Algebraic Structures and Their Applications Vol. 12 No. 2 (2025) pp 89-99.

Research Paper

SYMBOLIC REES ALGEBRA AND REDUCTION OF MODULES

PRITI SINGH*, SHARVAN KUMAR, AJIM UDDIN ANSARI AND SHIV DATT KUMAR

ABSTRACT. In this paper, we study the symbolic Rees algebras of modules by using the theory of reductions of modules. We extend several results of the symbolic Rees algebra of ideals to the symbolic Rees algebra of modules and prove the necessary condition for the symbolic Rees algebras of modules to be Noetherian. Several examples of the symbolic Rees algebra are provided.

1. INTRODUCTION

Algebraic geometry relates algebraic objects (e.g. rings, ideals, modules) to geometric objects (e.g. curves, surfaces, spaces). A good example of this is the Rees algebra of an ideal. The Rees algebra of an ideal I in a ring R is an algebraic object that is defined as the graded ring $R(I) = \oplus_{n=0}^{\infty} I^n$. Rees algebras were first used for studying valuations of ideals, they are perhaps best known for being coordinate rings of blow ups.

DOI: 10.22034/as.2024.21148.1700

MSC(2010): Primary: 13A30, 13C05, 13E05.

Keywords: Fiber cone, Rees algebra, Symbolic Rees algebra.

Received: 22 January 2024, Accepted: 25 September 2024.

*Corresponding author

When we are working with geometric objects such as curves and surfaces, one of the most desirable classes of objects is those that are smooth i.e., without singularities. A singularity of a curve can be thought of as a point, where the curve has more than one direction. It is desirable to have methods of resolving the singularities while preserving other properties of the curve. One of the methods to resolve singularities is the blow up of a space (scheme) in a closed subspace (subscheme). A blow up of a curve produces a new curve that is birational to the original, where singularities might have vanished. The blow up is computed by taking the projective spectrum of the Rees algebra of the corresponding ideal [5, 8, 9].

The symbolic Rees algebra is a central object in commutative algebra and algebraic geometry for their tight connection to the primary decomposition of ideals and the order of vanishing of polynomials. In general, this algebra is not finitely generated, even if the ring is Noetherian. It was first introduced by Rees [8] in order to answer Zariski′s conjecture to give a counter example by constructing a non-Noetherian symbolic Rees algebra. The necessary and sufficient condition for the symbolic Rees algebras of a prime ideal, $R_s(p)$ to be a Noetherian ring in three dimensional regular local ring with $ht(p) = 2$ is established by Huneke [6]. The finite generation problem on the R-algebra $R_s(\mathfrak{p})$ was observed by R. C. Cowsik [2] showing that if p is set theoretic complete intersection in R with $\dim A/p = 1$, then $R_s(\mathfrak{p})$ is a finitely generated R-algebra. Therefore it is important to know when the symbolic Rees algebra is Noetherian. Cowsik asked whether $R_s(p)$ is a Noetherian ring for every prime ideal in a regular local ring. In general, the answer is not affirmative. In fact, Roberts [9] showed that the answer is negative by constructing the examples of prime ideals in polynomial and power series rings whose symbolic Rees algebras are not Noetherian.

The properties of the symbolic powers of an ideal is the reduction of an ideal, a concept developed by Northcott and Rees [7]. An ideal J contained in I is called a reduction of I if $I^{r+1} = JI^r$ for some $r \geq 0$. An ideal $J \subset I$ is a minimal reduction if it is not contained in any reduction of I. Goto et al. [3] showed suitable symbolic powers of an ideal and using its minimal reduction, we get some information about the Noetherian property of $R_s(I)$. Several mathematicians [3, 4, 5, 12] carried out a study on non-Noetherian and Noetherian symbolic Rees algebras of ideals.

The main purpose of this paper is to construct non-trivial new examples of symbolic Rees algebra of modules. We have extended several results of the symbolic Rees algebra of ideals to the symbolic Rees algebra of modules and obtained the necessary conditions for the symbolic Rees algebras of modules to be Noetherian. Other results come from the fact that the symbolic Rees algebra of a module is a special case of the symbolic multi Rees ring, where the module is a direct sum of ideals.

