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Research Paper

ON AUTOMORPHISMS OF A CLASS OF NORMAL TRANSFORMATION SEMIGROUPS

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Abstract. In this paper, we are interested in the (as yet) unsolved problem of determining all subsemigroups of $\mathcal{T}(X)$ in which every automorphism is inner. We show that the semigroup of full transformations with restriction on a fixed set is bijective, *has an inner automorphism property* **(i.a.p)**.

1. Introduction

Transformation semigroups play a foundational role across various branches of mathematics and find extensive practical applications. Their unique ability to model dynamic systems through function composition makes them valuable in fields such as automata theory, combinatorial optimization, and mathematical analysis. By providing a structured framework for analyzing transformations, these semigroups enable researchers to classify and uncover the

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properties of numerous mathematical structures, leading to critical insights, particularly in computer science and optimization.

The study of automorphisms within transformation semigroups further deepens our understanding of these algebraic constructs, revealing inherent symmetries and manipulability. This exploration not only enhances theoretical comprehension but also connects abstract concepts to real-world applications, making automorphisms a key area of focus.

As research in this field progresses, the significance of automorphisms within transformation semigroups is likely to grow, illuminating connections with other mathematical areas such as order theory and topology. This ongoing inquiry promises to yield new insights and applications, affirming the role of transformation semigroups as a cornerstone in both mathematical theory and applied mathematics. Automorphisms of these structures play a significant role in understanding their symmetries and internal structure.

Let X be an arbitrary infinite set. The semigroup $\mathcal{T}(X)$ of full transformations on X consists of the functions from X to X, with composition as the semigroup operation. As established in [1], any semigroup S can be embedded in a transformation semigroup, making transformation semigroups a prototype for semigroups in general. Examining automorphisms of transformation semigroups is a crucial aspect of semigroup theory, with extensive work already accomplished in this area. For instance, Schreier [8] and Mal'cev [5] comprehensively described the automorphisms of the semigroup of all mappings from a set X to itself. In 1966, Magill [4] introduced and studied transformation semigroups on X that leave a subset Y invariant. Subsequently, Symons [13] explored classical concepts of these semigroups, and Sullivan [12] extended Symons' results to the class of constant-rich subsemigroups of $\mathcal{B}(X)$. Additionally, Levi [3] introduced the concept of $\mathcal{S}(X)$-normality, describing inner automorphisms within the class of $\mathcal{S}(X)$-normal semigroups, which are transformation semigroups closed under conjugation by permutations. Recently, Mir et al. [7] generalized the findings of Sullivan [12] and Levi [3] to $\mathcal{P}_{\mathcal{M}}(X)$, the semigroup of monotone partial transformations on a poset X.

A comprehensive study has been undertaken on the inner automorphisms of monoids of (partial) maps, with significant contributions from researchers who established fundamental results for several semigroups. The research have predominantly concentrated on particular instances such as the semigroups of all mappings, partial maps, and symmetric inverse semigroups. Recent improvements, notably the research by Mir et al. [7], have expanded these findings to encompass wider categories of transformation semigroups. Additionally, Mir and Alali, in [6], investigated the automorphisms of a semigroup S of centralizers of idempotent transformations with restricted range. However, the study of inner automorphisms in general monoids is still inadequately investigated, with only a limited number of general theorems and algorithms have been developed. More recently, Shah et al. in [9, 11, 10] extended the inner automorphism theorem from groups to monoids, introducing the concept of nearly complete monoids. Furthermore, they established necessary and sufficient conditions for a strong semilattice of groups to be classified as nearly complete.

In this paper, we introduce $\mathcal{S}(X,Y)$-normal semigroups and prove some general results on the automorphisms of these semigroups. We find a relationship between different types of ideals and finally we characterize the inner automorphisms of a class of $\mathcal{S}(X,Y)$-normal semigroups.

