_{i_j}), \dots, \alpha^{k+s}(x_{i_n})] \\
 &\quad + \sum_{l=j+1}^n \varepsilon(d_2, X_{i_j}) \varepsilon(d_1, X_{i_l}) [\alpha^{k+s}(x_{i_1}), \dots, D_2(\alpha^s(x_{i_l})), \dots, D_1(x_{i_j}), \dots, \alpha^{k+s}(x_{i_n})] \\
 &\quad \left. - \varepsilon(d_1, d_2) \sum_{j=1}^n \left(\sum_{l=1}^{j-1} \varepsilon(d_1, X_{i_j}) \varepsilon(d_2, X_{i_l}) \right) \right. \\
 &\quad \left. \cdot [\alpha^{k+s}(x_{i_1}), \dots, D_2(\alpha^s(x_{i_l})), \dots, D_1(x_{i_j}), \dots, \alpha^{k+s}(x_{i_n})] \right)
 \end{aligned}$$

$$\begin{aligned}
& + \sum_{l=j+1}^n \varepsilon(d_2, X_{i_j}) \varepsilon(d_1, X_{i_l}) [\alpha^{k+s}(x_{i_1}), \dots, D_1(\alpha^k(x_{i_l})), \dots, D_2(x_{i_j}), \dots, \alpha^{k+s}(x_{i_n})] \\
= & \sum_{i \in I} \left(\sum_{j=1}^n \varepsilon(d_1 + d_2, X_{i_j}) [\alpha^{k+s}(x_{i_1}), \dots, D_1 D_2(x_{i_j}), \dots, \alpha^{k+s}(x_{i_n})] \right. \\
& \left. - \sum_{j=1}^n \varepsilon(d_1 + d_2, X_{i_j}) [\alpha^{k+s}(x_{i_1}), \dots, \varepsilon(d_1, d_2) D_2 D_1(x_{i_j}), \dots, \alpha^{k+s}(x_{i_n})] \right) \\
= & \sum_{i \in I} \sum_{j=1}^n \varepsilon(d_1 + d_2, X_{i_j}) \left[\alpha^{k+s}(x_{i_1}), \dots, (D_1 D_2 - \varepsilon(d_1, d_2) D_2 D_1)(x_{i_j}), \dots, \alpha^{k+s}(x_{i_n}) \right] \\
= & \sum_{i \in I} \sum_{j=1}^n \varepsilon(d_1 + d_2, X_{i_j}) \left[\alpha^{k+s}(x_{i_1}), \dots, [D_1, D_2](x_{i_j}), \dots, \alpha^{k+s}(x_{i_n}) \right] \\
= & \delta_{[D_1, D_2]}(x).
\end{aligned}$$

□

Theorem 3.5. *If $(\mathcal{L}, [\cdot, \dots, \cdot], \alpha, \varepsilon)$ is a perfect n -Hom-Lie color algebra. Then the centralizer of $\text{Inn}(\mathcal{L})$ in $\mathcal{D}(\mathcal{L})$ is trivial. In particular, the center of the Hom-Lie color algebra $\mathcal{D}(\mathcal{L})$ is zero.*

Proof. Let $D \in \mathcal{D}_{\alpha^k}(\mathcal{L})$ of degree d and $D \in C_{\mathcal{D}(\mathcal{L})}(\text{Inn}(\mathcal{L}))$. Then

$$[D, \text{ad}_k(x_1, \dots, x_{n-1})] = 0, \quad \forall x_1, x_2, \dots, x_{n-1} \in \mathcal{H}(\mathcal{L}).$$

Now, for all $z \in \mathcal{H}(\mathcal{L})$, we have

$$\begin{aligned}
0 &= [D, \text{ad}_k(x_1, \dots, x_{n-1})](z) \\
&= D \circ \text{ad}_k(x_1, \dots, x_{n-1})(z) - \varepsilon(d, X) \text{ad}_k(x_1, \dots, x_{n-1}) \circ D(z) \\
&= D([x_1, \dots, x_{n-1}, \alpha^k(z)]) - \varepsilon(d, X)[[\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), \alpha^k(D(z))]].
\end{aligned}$$

From here, we obtain

$$\begin{aligned}
(3) \quad D([x_1, \dots, x_{n-1}, \alpha^k(z)]) &= \varepsilon(d, X)[\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), \alpha^k(D(z))] \\
&= \varepsilon(d, X_i)[\alpha^k(x_1), \dots, D(x_i), \dots, \alpha^k(x_{n-1}), \alpha^k(z)],
\end{aligned}$$

for $1 \leq i \leq n-1$.