2. Rees algebras of modules

Definition 2.1. Let R be a Noetherian ring and $Q = S^{-1}R$ be the total ring of quotient, where $S = R \setminus Z(R)$ and $Z(R) = \{x \in R \mid x\ y = 0 \text{ for some } 0 \neq\ y \in R\}$. Suppose E is a finitely generated R-module with rank $e > 0$, (which means $E \otimes_R Q \simeq Q^e$). Suppose $E \subset G \simeq R^e$. Then the tensor algebra $T(E) = \oplus_{n=0}^{\infty} T^n(E) = R \oplus E \oplus E \otimes E \oplus \cdots$ and $C(E) :=$ two sided ideal of $T(E)$ generated by $\{x \otimes y - y \otimes x : x, y \in E\}$ and $Sym(E) = \dfrac{T(E)}{C(E)}$ is the symmetric algebra of E. Let $\phi : E \longrightarrow G$ be homomorphisms, where G runs over all free modules, $Sym(\phi) : Sym(E) \longrightarrow Sym(G)$ and $L_\phi = Ker(Sym(\phi))$. Then the Rees algebra $R(E)$ of E is $Sym(E)$ modulo its R-torsion submodule i.e.,

$$R(E) = \frac{Sym(E)}{ker(Sym(\phi))} = \frac{Sym(E)}{\cap_\phi L_\phi},$$

where intersection is taken over all homomorphisms $\phi : E \longrightarrow G$ and G runs over all free modules. We denote the n^{th} component of the Rees algebra of E by E^n. Therefore, the Rees algebra $R(E) = \oplus_{n \geq 0} E^n$ is a standard graded algebra over R with $E^0 = R$ and $E^1 = E$. Note that if E has rank, then $Ker(Sym(\phi)) = Tor_R(Sym(E))$.

Definition 2.2. The fiber cone of a module over a Noetherian local ring (R, m) is defined as $F(E) = R/m \otimes_R R(E) = \bigoplus_{n \geq 0} \dfrac{E^n}{mE^n}$. The analytic spread of E, denoted as $l(E)$ is the Krull dimension of $F(E)$.

Definition 2.3. Let $U \subseteq E$ be a submodule of E. Then the module U is said to be a reduction of E if $E^{n+1} = U.E^n$ for some $n \geq 0$.

The least integer n for which $E^{n+1} = U.E^n$ is called the reduction number of E. A submodule U of E is said to be a minimal reduction of E if it is minimal with respect to among all the reductions of E.

Definition 2.4. Let $E \subset G \simeq R^e$, $e > 0$ be an R-submodule of G with rank e but not free. Then E is said to be an ideal module if the double dual E^{**} is a free R-module, where $E^* := Hom_R(E, R)$.

An ideal module E is said to be equimultiple module if $l(E) = ht(\mathbb{F}_e(E)) + e - 1$, where $\mathbb{F}_e(E)$ is the e-th Fitting invariant of E.

Example 2.5. Let $R = k[[x, y, z]]$ be a ring over a field k and $E = (zx, zy)$ be an R-module. Note that E is not a free R module. Since $E \simeq I = (x, y)$ (as R-modules), $E^{**} \simeq I^{**}$. So that E^{**} is a free R- module, for I is an ideal of grade 2. Thus E is an ideal module.

3. Interpretation of symbolic Rees algebra of prime ideal

If K is an algebraically closed field, $S = K[x_1, \ldots, x_n]$ the polynomial ring in n variables, $P \subset S$ a prime ideal, $X \subset K^n$ the irreducible algebraic set corresponding to P, and $m_a =$

$(x_1 - a_1, \ldots, x - a_n)$ the maximal ideal corresponding to the point $a = (a_1, a_2, \ldots, a_n) \in X$, then $P^{(n)} = \bigcap_{a\in X} m_a^n = \{f \in S \mid f$ vanishes to order $\le n$ at every point of $X\}$ = Taylor series of f around a begins with term of order $\ge n$.

Definition 3.1. The symbolic Rees algebra of I is the graded ring

$$R_s(I) = \bigoplus_{n\ge 0} I^{(n)} t^n \subseteq R[t],$$

where $I^{(n)} = \{r \in R \mid rs \in I^n$ for some $s \in R - \cup P$ where$P \in Min(\frac{R}{I})I\} = \bigcap_{p\in Min(\frac{R}{I})}(I^n)_p \cap R$ and t is an indeterminate.