We consider a semigroup defined as for any fixed non-empty subset Y of X,

$$\mathcal{P}_{\mathcal{S}}(X,Y) = \{\alpha \in \mathcal{T}(X) : \ \alpha|_Y \in \mathcal{S}(Y)\}.$$

These semigroups were introduced by Laysirikul [2] in 2016. He studied the regularity and characterize the complete regular elements of $\mathcal{P}_{\mathcal{S}}(X,Y)$ and find the relationship between $\mathcal{P}_{\mathcal{S}}(X,Y)$, the subsemigroups of $\mathcal{P}_{\mathcal{S}}(X,Y)$ and $\mathcal{S}(X,Y)$. Also, let $\mathcal{S}(X,Y) = \{\alpha \in \mathcal{S}(X) : \ \alpha|_Y \in \mathcal{S}(Y)\}$, then one can easily seen that $\mathcal{S}(X,Y)$ forms a subgroup of $\mathcal{S}(X)$. A subsemigroup S of $\mathcal{P}_{\mathcal{S}}(X,Y)$ is said to have an inner automorphism property **(i.a.p)** if every automorphism ϕ of S takes the form $\phi(s) = hsh^{-1}$ for all $s \in S$, where h is some fixed element of $\mathcal{S}(X,Y)$ (in such a case we say ϕ is inner induced by h). By $(x\ y)$ we mean permutation of order 2 and $\alpha|_Y$ is restriction of α to Y. In this paper, we show the every normal subsemigroup S of $\mathcal{P}_{\mathcal{S}}(X,Y)$ which is not contained in $\mathcal{O}(X)$ has **i.a.p**.

Section 2 offers a comprehensive overview of the foundational concepts and general results in normal semigroups that form the basis for the discussions and findings in this paper. In Section 3, we introduce and rigorously prove key results concerning point and function right ideals, establishing a relationship between these structures. Finally, in Section 4, we demonstrate that each automorphism ϕ of S permutes point right ideals and provide a characterization of the inner automorphisms within a class of $\mathcal{S}(X,Y)$-normal semigroups.

2. General Results on Normal Semigroups

Definition 2.1. A semigroup S of transformations on X is said to be $\mathcal{S}(X,Y)$-normal if for all $\alpha \in S$, $g \in \mathcal{S}(X,Y)$, $g\alpha g^{-1} \in S$.

It is clear that for all $\alpha \in \mathcal{P}_{\mathcal{S}}(X,Y)$, $g \in \mathcal{S}(X,Y)$, $g\alpha g^{-1} \in \mathcal{P}_{\mathcal{S}}(X,Y)$, so $\mathcal{P}_{\mathcal{S}}(X,Y)$ is $\mathcal{S}(X,Y)$-normal. Now we have the following lemma.

Lemma 2.2. *If S is $\mathcal{S}(X)$-normal semigroup, then S is $\mathcal{S}(X,Y)$-normal semigroup.*

Proof. Let S is $\mathcal{S}(X)$-normal then for all $\alpha \in S$ and $g \in \mathcal{S}(X)$, $g^{-1}\alpha g \in S$. In particular, $g^{-1}\alpha g \in S$ for all $g \in \mathcal{S}(X,Y)$, therefore S is $\mathcal{S}(X,Y)$-normal. $\square$

The converse of above lemma is not true and is illustrated in the following example.

Example 2.3. *Let $X = \mathbb{N}$ and consider $Y = \{1, 2\}$, then it is clear that, $\mathcal{S}(X, Y) = \{\alpha \in \mathcal{S}(X) : \alpha|_Y = id_Y$ or $\alpha|_Y = (1\ 2)\}$. Consider a semigroup*

$$S = \mathcal{S}(X, Y) \cup \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{smallmatrix}\right)\}.$$

Since $\mathcal{S}(X, Y)$ is subgroup of S, so for any $\alpha \in S$ and $g \in \mathcal{S}(X, Y)$, $g^{-1}\alpha g \in S$. This implies, S is $\mathcal{S}(X, Y)$-normal. But S is not $\mathcal{S}(X)$-normal, as $g = (2\ 3) \in \mathcal{S}(X)$, $\alpha = (1\ 2) \in S$, and $g^{-1}\alpha g \notin S$.

So in view of Lemma 2.2 it is now natural to introduce $\mathcal{S}(X, Y)$-*normal* semigroups and describe their automorphisms.