Next, let $x_1, \dots, x_{n-1}, y_1, \dots, y_n \in \mathcal{H}(\mathcal{L})$ and taking into account Eq. (3), we get

$$\begin{aligned}
& D([x_1, \dots, x_{n-1}, [y_1, \dots, y_n]]) \\
&= \sum_{i=1}^{n-1} \varepsilon(d, X_i)[\alpha^k(x_1), \dots, D(x_i), \dots, \alpha^k(x_{n-1}), [\alpha^k(y_1), \dots, \alpha^k(y_n)]]
\end{aligned}$$

$$\begin{aligned}
 & + \sum_{j=1}^n \varepsilon(d, X + Y_j) [\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), [\alpha^k(y_1), \dots, D(y_j), \dots, \alpha^k(y_n)]] \\
 & = (n - 1)D([x_1, \dots, x_{n-1}, [y_1, \dots, y_n]]) + nD([x_1, \dots, x_{n-1}, [y_1, \dots, y_n]]) \\
 & = (2n - 1)D([x_1, \dots, x_{n-1}, [y_1, \dots, y_n]]),
 \end{aligned}$$

so we obtain $(2n - 2)D([x_1, \dots, x_{n-1}, [y_1, \dots, y_n]]) = 0$. That is,

$$D([x_1, \dots, x_{n-1}, [y_1, \dots, y_n]]) = 0.$$

Taking into account $[\mathcal{L}, \mathcal{L}, \dots, \mathcal{L}] = \mathcal{L}$, we conclude that $D = 0$. \square

4. TRIPLE DERIVATIONS OF DOUBLE DERIVATION ALGEBRA $\mathcal{D}(\mathcal{L})$

In the final section, we are particularly interested in the triple derivations of the Hom-Lie color algebra $(\mathcal{D}(\mathcal{L}), [., .], \tilde{\alpha}, \varepsilon)$ consisting of double derivations algebra as well as in the triple derivations of the Hom-Lie color algebra $(\text{Inn}(\mathcal{L}), [., .], \tilde{\alpha}, \varepsilon)$ consisting of inner derivations algebra of a perfect n -Hom-Lie color algebra \mathcal{L} .

Recall that, a linear map D of a multiplicative Hom-Lie color algebra $(L, [., .], \alpha, \varepsilon)$ is called an α^k -triple derivation if it satisfies

(1) $D \circ \alpha = \alpha \circ D$.

(2) For all $x, y, z \in \mathcal{H}(L)$,

(4) $D([x, [y, z]]) = [D(x), [\alpha^k(y), \alpha^k(z)]] + \varepsilon(d, x)[\alpha^k(x), [D(y), \alpha^k(z)]]$

(5) $+ \varepsilon(d, x + y)[\alpha^k(x), [\alpha^k(y), D(z)]]$.

Let us denote the set of all α^k -triple derivations of L by $TDer_{\alpha^k}(L)$ and denote $TDer(L) = \bigoplus_{k \geq 0} TDer_{\alpha^k}(L)$, which is a Hom-Lie color algebra and it is called the triple derivation algebra of L

In the following we suppose that \mathcal{L} is a centerless perfect n -Hom-Lie color algebra and $L = (\text{Inn}(\mathcal{L}), [., .], \tilde{\alpha} = D \circ \alpha, \varepsilon)$ is a perfect Hom-Lie color algebra. Then we have the following results.