We generalize this construction to a finitely generated R-module E having rank. The symbolic Rees algebras of a module E

$$R_s(E) = \bigoplus_{n\ge 0} E^{(n)} \subseteq R[t_1, \ldots, t_e],$$

where $E^{(n)} = \bigcap_{p\in Min(\frac{G}{E})}(E^n)_p \cap G^n$, and $Min(G/E)$ denotes the set of minimal elements of $Ass(G/E)$.

A special case is the Rees algebra of a module $E = I_1 \oplus \cdots \oplus I_e$, where $I_1, \ldots, I_e$ are R-ideals. Then the symbolic Rees algebra is the multi-symbolic Rees algebra

$$R_s(I_1, \ldots, I_e) = \bigoplus_{n_1,\ldots,n_e\in\mathbb{N}} (I_1^{n_1} \ldots I_e^{n_e})^{**} t_1^{n_1} t_2^{n_2} \ldots t_e^{n_e},$$

where $(I_1^{n_1} \ldots I_e^{n_e})^{**}$ is the reflexive hull of $I_1^{n_1} \ldots I_e^{n_e}$ (see [10]) and

$$E^{(n)} \simeq \bigoplus_{n_1+\cdots+n_e=n} (I_1^{n_1} \ldots I_e^{n_e})^{**}.$$

Remark 3.2. If E is a free R-module of rank e, then the symbolic Rees algebra $R_s(E) \simeq R[t_1, \ldots, t_e]$, for $R(E) \simeq R[t_1, \ldots, t_e]$ and $R(E) \subset R_s(E) \subset R[t_1, \ldots, t_e]$. In general, if E is an R-module of rank e and G is a free R-module of rank r, then

$$R_s(E \oplus G) \simeq R_s(E)[t_1, \ldots, t_r].$$

Example 3.3. Consider $G = F^e$ as F-module, where F is a field and $E = \{(x_1, x_2, \ldots, x_e) | x_1 + x_2 + \cdots + x_e = 0\}$ is a submodule of G of dimension $e - 1$. Let $\phi : E \longrightarrow G$ be an embedding. Symmetric algebra of E and G are as follows: $Sym(E) = \oplus_{k\ge 0} S^k(E) = \oplus_{k\ge 0} E^k$ and note that $dim S^k(E) = {}^{k+e-2}C_{e-2} = d_k$. In fact $E^k = S^k(E) = F^{d_k}$, $\forall\ k \ge 2$. This implies $Sym(E) = F \oplus F^{e-1} \oplus F^{d_2} \oplus F^{d_3} \oplus \cdots$. Further $Sym(G) = \oplus_{k\ge 0} S^k(G) = \oplus_{k\ge 0} G^k$, $dim\ S^k(G) = {}^{k+e-1}C_{e-1} = l_k$ and so $Sym(G) = F \oplus F^e \oplus F^{l_2} \oplus F^{l_3} \oplus \cdots$.

Also $Sym(\phi) : Sym(E) \longrightarrow Sym(G)$ given by $Sym(\phi)|_{S^n(E)}(x_1 \otimes x_2 \otimes \ldots \otimes x_n + C^n(E)) = \phi(x_1) \otimes \phi(x_2) \otimes \ldots \otimes \phi(x_n) + C^n(G) = x_1 \otimes x_2 \otimes \cdots \otimes x_n + C^n(G)$ and $Sym(\phi)|_F = I_F$ = identity map on F. Therefore $Ker(Sym(\phi)) = \{0\}$. Thus Rees algebra of $E= R(E) = \dfrac{Sym(E)}{Ker(\phi)} = \oplus_{n\geq 0}E^n$, where $E^n \cong F^{{}^{n+e-2}C_{e-2}}$. Here E^n and G^n are called n-th symmetric powers of E and G respectively. Note that minimal elements of $Ass(G/E)$ and $Supp(G/E)$ coincide. In fact, $Ass(G/E) = Supp(G/E) = \{0\}$ and $\{0\} = Ann((1,1,\ldots,1)+E)$ and localization $(G/E)_{\{0\}} \neq 0$. Thus $(E^n)_{\{0\}} = \left\{\dfrac{x}{t}|x \in E^n, t \in F-\{0\}\right\}$, e.g., if V is a vector space, then $V_{\{0\}} = V$. So $(E^n)_{\{0\}} \cong E^n$ and $E^{(n)} = (E^n)_{\{0\}} \cap G^n = E^n$, for $E^n \subset G^n$. Thus symbolic Rees algebra of E is $R_s(E) = \oplus_{n\geq 0}E^{(n)} = \oplus_{n\geq 0}E^n = \oplus_{n\geq 0}F^{{}^{n+e-2}C_{e-2}}$.