Definition 2.4. If S is an arbitrary semigroup of transformations on X then

$$\Delta(S) = \{im\ \alpha : \alpha \in S\}.$$

Definition 2.5. We say that $\Delta(S)$ is $\mathcal{S}(X, Y)$-*normal* if for each $h \in \mathcal{S}(X, Y)$

$$h(\Delta(S)) = \Delta(S),$$

(by $h(\Delta(S))$ we mean $\{h(A) : A \in \Delta(S)\}$.

Lemma 2.6. *If S is a $\mathcal{S}(X, Y)$-normal semigroup, then $\Delta(S)$ is $\mathcal{S}(X, Y)$-normal.*

Proof. Let $B \in h(\Delta(S))$ there exists $A \in \Delta(S)$ such that $h(A) = B$. Therefore

$$\begin{aligned} B &= h(\alpha(X)) \text{ for some } \alpha \in S \\ &= h\alpha h^{-1}h(X)\ (h^{-1}h = id_X) \\ &= (h\alpha h^{-1})h(X) \\ &= gh(X)\ (h\alpha h^{-1} = g \in S) \\ &= g(X) \in \Delta(S) \text{ (as } h \text{ is onto)}, \end{aligned}$$

hence $h(\Delta(S)) \subseteq \Delta(S)$. Now let $B \in \Delta(S)$ therefore,

$$\begin{aligned} B &= \alpha(X) \text{ for some } \alpha \in S \\ &= hh^{-1}\alpha(X) \text{ (as } hh^{-1} = id_X) \\ &= h(h^{-1}\alpha(X)) \\ &= hg(X) \in h\Delta(S) \text{ (as } h^{-1}\alpha = g), \end{aligned}$$

that is, $\Delta(S) \subseteq h(\Delta(S))$, that is, $\Delta(S) = h(\Delta(S))$. Hence $\Delta(S)$ is $\mathcal{S}(X, Y)$-normal. □

We denote the semigroup of all onto mappings by $\mathcal{O}(X)$.

Lemma 2.7. *If S is $\mathcal{S}(X,Y)$-normal subsemigroup of $\mathcal{P}_{\mathcal{S}}(X,Y)$ such that $S \nsubseteq \mathcal{O}(X)$, then for given $x, y \in X \setminus Y$ with $x \neq y$ there exists an $A \in \Delta(S)$ with $y \in A$, $x \in X \setminus A$.*

Proof. Since $S \nsubseteq \mathcal{O}(X)$, then there exists $\alpha \in S$ such that $\alpha(X) = A \subsetneq X$. Take any $x, y \in X \setminus Y$ with $x \neq y$. If $y \in A$ and $x \in X \setminus A$, then we are done, otherwise the following cases will arise:

Case (i). If $x, y \in A$, we choose $h = (x\ z)$, where $z \notin A$, then we have $h(A) = (x\ z)(A) = (A \cup \{z\}) \setminus \{x\} = A' \in \Delta(S)$, thus for given $x, y \in X$ with $x \neq y$ there exists an $A' \in \Delta(S)$ with $y \in A'$, $x \in X \setminus A'$.

Case (ii). If $x, y \notin A$, since by definition of S we have $Y \subsetneq A$ so we choose $h = (y\ a)$, where $a \in A \setminus Y$, then we have $h(A) = (y\ a)(A) = (A \cup \{y\}) \setminus \{a\} = A' \in \Delta(S)$, thus the required set A' in $\Delta(S)$ exists. □

3. Relationship between Point and Function Right Ideals

From now on, let S denote the subsemigroup of $\mathcal{P}_{\mathcal{S}}(X,Y)$ that contains $\mathcal{S}(X,Y)$, with the condition that $S \nsubseteq \mathcal{O}(X)$. Clearly, S is $\mathcal{S}(X,Y)$-normal. We begin with the following lemma.

Lemma 3.1. *For any $x \in X \setminus Y$ the set*

$$\mathfrak{I}_x = \{r \in S : x \in X \setminus \Delta(r)\},$$

is a right ideal of S.