Lemma 4.1. *If $(\mathcal{L}, [., \dots, .], \alpha, \varepsilon)$ is a perfect n -Hom-Lie color algebra. Then every triple derivation D of $(\mathcal{D}(\mathcal{L}), [., .], \tilde{\alpha}, \varepsilon)$ keeps $L = \text{Inn}(\mathcal{L})$ invariant. Furthermore, if $D(L) = 0$, then $D = 0$.*

Proof. First, we show that $D(L) \subseteq L$, for $D \in TDer_{\tilde{\alpha}^k}(\mathcal{D}(\mathcal{L}))$. Since $L = \{\text{ad}_s(x_1, \dots, x_{n-1}) : \alpha(x_i) = x_i \in \mathcal{L}, i = 1, \dots, n - 1\}$ is a perfect Hom-Lie color algebra, for

all $x_1, \dots, x_{n-1} \in \mathcal{H}(\mathcal{L})$ there exist $x_{1_i}, \dots, x_{n-1_i}, y_{1_{i_j}}, \dots, y_{n-1_{i_j}} \in \mathcal{L}$ and $z_{1_{i_j}}, \dots, z_{n-1_{i_j}} \in \mathcal{H}(\mathcal{L})$ such that

$$\begin{aligned} \text{ad}_s(x_1, \dots, x_{n-1}) &= \sum_{i \in I} [\text{ad}_s(x_{1_i}, \dots, x_{n-1_i}), \text{ad}_s(y_{1_i}, \dots, y_{n-1_i})] \\ &= \sum_{i \in I} \left([\text{ad}_s(x_{1_i}, \dots, x_{n-1_i}), \sum_{j \in J} [\text{ad}_s(y_{1_{i_j}}, \dots, y_{n-1_{i_j}}), \text{ad}_s(z_{1_{i_j}}, \dots, z_{n-1_{i_j}})]] \right), \end{aligned}$$

for some finite index sets I and J . Now, by definition of triple derivation we have

$$\begin{aligned} &D(\text{ad}_s(x_1, \dots, x_{n-1})) \sum_{i \in I} \sum_{j \in J} D \left([\text{ad}_s(x_{1_i}, \dots, x_{n-1_i}), [\text{ad}_s(y_{1_{i_j}}, \dots, y_{n-1_{i_j}}), \text{ad}_s(z_{1_{i_j}}, \dots, z_{n-1_{i_j}})]] \right) \\ &= \sum_{i \in I} \sum_{j \in J} \left([D(\text{ad}_s(x_{1_i}, \dots, x_{n-1_i})), \right. \\ &\quad [\text{ad}_s(\tilde{\alpha}^k(y_{1_{i_j}}, \dots, y_{n-1_{i_j}}), \tilde{\alpha}^k(z_{1_{i_j}}, \dots, z_{n-1_{i_j}}))] \\ &\quad + \varepsilon(d, X_i) [\text{ad}_s(\tilde{\alpha}^k(x_{1_i}), \dots, \tilde{\alpha}^k(x_{n-1_i})), \\ &\quad [D(\text{ad}_s(\tilde{\alpha}^k(y_{1_{i_j}}, \dots, y_{n-1_{i_j}})), \text{ad}_s(\tilde{\alpha}^k(z_{1_{i_j}}, \dots, z_{n-1_{i_j}}))] \\ &\quad + \varepsilon(d, X_i + Y_{i_j}) [\text{ad}_s(\tilde{\alpha}^k(x_{1_i}), \dots, \tilde{\alpha}^k(x_{n-1_i})), \\ &\quad \left. [\text{ad}_s(\tilde{\alpha}^k(y_{1_{i_j}}, \dots, y_{n-1_{i_j}}), D(\text{ad}_s(z_{1_{i_j}}, \dots, z_{n-1_{i_j}}))] \right)]. \end{aligned}$$

Thanks to Theorem 2.8, we obtain $D(\text{ad}_s(x_1, \dots, x_{n-1}))$ is an inner $\tilde{\alpha}^{k+s}$ -derivation and so $D(L) \subseteq L$.