Example 3.4. Let $E = \mathbb{Z}_2$, $R = \mathbb{Z}_4$. Then E is a finitely generated R-module, which is not free. Any $\mathbb{Z}_4$ module is of the form $(\oplus^k \mathbb{Z}_2) \oplus (\oplus^m \mathbb{Z}_4)$, for $k \neq m$. $Sym(E) = \mathbb{Z}_4 \oplus \mathbb{Z}_2^{\otimes 2} \oplus \mathbb{Z}_2^{\otimes 3} \oplus \cdots = \mathbb{Z}_4 \oplus \mathbb{Z}_2^{\otimes \mathbb{N}}$. Since $C(E) = 0$ and $\mathbb{Z}_2 \otimes_{\mathbb{Z}_4} \mathbb{Z}_2 = \mathbb{Z}_2$. This implies $\mathbb{Z}_2^{\otimes n} = \mathbb{Z}_2$. In fact $\mathbb{Z}_2^{\otimes n} = \{\bar{0}\otimes\bar{0}\otimes\cdots\otimes\bar{0}, \bar{1}\otimes\bar{1}\otimes\cdots\otimes\bar{1}\}$ for all $n \geq 1$. Let F be a free $\mathbb{Z}_4$-module. Then $F = \oplus_{i\in J}\mathbb{Z}_4$ for some $J \subset \mathbb{N}$. Any homomorphism $\phi : \mathbb{Z}_2 \longrightarrow \oplus_{i\in J}\mathbb{Z}_4$ sends $\bar{1} \to (a_i)_{i\in J}$, where a_i's are $\bar{2}$ finitely many times and rest are $\bar{0}$, for $o(\bar{1}) = 2$ and so $o(a_i)$ should be 2. Now consider $Sym(\phi) : Sym(\mathbb{Z}_2) \longrightarrow Sym(\oplus_{i\in J}\mathbb{Z}_4) \cong \mathbb{Z}_4^{\mathbb{N}}$. Thus $Sym(\phi)_{|_{\mathbb{Z}_4}} = Id_{\mathbb{Z}_4}$ and $Sym(\phi)|\mathbb{Z}_4^{\otimes n} = 0$, for every $n \geq 2$. Thus $L_\phi = Ker(Sym(\phi)) = \oplus_{n\geq 2}{\mathbb{Z}_2}^{\otimes n}$ for all non-zero homomorphism ϕ. The same will be obtained for the case of zero homomorphism also. Hence $\cap_\phi L_\phi = \oplus_{n\geq 2}\mathbb{Z}_2^{\otimes n}$. Consequently, the Rees algebra of E is $\mathbb{Z}_4 \oplus \mathbb{Z}_2$.

Example 3.5. Let $A = R/(XY - ZW)R$ be a ring, where $R = k[[X,Y,Z,W]]$ is a formal power series ring over a field k. Assume that $E = (y,z,w)A$ is an A-module, where $x = X + (XY - ZW)$, $y = Y + (XY - ZW)$, $z = Z + (XY - ZW)$, $w = W + (XY - ZW)$. Then the symbolic Rees algebra is $R_s(E) = A[yt, zt, wt, yt^2]$.

Example 3.6. Let $R = k[a,b,x,y]$ be a ring and $E = (ax, ay) \oplus (ax, by)$ be an R-module. Then the symbolic Rees algebra is isomorphic to the Segre product of two polynomial rings $k[X_1, X_2, X_3] \times k[Y_1, Y_2, Y_3]$.