Proof. Suppose $\mathfrak{I}_x$ is not right ideal of S, then there exists $s \in S$ and $r \in \mathfrak{I}_x$ such that $rs \notin \mathfrak{I}_x$, that is, $x \in \Delta(rs)$, this implies for any $y \in X$ we have $rs(y) = x$ then $r(z) = x$ where $z = s(y) \in X$. Which gives $x \in \Delta(r)$, that is, $r \notin \mathfrak{I}_x$ a contradiction. Hence $\mathfrak{I}_x$ is right ideal of S. □

Definition 3.2. For any $x \in X \setminus Y$, the ideal $\mathfrak{I}_x = \{r \in S : x \in X \setminus \Delta(r)\}$ is called a point right ideal of S.

Lemma 3.3. *Given $x, y \in X \setminus Y$, then $\mathfrak{I}_x \subseteq \mathfrak{I}_y$ if and only if $x = y$.*

Proof. Suppose $x \neq y$ by Lemma 2.7 we can choose an $A \in \Delta(S)$ with $y \in A$, $x \in X \setminus A$. If $\alpha \in S$ with $\Delta(\alpha) = A$, then $x \in X \setminus \Delta(\alpha)$, that is, $\alpha \in \mathfrak{I}_x$. Since $y \in \Delta(\alpha)$, implies $\alpha \notin \mathfrak{I}_y$. That is, $\alpha \in \mathfrak{I}_x \setminus \mathfrak{I}_y$, and so $\mathfrak{I}_x \nsubseteq \mathfrak{I}_y$, which is a contradiction. Hence $x = y$. The converse is obvious. □

Consider a set $\mathcal{A} = \{\mathfrak{I}_x : x \in X \setminus Y\} \cup Y$. Now define a map $\theta : X \to \mathcal{A}$ defined by

$$\theta(x) = \begin{cases} \mathfrak{I}_x, \text{ if } x \in X \setminus Y, \\ g(x), \text{ where } g \in \mathcal{S}(Y). \end{cases}$$

Lemma 3.4. *θ defined above is a bijection of X onto $\mathcal{A}$.*

Proof. θ is clearly onto and Lemma 3.3 ensures that θ is one-one. □

Lemma 3.5. *For any $\alpha_1, \alpha_2 \in S$ if $\alpha_1 \neq \alpha_2$ then the set*

$$\mathfrak{I}_{\alpha_1,\alpha_2}=\{r \in S : \alpha_1 r = \alpha_2 r\},$$

is a right ideal of S.

Proof. For every $s \in S$ and $r \in \mathfrak{I}_{\alpha_1,\alpha_2}$, we have $\alpha_1 r = \alpha_2 r$, this implies $\alpha_1(rs) = \alpha_2(rs)$, that is, $rs \in \mathfrak{I}_{\alpha_1,\alpha_2}$. Hence $\mathfrak{I}_{\alpha_1,\alpha_2}$ is a right ideal of S. □

Definition 3.6. Given distinct $\alpha_1, \alpha_2 \in S$, the ideal

$$\mathfrak{I}_{\alpha_1,\alpha_2}=\{r \in S : \alpha_1 r = \alpha_2 r\},$$

is called a function right ideal of S.

Lemma 3.7. *Given α_1 and α_2 in S, we have*

$$r \in \mathfrak{I}_{\alpha_1,\alpha_2} \Leftrightarrow \Delta(r) \subseteq \{x \in X : \alpha_1(x) = \alpha_2(x)\}.$$

Proof. Let $r \in \mathfrak{I}_{\alpha_1,\alpha_2}$, then we have $\alpha_1 r = \alpha_2 r$. Let $y \in \Delta(r)$, then there exists some $z \in X$ such that $r(z) = y$. Since $\alpha_1 r = \alpha_2 r$, so we have $\alpha_1 r(z) = \alpha_2 r(z)$, that is, $\alpha_1(y) = \alpha_2(y)$, implies that, $y \in \{x \in X : \alpha_1(x) = \alpha_2(x)\}$. Thus we have $\Delta(r) \subseteq \{x \in X : \alpha_1(x) = \alpha_2(x)\}$. Conversely, suppose $\Delta(r) \subseteq \{x \in X : \alpha_1(x) = \alpha_2(x)\}$, this implies, for any $y \in \Delta(r)$, we have $\alpha_1(y) = \alpha_2(y)$, that is, $\alpha_1 r = \alpha_2 r$, implies, $r \in \mathfrak{I}_{\alpha_1,\alpha_2}$. □