Next, let D be an $\tilde{\alpha}^k$ -triple derivation of $\mathcal{D}(\mathcal{L})$ which satisfies $D(L) = 0$. Then for each double derivation d of \mathcal{L} , again by perfectness of L we have

$$\begin{aligned} [D(d), \text{ad}_s(x_1, \dots, x_{n-1})] &= \sum_{i \in I} [D(d), [\text{ad}_s(x_{1_i}, \dots, x_{n-1_i}), \text{ad}_s(y_{1_i}, \dots, y_{n-1_i})]] \\ &= \sum_{i \in I} \left(D([d, [\text{ad}_s(x_{1_i}, \dots, x_{n-1_i}), \text{ad}_s(y_{1_i}, \dots, y_{n-1_i})]]) \right. \\ &\quad - \varepsilon(d, e) [\tilde{\alpha}^k(d), [D(\text{ad}_s(x_{1_i}, \dots, x_{n-1_i})), \tilde{\alpha}^k(\text{ad}_s(y_{1_i}, \dots, y_{n-1_i}))]] \\ &\quad \left. - \varepsilon(d, e + X_i) [\tilde{\alpha}^k(d), [\tilde{\alpha}^k(\text{ad}_s(x_{1_i}, \dots, x_{n-1_i})), D(\text{ad}_s(y_{1_i}, \dots, y_{n-1_i}))]] \right). \end{aligned}$$

Since $[d, L] \subseteq L$ (thanks to Theorem 2.8), we get $D([d, \text{ad}_s(x_{1_i}, \dots, x_{n-1_i})]) = 0$. Thus, we can obtain $[D(d), \text{ad}_s(x_{1_i}, \dots, x_{n-1_i})] = 0$. Therefore, $D(d)$ is in the centralizer of L in $\mathcal{D}(\mathcal{L})$, and by Proposition 3.5, $D(d) = 0, \forall d \in \mathcal{D}(\mathcal{L})$. Hence, $D = 0$. The proof is completed. \square

Lemma 4.2. *Let $(\mathcal{L}, [., \dots, .], \alpha, \varepsilon)$ be a centerless perfect n -Hom-Lie color algebra. If $D \in T\text{Der}(\text{Der}(\mathcal{L}))$, then there exists $\Delta \in \text{Der}(\mathcal{L})$ such that $D(\text{ad}_s(x)) = \text{ad}_s(\Delta(x))$ for all $x \in \mathcal{H}(\mathcal{L})$.*

Proof. For any $D \in TDer_{\tilde{\alpha}^k}(Der(\mathcal{L}))$, by Lemma 4.1, we can obtain $D(ad_s(x)) \in Inn_{\tilde{\alpha}^{k+s}}(\mathcal{L})$ for all $x \in \mathcal{H}(\mathcal{L})$. Now, let $y \in \mathcal{H}(\mathcal{L})$ such that $D(ad_s(x)) = ad_{k+s}(y)$. Note that y is unique, thanks to $Z(\mathcal{L}) = 0$. Define the map $\Delta : \mathcal{L} \rightarrow \mathcal{L}$ by $\Delta(x) = y$ which is an endomorphism. For $x_1, x_2, x_3 \in \mathcal{H}(\mathcal{L})$, we have

$$\begin{aligned} ad_{k+s}\left(\Delta([x_1, x_2], x_3)\right) &= D\left(ad_s([x_1, x_2], x_3)\right) \\ &= D\left([\text{ad}_s(x_1), \text{ad}_s(x_2)], \text{ad}_s(x_3)\right) \\ &= [D(\text{ad}_s(x_1)), \tilde{\alpha}^k(\text{ad}_s(x_2)), \tilde{\alpha}^k(\text{ad}_s(x_3))] \\ &\quad + \varepsilon(d, x_1)[[\tilde{\alpha}^k(\text{ad}_s(x_1)), D(\text{ad}_s(x_2))], \tilde{\alpha}^k(\text{ad}_s(x_3))] \\ &\quad + \varepsilon(d, x_1 + x_2)[[\tilde{\alpha}^k(\text{ad}_s(x_1)), \tilde{\alpha}^k(\text{ad}_s(x_2))], D(\text{ad}_s(x_3))] \\ &= [\text{ad}_{k+s}(\Delta(x_1)), \tilde{\alpha}^k(\text{ad}_s(x_2)), \tilde{\alpha}^k(\text{ad}_s(x_3))] \\ &\quad + \varepsilon(d, x_1)[[\tilde{\alpha}^k(\text{ad}_s(x_1)), \text{ad}_{k+s}(\Delta(x_2))], \tilde{\alpha}^k(\text{ad}_s(x_3))] \\ &\quad + \varepsilon(d, x_1 + x_2)[[\tilde{\alpha}^k(\text{ad}_s(x_1)), \tilde{\alpha}^k(\text{ad}_s(x_2))], \text{ad}_{k+s}(\Delta(x_3))] \\ &= \text{ad}_{k+s}\left([\Delta(x_1), [\alpha^k(x_2), \alpha^k(x_3)]] + \varepsilon(d, x_1)[\alpha^k(x_1), [\Delta(x_2), \alpha^k(x_3)]]\right. \\ &\quad \left. + \varepsilon(d, x_1 + x_2)[\alpha^k(x_1), [\alpha^k(x_2), \Delta(x_3)]]\right). \end{aligned}$$