4. Noetherian symbolic Rees algebra of a module

For proving the Noetherian property of the symbolic Rees algebras, we need the following results to prove Theorem 4.6 and Proposition 4.10.

Proposition 4.1. *Let R be a ring and $E \subset G \simeq R^e$, $e > 0$ be an R-module. Then*

(1) $E^n \subset E^{(n)}$ *for all* $n > 0$.
(2) $E^{(n)m} \subset E^{(nm)}$ *and* $E^{(n)}.E^{(m)} \subset E^{(n+m)}$ *for all* m, $n \geq 1$, *where the product is taken in* $R_s(E)$.
(3) $E^{(mn)} \subset E^{(m)(n)}$ *for all* m, $n \geq 1$.
(4) $E^r \subseteq E^{(m)}$ *if and only if* $m \leq r$.

Proof. (1) Denote $X = Min(G/E)$. Let $x \in E^n$. Then $xR_p \subset E^nR_p$ for each prime ideal $p \in X$. Therefore, $xR_p \subset \bigcap_{p \in X} E^nR_p$. Since $E^n \subset G^n$, $x \in G^n$. So that $x \in \bigcap_{p\in X} E^nR_p \cap G^n = \bigcap_{p\in X}(E^n)_p \cap G^n$ and $x \in E^{(n)}$. Hence $E^n \subset E^{(n)}$ for all $n \geq 1$.

(2) Let $x \in E^{(n)m}$. Then $x \in [\bigcap_{p\in X}(E^n)_p \cap G^n]^m = [\bigcap_{p\in X}(E_p)^n \cap G^n]^m \subset \bigcap_{p\in X}(E_p)^{nm} \cap G^{nm} = \bigcap_{p\in X}(E^{nm})_p \cap G^{nm}$. Therefore, $E^{(n)m} \subset E^{(nm)}$.
Let $x \in E^{(n)}.E^{(m)}$. Then $x = \alpha\beta$, where $\alpha \in E^{(n)}$ and $\beta \in E^{(n)}$. So that, $x \in (\bigcap_{p\in X}(E^n)_p \cap G^n).(\bigcap_{p\in X}(E^m)_p \cap G^m) \subset \bigcap_{p\in X}(E^{n+m})_p \cap G^{n+m}$, where product is taken in $R_s(E)$. Therefore, $x \in E^{(n+m)}$ and $E^{(n)}.E^{(m)} \subset E^{(n+m)}$.

(3) Let $x \in E^{(mn)}$. Then $x \in \bigcap_{p\in X}(E_p)^{mn} \cap G^{mn}$ for all m, $n \geq 1$. By (1), $E^m \subset E^{(m)}$ and $E^{mn} \subset [E^{(m)}]^n \subset E^{(m)(n)}$. This implies that $x \in E^{(m)(n)}$.

(4) Let $m \leq r$. Then $E^r \subseteq E^m \subseteq E^{(m)}$. Conversely, let $r < m$ and $E^r \subseteq E^{(m)}$. Then $E^m \subseteq E^r$ and $E^{(m)} \subseteq E^{(r)}$. Since $E^r \subseteq E^{(m)}$, $E^{(r)} \subseteq E^{(m)}$. Therefore, $E^{(r)} = E^{(m)}$ and then there is a minimal associated prime ideal of E such that $E^rR_p = E^mR_p = E^rR_p \cdot E^sR_p$, where $r + s = m$. By Nakayama lemma, $E^rR_p = 0$. Then $E = 0$ is the contradiction, for E has rank. Thus $m \leq r$.

□

Proposition 4.2. *Let R be a Noetherian ring, E a finitely generated torsion free R-module having rank and A be a flat R-algebra. Then we have a natural isomorphism of a graded A-algebra $R_s(E \otimes_R A) \simeq R_s(E) \otimes_R A$.*

Proof. It is enough to show that $E^{(n)} \otimes_R A \simeq (E \otimes_R A)^{(n)}$ for $n \geq 1$. Suppose $E^n \hookrightarrow E^{(n)}$ for any $n \geq 1$. Since A is a flat R-algebra, $E^n \otimes_R A \hookrightarrow E^{(n)} \otimes_R A$. Consider the following diagram

$$\begin{array}{ccc} E^n \otimes_R A & \xrightarrow{\beta} & E^{(n)} \otimes_R A \\ \downarrow f & & \downarrow g \\ (E \otimes_R A)^n & \xrightarrow{\alpha} & (E \otimes_R A)^{(n)}. \end{array}$$

Note that the diagram commutes. Therefore $E^{(n)} \otimes_R A \simeq (E \otimes_R A)^{(n)}$, for $(E \otimes_R A)^n \simeq E^n \otimes_R A$.