The following notation applies to an arbitrary $\mathcal{S}(X, Y)$-normal semigroup S.
Let α_1, α_2 be distinct transformations in S then

$$\mathfrak{D}_{\alpha_1,\alpha_2} = \{x \in X : \alpha_1(x) \neq \alpha_2(x)\}.$$

We now establish the relationships between point right ideals and function right ideals of an $\mathcal{S}(X, Y)$-normal subsemigroup S of $\mathcal{P}_{\mathcal{S}}(X, Y)$, where S is not contained in $\mathcal{O}(X)$.

Proposition 3.8. *Let $\alpha_1, \alpha_2 \in S$ with $\mathfrak{I}_{\alpha_1,\alpha_2} \neq \emptyset$. Then*

$$\mathfrak{I}_{\alpha_1,\alpha_2} = \bigcap_{x \in \mathfrak{D}_{\alpha_1,\alpha_2}} \mathfrak{I}_x.$$

Proof. Let $r \in \mathfrak{I}_{\alpha_1,\alpha_2}$, that is $\alpha_1 r = \alpha_2 r$. If $x \in \mathfrak{D}_{\alpha_1,\alpha_2}$, then $\alpha_1(x) \neq \alpha_2(x)$, therefore $x \in X \setminus \Delta(r)$, that is, $r \in \mathfrak{I}_x$. Since this is true for each $x \in \mathfrak{D}_{\alpha_1,\alpha_2}$, so we have

$$r \in \bigcap_{x \in \mathfrak{D}_{\alpha_1,\alpha_2}} \mathfrak{I}_x,$$

or

$$\mathfrak{I}_{\alpha_1,\alpha_2} \subseteq \bigcap_{x \in \mathfrak{D}_{\alpha_1,\alpha_2}} \mathfrak{I}_x.$$

Conversely, if

$$r \in \bigcap_{x \in \mathfrak{D}_{\alpha_1,\alpha_2}} \mathfrak{I}_x,$$

then for each y in $\Delta(r)$ we have $y \in X \setminus \mathfrak{D}_{\alpha_1,\alpha_2}$ therefore $\alpha_1(y) = \alpha_2(y)$ and hence $\alpha_1 r = \alpha_2 r$, that is, $r \in \mathfrak{I}_{\alpha_1,\alpha_2}$ and so

$$\bigcap_{x \in \mathfrak{D}_{\alpha_1,\alpha_2}} \mathfrak{I}_x \subseteq \mathfrak{I}_{\alpha_1,\alpha_2}.$$

This proves the desired equality. □

We now show that there always exist distinct elements $\alpha_1, \alpha_2 \in S$ such that $\mathfrak{I}_{\alpha_1,\alpha_2}$ is non-empty. However, in some cases, $\mathfrak{I}_{\alpha_1,\alpha_2}$ may still be empty. For instance, if α_1 and α_2 are chosen such that they are never equal, then $\mathfrak{I}_{\alpha_1,\alpha_2} = \emptyset$.

Proposition 3.9. *Given $x \in X \setminus Y$ there exists $\alpha_1, \alpha_2 \in S$ such that $\mathfrak{I}_x = \mathfrak{I}_{\alpha_1,\alpha_2}$.*

Proof. Let $x \neq w \in X$ then as S contains non-constant maps there exists $\alpha \in S$ such that $\alpha(x) = y \neq \alpha(w)$, suppose $z \notin \Delta(\alpha)$ and $z \neq x$. Take $\beta = (x\ z)\ \alpha\ (x\ z)$ in S then

$$\begin{aligned}\beta(z) &= (x\ z)\ \alpha(x\ z)(z)\\ &= (x\ z)\ \alpha((x\ z)(z))\\ &= (x\ z)\ \alpha(x)\\ &= y.\end{aligned}$$

Hence, we get $\beta(z) = y$ and $z \notin \Delta(\beta)$, for otherwise $z \in \Delta(\alpha)$ a contradiction. We let

$$\lambda = (y\ z),\ \alpha_1 = \beta\alpha \text{ and } \alpha_2 = \lambda\beta\lambda^{-1}\alpha.$$