Since $Z(\mathcal{L}) = 0$, we obtain

$$\begin{aligned} \Delta\left([x_1, x_2], x_3\right) &= [[\Delta(x_1), \alpha^k(x_2)], \alpha^k(x_3)] + \varepsilon(d, x_1)[[\alpha^k(x_1), \Delta(x_2)], \alpha^k(x_3)] \\ &\quad + \varepsilon(d, x_1 + x_2)[[\alpha^k(x_1), \alpha^k(x_2)], \Delta(x_3)]. \end{aligned}$$

That is, $\Delta \in TDer_{\tilde{\alpha}^k}(\mathcal{L})$. By Proposition 3.3, $\Delta \in Der(\mathcal{L})$. \square

Theorem 4.3. Let $(\mathcal{L}, [., \dots, .], \alpha, \varepsilon)$ be a centerless perfect n -Hom-Lie color algebra. Then the triple derivation algebra of $Inn(\mathcal{L})$ coincides with the derivation algebra of $Inn(\mathcal{L})$.

Proof. We have $L = (Inn(\mathcal{L}), [., .], \tilde{\alpha}, \varepsilon)$ is a perfect Hom-Lie color algebra with zero center. For any $D \in TDer_{\tilde{\alpha}^k}(L)$, by Lemma 4.2 there exists $\Delta \in Der(\mathcal{L})$ such that $D(ad_s(x)) = ad_s(\Delta(x))$ for all $x \in \mathcal{L}$. As well known that $ad_s(\Delta(x)) = [\Delta, ad_s(x)]$. We also have

$$D(ad_s(x)) = ad_s(\Delta(x)) = [\Delta, ad_s(x)] = ad_s(\Delta)(ad_s(x)).$$

Hence, $(D - ad_s(\Delta(x)))(ad_s(x)) = 0$. By Lemma 4.1, $D = ad_s(\Delta(x))$. Therefore, $TDer(L)$ equal to derivation algebra of $Inn(\mathcal{L})$. \square

Theorem 4.4. *Let $(\mathcal{L}, [\cdot, \dots, \cdot], \alpha, \varepsilon)$ be a perfect n -Hom-Lie color algebra. Then the triple derivation algebra of Hom-Lie color algebra $(Der(\mathcal{L}), [\cdot, \cdot], \tilde{\alpha}, \varepsilon)$ is equal to the derivation algebra $Der(\mathcal{L})$.*

Proof. Suppose that D is a $\tilde{\alpha}^k$ -triple derivation of $Der(\mathcal{L})$, then D is an $\tilde{\alpha}^k$ -triple derivation of $L = Inn(\mathcal{L})$ by Lemma 4.2, we get D is an $\tilde{\alpha}^k$ -derivation of $L = Inn(\mathcal{L})$, thanks to Theorem 4.3. Since L is a perfect Hom-Lie color algebra, for any $x_1, \dots, x_{n-1} \in \mathcal{H}(\mathcal{L})$, there exist $x_{1_i}, \dots, x_{n-1_i}, y_{1_i}, \dots, y_{n-1_i} \in \mathcal{H}(\mathcal{L})$ such that

$$\text{ad}_s(x_1, \dots, x_{n-1}) = \sum_{i \in I} [\text{ad}_s(x_{1_i}, \dots, x_{n-1_i}), \text{ad}_s(y_{1_i}, \dots, y_{n-1_i})],$$

for some finite index set I . Then for all $d_1, d_2 \in Der(\mathcal{L})$ of degrees λ_1, λ_2 respectively, we have