□

Corollary 4.3. *Let $p \in spec(R)$. Then $R_s(E_p) \simeq R_s(E) \otimes_R R_p$.*

Proposition 4.4. *Let R be a Noetherian ring with $dim(R) = d > 0$ and $E \subset G \simeq R^e$, $e > 0$ be an R-module with $dim(G/E) > 0$. Then*

(1) *depth*$\left(\dfrac{G^n}{E^{(n)}}\right) \geq 1$ *for any* $n \geq 1$.

(2) *If* R *is a Cohen-Macaulay, then* $1 \leq depth(E^{(n)}) \leq depth(G^n/E^{(n)}) + 1 \leq d-1$.

Proof. (1) Claim: $Supp\ (G/E) \subseteq Supp\ (G^n/E^n)$. For $n = 1$, it is obvious. Now suppose, for $n > 1$, $p \notin Supp(G^n/E^n)$. Then $\left(\dfrac{G^n}{E^n}\right)_p = 0$. This implies $\dfrac{(G^n)_p}{(E^n)_p} = \dfrac{(G_p)^n}{(E_p)^n} = 0$. Thus $G_p^n = E_p^n$. Note that $E_p^{n-1} \subseteq G_p^{n-1}$ and so $E_pE_p^{n-1} \subseteq E_p \cdot G_p^{n-1}$, i.e. $E_pG_p^{n-1} \subseteq E_p \cdot G_p^{n-1} \subseteq G_p \cdot G_p^{n-1} = G_p^n = E_p^n$, so equality holds throughout. Therefore, E_p is a reduction of G_p. Note that if G_p is a free R_p-module, then G_p has no proper reduction. Thus $E_p = G_p$ and so $p \notin Supp(G/E)$. Hence $Supp\ (G/E) \subseteq Supp\ (G^n/E^n)$. Conversely, suppose $p \notin Supp\ (G/E)$. Then $(G/E)_p = 0$. This implies $\dfrac{G_p}{E_p} = 0$ so $G_p = E_p$. Thus $\dfrac{G_p^n}{E_p^n} = 0$. Hence $p \notin Supp\,(G^n/E^n)$. Now we show that $Min\ (G/E) = Min\ (G^n/E^n)$. Let $p \in Min(G^n/E^n)$. Then by definition of $Min(G^n/E^n)$, $p \in Supp(G^n/E^n) = Ass(G^n/E^n)$ for all $n \geq 1$. Since $Supp(G^n/E^n) = Supp(G/E)$, $p \in Supp(G/E)$. Therefore, $p \in Ass(G/E) = Supp(G/E)$ and $p \in X$. Conversely, let $p \in Min(G/E)$. Then by definition of $Min(G/E)$, $p \in Supp(G/E) = Ass(G/E)$ and $Supp(G^n/E^n) = Supp(G/E)$, $p \in Ass(G^n/E^n) = Supp(G^n/E^n)$. Hence $p \in Min(G^n/E^n)$. Similarly $Ass\ (G/E) = Ass\ (G^n/E^n)$. By definition $E^{(n)} = \cap_{P \in Min(G/E)}(E^n)_P \cap G^n$ and $dim(G/E) > 0$, $depth\left(\dfrac{G^n}{E^{(n)}}\right) \geq 1$ for all $n \geq 1$.

(2) Since $E^{(n)} \subset G^n \simeq R^{\binom{n+e-1}{e-1}}$, $depth(G^n) = depth(R) = dim(R) = d > 0$. Consider a short exact sequence of R-modules

$$0 \to E^{(n)} \to G^n \to G^n/E^{(n)} \to 0.$$

Then applying the depth lemma in this exact sequence, we have

$$depth(E^{(n)}) \geq min\{depth(G^n), depth(G^n/E^{(n)}) + 1\}.$$

By (1) $depth(G^n/E^{(n)}) > 0$, $1 \leq depth(E^{(n)}) \leq depth(G^n/E^{(n)}) + 1 \leq d-1$.