Then for each $u \neq x$ in X we have,

$$\begin{aligned}\alpha_1(u) &= \beta\alpha(u)\\ &= \beta\lambda^{-1}\alpha(u) \text{ (as } u \neq x \text{ and } z \notin \Delta(\alpha))\\ &= \lambda\beta\lambda^{-1}\alpha(u)\\ &= \alpha_2(u).\end{aligned}$$

Hence $\alpha_1(u) = \alpha_2(u)$. While

$$\alpha_2(x) = \lambda\beta\lambda^{-1}\alpha(x) = \lambda\beta\lambda^{-1}(y) = \lambda\beta(z) = \lambda(y) = z \neq \beta(y) = \beta\alpha(x) = \alpha_1(x).$$

Hence $\alpha_1(x) \neq \alpha_2(x)$ and so $\mathfrak{D}_{\alpha_1,\alpha_2}= \{x\}$, exactly as required. □

Proposition 3.10. *Given f_1 and f_2 in S, $\mathfrak{I}_{\alpha_1,\alpha_2}$ is maximal function right ideal if and only if $|\mathfrak{D}_{\alpha_1,\alpha_2}|$=1.*

Proof. Suppose $\mathfrak{I}_{\alpha_1,\alpha_2}$ is a maximal function right ideal, while $x, y \in \mathfrak{D}_{\alpha_1,\alpha_2}$, $x \neq y$. Then

$$\begin{aligned}\mathfrak{I}_{\alpha_1,\alpha_2} &= \bigcap_{x\in\mathfrak{D}_{\alpha_1,\alpha_2}} \mathfrak{I}_x \text{ (by Proposition 3.8)}\\ &\subseteq \mathfrak{I}_x \cap \mathfrak{I}_y\\ &\subsetneqq \mathfrak{I}_x \text{ (by Lemma 3.3).}\end{aligned}$$

It follows from Proposition 3.9 that there exists β_1 and β_2 with $\mathfrak{I}_{\beta_1,\beta_2} = \mathfrak{I}_x$, and so $\mathfrak{I}_{\alpha_1,\alpha_2} \subsetneqq \mathfrak{I}_x = \mathfrak{I}_{\beta_1,\beta_2}$, a contradiction to the maximality of $\mathfrak{I}_{\alpha_1,\alpha_2}$. Hence $|\mathfrak{D}_{\alpha_1,\alpha_2}|$=1.
Conversely, suppose $\mathfrak{D}_{\alpha_1,\alpha_2}$=$\{x\}$, for some $x \in X$, while there exists β_1, β_2 in S such that

$$\mathfrak{I}_{\beta_1,\beta_2} \supseteq \mathfrak{I}_{\alpha_1,\alpha_2}.$$

Since

$$\mathfrak{I}_{\beta_1,\beta_2} = \bigcap_{y\in\mathfrak{D}_{\alpha_1,\alpha_2}} \mathfrak{I}_y \text{ (by Proposition 3.8).}$$

We have

$$\bigcap_{y\in\mathfrak{D}_{\beta_1,\beta_2}} \mathfrak{I}_y = \mathfrak{I}_{\beta_1,\beta_2} \supseteq \mathfrak{I}_{\alpha_1,\alpha_2} = \mathfrak{I}_x,\ \text{(by Proposition 3.8)}$$

and so Lemma 3.3 ensures $\mathfrak{D}_{\alpha_1,\alpha_2}$=$\{x\}$, that is,

$$\mathfrak{I}_{\beta_1,\beta_2} = \mathfrak{I}_x = \mathfrak{I}_{\alpha_1,\alpha_2}.$$

□

Next we have the corollary which follows from Proposition 3.8 and 3.9.

Corollary 3.11. *Given α_1 and α_2 in S, $\mathfrak{I}_{\alpha_1,\alpha_2}$ is maximal function right ideal if and only if $\mathfrak{I}_{\alpha_1,\alpha_2} = \mathfrak{I}_x$, some $x \in X \setminus Y$.*

4. Inner Automorphisms of a Class of Normal Semigroups

In this section we first show that each automorphism ϕ of S permutes point right ideals.