$$\begin{aligned} D\left([d_1, d_2], \text{ad}_s(x_1, \dots, x_{n-1})\right) &= D\left([d_1, d_2], \sum_{i \in I} [\text{ad}_s(x_{1_i}, \dots, x_{n-1_i}), \text{ad}_s(y_{1_i}, \dots, y_{n-1_i})]\right) \\ &= \left[D([d_1, d_2]), \sum_{i \in I} [\text{ad}_s(\tilde{\alpha}^k(x_{1_i}), \dots, \tilde{\alpha}^k(x_{n-1_i})), \text{ad}_s(\tilde{\alpha}^k(y_{1_i}), \dots, \tilde{\alpha}^k(y_{n-1_i}))]\right] \\ &\quad - \varepsilon(d, \lambda_1 + \lambda_2) \left[\tilde{\alpha}^k([d_1, d_2]), \sum_{i \in I} [D(\text{ad}_s(x_{1_i}, \dots, x_{n-1_i})), \text{ad}_s(\tilde{\alpha}^k(y_{1_i}), \dots, \tilde{\alpha}^k(y_{n-1_i}))]\right] \\ &\quad - \varepsilon(d, \lambda_1 + \lambda_2 + X_i) \left[\tilde{\alpha}^k([d_1, d_2]), \sum_{i \in I} [\text{ad}_s(\tilde{\alpha}^k(x_{1_i}), \dots, \tilde{\alpha}^k(x_{n-1_i})), D(\text{ad}_s(y_{1_i}, \dots, y_{n-1_i}))]\right] \\ &= \left[D([d_1, d_2]), \tilde{\alpha}^k(\text{ad}_s(x_1, \dots, x_{n-1}))\right] + \varepsilon(d, \lambda_1 + \lambda_2) \left[\tilde{\alpha}^k[d_1, d_2], D(\text{ad}_s(x_1, \dots, x_{n-1}))\right] \\ &= \left[D([d_1, d_2]), \tilde{\alpha}^k(\text{ad}_s(x_1, \dots, x_{n-1}))\right] - \varepsilon(\lambda_1 + \lambda_2, X) D\left([\text{ad}_s(x_1, \dots, x_{n-1}), [d_1, d_2]]\right) \\ &\quad + \varepsilon(d, X) \varepsilon(\lambda_1 + \lambda_2, X) \left[\tilde{\alpha}^k(\text{ad}_s(x_1, \dots, x_{n-1})), [D(d_1), \tilde{\alpha}^k(d_2)]\right] \\ &\quad + \varepsilon(d, \lambda_1 + X) \varepsilon(\lambda_1 + \lambda_2, X) \left[\tilde{\alpha}^k(\text{ad}_s(x_1, \dots, x_{n-1})), [\tilde{\alpha}^k(d_1), D(d_2)]\right] \\ &= \left[D([d_1, d_2]), \tilde{\alpha}^k(\text{ad}_s(x_1, \dots, x_{n-1}))\right] + D\left([d_1, d_2], \text{ad}_s(x_1, \dots, x_{n-1})\right) \\ &= \left[[D(d_1), \tilde{\alpha}^k(d_2)], \tilde{\alpha}^k(\text{ad}_s(x_1, \dots, x_{n-1}))\right] - \varepsilon(d, \lambda_1) \left[[\tilde{\alpha}^k(d_1), D(d_2)], \tilde{\alpha}^k(\text{ad}_s(x_1, \dots, x_{n-1}))\right]. \end{aligned}$$

Thus,

$$D([d_1, d_2]) - [D(d_1), \tilde{\alpha}^k(d_2)] - \varepsilon(d, \lambda_1) [[\tilde{\alpha}^k(d_1), D(d_2)], \tilde{\alpha}^k(\text{ad}_s(x_1, \dots, x_{n-1}))] = 0.$$

Now, Proposition 3.5 gives us

$$D([d_1, d_2]) = [D(d_1), \tilde{\alpha}^k(d_2)] - \varepsilon(d, \lambda_1) [\tilde{\alpha}^k(d_1), D(d_2)].$$

That is D is an $\tilde{\alpha}^k$ -derivation of Hom-Lie color algebra $Der(\mathcal{L})$. \square

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