□

Lemma 4.5. *Let* R *be a ring and* $E \subset G \simeq R^e$ *be a submodule of* G. *If the symbolic Rees algebra* $R_s(E)$ *is a Noetherian ring, then there exist* $k > 0$ *such that* $E^{(k)n} = E^{(kn)}$ *for all* $n \geq 1$.

Proof. The inclusion $\subseteq$ is clear (see Proposition 4.1). Conversely, we have to show that there exists $k > 0$ such that $E^{(kn)} \subseteq E^{(k)n}$ for all $n \geq 1$. Let the symbolic Rees algebra $R_s(E)$ be a Noetherian ring. Then $A_+ = \oplus_{n\geq 1} E^{(n)}$ is a finitely generated ideal of $R_s(E)$. Suppose $A_+ = (x_1, \ldots, x_s)$, where $x_i \in E^{(r_i)}$ for $i = 1, \ldots, s$. Suppose $r = lcm(r_1, \ldots, r_s)$ and $k = rs$. Note that $E^{(m)}$ is R-linear combination of monomials of the form $x_1^{u_1}.\ldots.x_s^{u_s}$, where $u_1 r_1 + \cdots + u_s r_s = m$. There are two cases:

Case 1. If $m < k$, then $u_1 r_1 + \ldots u_s r_s < k = rs$. Since $r = lcm(r_1, \ldots, r_s)$, $m < k$ is not possible.

Case 2. If $m \geq k$, then $u_i r_i \geq r$ for some $i = 1, \ldots, s$. Suppose contrary that $u_i r_i < r$ for all $i = 1, \ldots, s$. Then $m = u_1 r_1 + \ldots, u_s r_s < rs = k$, which is a contradiction, for $m \geq k$. Therefore, $u_i r_i \geq r$ for some $i = 1, \ldots, s$. Suppose $r = v r_i$. Then

$$x_1^{u_1}.\ldots.x_s^{u_s} = (x_1^{u_1}.\ldots.x_i^{u_i - v} x_s^{u_s}) x_i^v.$$

Since $x_i \in E^{(r_i)}$ and $x_i^v \in E^{(r_i)v} \subset E^{(r_i v)}$ (see Proposition 4.1), $x_i^v \in E^{(r)}$ and $(x_1^{u_1}.\ldots.x_i^{u_i-v} x_s^{u_s}) \in E^{(m-r)}$. Thus

$$E^{(m)} \subseteq E^{(m-r)}.E^{(r)} \text{ for } m \geq k. \tag{1}$$

By equation (1), for any positive integer l,

$$\begin{aligned} E^{(k+rl)} &\subseteq E^{(k+rl-r)}.E^{(r)} \text{ for } k + rl > k \\ &= E^{(k+r(l-1))}.E^{(r)} \\ &\subseteq E^{(k+r(l-1)-r)}.E^{(r)}E^{(r)} \text{by equation (1)} \\ &= E^{(k+r(l-2))}.E^{(r)2} \\ &\subseteq E^{(k+r(l-l))}.E^{(r)l}. \end{aligned}$$

Therefore, $E^{(k+rl)} \subseteq E^{(k)}.E^{(r)l} \subseteq E^{(k)}.E^{(rl)}$ (see Proposition 4.1). Now we show that $E^{(nk)} \subseteq E^{(k)n}$ for all $n \geq 1$ by induction on n. If $n = 1$, then the result holds trivially. Suppose the result is true for $n - 1$. Then $E^{((n-1)k)} \subseteq E^{(k)n-1}$. Now,

$$\begin{aligned} E^{(nk)} &= E^{(k+(n-1)k))} \\ &\subseteq E^{(k)}.E^{((n-1)k)}, \text{ for } E^{(k+rl)} \subseteq E^{(k)}.E^{(rl)} \\ &\subseteq E^{(k)}.E^{(k)n-1} \text{ by assumption} \\ &= E^{(k)n}. \end{aligned}$$