Proposition 4.1. *Given $x \in X \setminus Y$ and $\phi \in Aut(S)$, $\phi(\mathfrak{I}_x) = \mathfrak{I}_u$ for some $u \in X \setminus Y$.*

Proof. Choose α_1 and α_2 in S such that $\mathfrak{I}_{\alpha_1,\alpha_2} = \mathfrak{I}_x$ (Proposition 3.9), then

$$\begin{aligned}
\phi(\mathfrak{I}_x) &= \phi(\mathfrak{I}_{\alpha_1,\alpha_2}) \\
&= \phi(\{r : \alpha_1 r = \alpha_2 r\}) \\
&= \{\phi(r) : \phi(\alpha_1 r) = \phi(\alpha_2 r)\} \\
&= \{\phi(r) : \phi(\alpha_1)\phi(r) = \phi(\alpha_2)\phi(r)\} \\
&= \{r' : \phi(\alpha_1) r' = \phi(\alpha_2) r'\}. \\
&= \mathfrak{I}_{\phi(\alpha_1),\phi(\alpha_2)}.
\end{aligned}$$

Now Corollary 3.11 ensures $\mathfrak{I}_{\alpha_1,\alpha_2}$ is a maximal function right ideal, hence $\mathfrak{I}_{\phi(\alpha_1),\phi(\alpha_2)}(= \phi(\mathfrak{I}_{\alpha_1,\alpha_2}))$ is a maximal function right ideal, so there exists $u \in X$ such that $\mathfrak{I}_{\phi(\alpha_1),\phi(\alpha_2)} = \mathfrak{I}_u$ (by Corollary 3.11) and thus $\phi(\mathfrak{I}_x) = \mathfrak{I}_{\phi(\alpha_1),\phi(\alpha_2)} = \mathfrak{I}_u$. □

Lemma 4.2. *The mapping $\eta : \mathcal{A} \to \mathcal{A}$ defined by*

$$\eta(s) = \begin{cases} \phi(s), \ \textit{if } s \in \{\mathfrak{I}_x;\ x \in X \setminus Y\}, \\ g(s), \ \textit{where } g \in \mathcal{S}(Y). \end{cases}$$

is a bijection.

Proof. That η is a mapping in the content of Proposition 4.1. Similarly by considering the automorphism ϕ^{-1} we define a map $\zeta : \mathcal{A} \to \mathcal{A}$ via

$$\zeta(s) = \begin{cases} \phi^{-1}(s), \ \text{if } s \in \{\mathfrak{I}_x;\ x \in X \setminus Y\}, \\ g^{-1}(s), \ \text{where } g^{-1} \in \mathcal{S}(Y). \end{cases}$$

Certainly, ζ is the inverse of η and so η is a bijection.□

Definition 4.3. Define a map $h : X \to X$ defined by

$$h(x) = \begin{cases} u, \text{ where } \eta(\mathfrak{I}_x) = \mathfrak{I}_u, \text{ for each } x \in X \setminus Y, \\ g(x), \text{ where } g \in \mathcal{S}(Y). \end{cases}$$

Remark 4.4. *By Lemma 3.4 and Lemma 4.2, it is clear that $h = \theta^{-1}\eta\theta$, and so Lemma 3.4 ensures h is a bijection of X and since each of θ and η is a bijection on Y, it follows that $h|_Y \in \mathcal{S}(Y)$, that is, $h \in \mathcal{S}(X, Y)$. We call h the bijection associated with ϕ.*

Lemma 4.5. *Given $\alpha \in S$ and $\phi \in Aut(S)$, $\Delta(\phi(\alpha)) = h(\Delta(\alpha))$.*

Proof. Let $h \in \mathcal{S}(X, Y)$, then $h(X \setminus \Delta(\alpha)) = X \setminus h(\Delta(\alpha))$. Now we have the following cases:
Case (i). If $x \in X \setminus \Delta(\alpha)$, that is $\alpha \in \mathfrak{I}_x$, then

$$\phi(\alpha) \in \phi(\mathfrak{I}_x) = \eta(\mathfrak{I}_x) = \mathfrak{I}_u = \mathfrak{I}_{h(x)},$$

so

$$h(x) \in X \setminus \Delta(\phi(\alpha)),$$

or

$$h(X \setminus \Delta(\alpha)) \subseteq X \setminus \Delta(\phi(\alpha)).$$

Thus

$$h(\Delta(\alpha)) \subseteq \Delta(\phi(\alpha)).$$

For the reverse inclusion, observe that $h^{-1} = \theta^{-1}\eta^{-1}\theta$ is a bijection associated with ϕ^{-1} and by above inclusion we have for given $\beta \in S$,

$$h^{-1}(\Delta(\beta)) \subseteq \Delta(\phi^{-1}(\beta)).$$

Specifically, by setting $\beta = \phi(\alpha)$, we obtain

$$h^{-1}(\Delta(\phi(\alpha))) \subseteq \Delta(\phi^{-1}(\phi(\alpha))),$$

or

$$\Delta(\phi(\alpha)) \subseteq h(\Delta(\alpha)),$$

and the equality follows.
Case (ii). Let $\Delta(\alpha) = X$ and $\phi(\alpha) = \beta$, then we have $h(\Delta(\alpha)) = X = \Delta(\alpha)$. If $\Delta(\beta) \subsetneq X$,

then by Case (i), it follows that

$$\begin{aligned} h(\Delta(\beta)) &= \Delta(\phi^{-1}(\beta)) \text{ (where } \phi^{-1} \in \text{ Aut}(S)) \\ &= \Delta(\alpha) \\ &= X, \end{aligned}$$

this implies $\Delta(\beta) = X$, thus $\Delta(\phi(\alpha)) = \Delta(\beta) = X = h(\Delta(\alpha))$ and the equality follows in this case. □

Now we have the main theorem of this section in which we show that every subsemigroup of $\mathcal{P}_{\mathcal{S}}(X, Y)$ containing $\mathcal{S}(X, Y)$ is nearly complete.

Theorem 4.6. *Let S be a subsemigroup of $\mathcal{P}_{\mathcal{S}}(X, Y)$ containing $\mathcal{S}(X, Y)$ such that $S \nsubseteq \mathcal{O}(X)$, then S is nearly complete, that is, for some $h \in \mathcal{S}(X, Y)$*

$$\phi(\alpha) = h\alpha h^{-1}, \textit{ for each } \alpha \in S.$$

Proof. Consider the bijection h associated with ϕ as defined to Lemma 4.5. Take an arbitrary $\alpha \in S$, $x \in X$ and let $\alpha(x) = u$ for some $u \in X$. Choose A in $\Delta(S)$ with $A \neq X$ and $x \in A$. Let $z \notin A$ and $B = (A \setminus \{x\}) \cup \{z\} \in \Delta(S)$ (Lemma 2.7). Choose γ and δ in S such that $\Delta(\gamma) = A$ and $\Delta(\delta) = B$.
Now

$$\Delta(\gamma) \setminus \Delta(\delta) = A \setminus B = \{x\},$$

thus

$$\Delta(\alpha\gamma) \setminus \Delta(\alpha\delta) = \{\alpha(A \setminus B)\} = \{\alpha(x)\} = \{u\}.$$

By using Lemma 4.5 we have,

$$\begin{aligned} \Delta(\phi(\gamma)) \setminus \Delta(\phi(\delta)) &= h(\Delta(\gamma) \setminus \Delta(\delta)) \\ &= \{h(A \setminus B)\} \\ &= \{h(x)\}, \end{aligned}$$

and

$$\Delta(\phi(\alpha\gamma)) \setminus \Delta(\phi(\alpha\delta)) = \{h(u)\}.$$

However

$$\Delta(\phi(\alpha\gamma)) \setminus \Delta(\phi(\alpha\delta)) = \Delta(\phi(\alpha)\phi(\gamma)) \setminus \Delta(\phi(\alpha)\phi(\delta)) = \{\phi(\alpha)h(x)\}.$$

So

$$\phi(\alpha)h(x) = h(u) = h\alpha(x),$$

that is,

$$\phi(\alpha) = h\alpha h^{-1}.$$

Hence the Theorem.$\square$

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