Therefore, $E^{(nk)} \subseteq E^{(k)n}$ for all $n \geq 1$. □

Theorem 4.6. *Let (R, m) be a Noetherian local ring with $dim(R) = d > 0$ and $G \simeq R^e$ be a free R-module with rank $e > 0$. Assume that $E \subset G$ is an R-submodule of G and the symbolic Rees algebra $R_s(E)$ is a Noetherian ring. Then there exists $k > 0$ such that $l(E^{(k)}) \leq d+e-2$.*

Proof. Since the symbolic Rees algebra $R_s(E)$ is a Noetherian ring and Lemma 4.5 there exists $k > 0$ such that $E^{(k)n} = E^{(kn)}$ for any $n > 0$. So that there exists $k > 0$ such that $l(E^{(k)}) \leq d + e - 1 - inf_{n\geq 1} depth\left(\frac{G^{nk}}{E^{(k)n}}\right)$ (Theorem 1.1, [1]). By Proposition 4.4, $depth\left(\frac{G^{nk}}{E^{(k)n}}\right) \geq 1$. Therefore, $l(E^{(k)}) \leq d + e - 2$. □

Following example shows that $l(E^{(k)}) \leq d+e-2$ for some $k > 0$ but we do not always have a reduction generated by $d+e-2$ elements, even when $R_s(E)$ is a Noetherian ring and R/m is an infinite residue field.

Example 4.7. Consider the formal power series ring $R = K[[x, y, z]]$ over an infinite field K. Let p be the prime ideal defining the space monomial curve: $x = t^3,\ y = t^4,\ z = t^5$. Then $R_s(p) = R(p)$ is a Noetherian ring, $l(p) = 3$ and $l(p^{(k)}) \leq d+e-2 = 3+1-2 = 2$. Note that p has a minimal reduction generated by 3 elements. So, we can not find a reduction generated by 2 elements.

Lemma 4.8. *Let R be a Noetherian ring and $E^n \subset E^{(n)}$ finitely generated R-modules such that $grade\big(E^{(n)}/E^n\big) \geq 2$ for all $n > 0$. Then $(E^n)^{**} \simeq (E^{(n)})^{**}$.*

Proof. Consider the short exact sequence of R-modules

$$0 \to E^n \to E^{(n)} \to E^{(n)}/E^n \to 0.$$

Dualizing above short exact sequence of R-modules, we have

$$0 \to \big(E^{(n)}/E^n\big)^* \to (E^{(n)})^* \to (E^n)^* \to Ext^1_R\big(E^{(n)}/E^n, R\big).$$

Since $grade\big(E^{(n)}/E^n\big) \geq 2$, $E^{(n)}/E^n = Ext^1_R\big(E^{(n)}/E^n, R\big) = 0$. So that $(E^n)^* \simeq (E^{(n)})^*$ and $(E^n)^{**} \simeq (E^{(n)})^{**}$. □

Remark 4.9. (1) Let E be a non-zero module over a Noetherian ring R. Then E is an ideal module if and only if $E \subset G \simeq R^e$ and $grade(G/E) \geq 2$.

(2) If E is an ideal module, then E^n is an ideal module for $n > 0$.

(3) Let E^n be an ideal module and $E^n \subset E^{(n)}$ with $grade\big(E^{(n)}/E^n\big) \geq 2$. Then $E^{(n)}$ is an ideal module (Lemma 4.8) for any n.

Proposition 4.10. *Let $E \subset G \simeq R^e$ be an ideal module with $e > 0$, $R_s(E)$ be a Noetherian ring with grade$\big(E^{(n)}/E^n\big) \geq 2$ and depth$(G^n/E^n) = d - ht(\mathbb{F}_e(E))$ for infinitely many n. Then $E^{(k)}$ is a equimultiple module for some $k > 0$.*

Proof. Let $R_s(E)$ be a Noetherian ring. Then by Lemma 4.5, there exists $k > 0$ such that $E^{(k)n} = E^{(kn)}$ for any $n > 0$. Note that if E is an ideal module, then $E^{(k)}$ is an ideal module for any $k > 0$ (Remark 4.9). Therefore, taking k large enough we may assume that $depth(G^{nk}/E^{(n)k}) = d - ht(\mathbb{F}_e(E^{(k)}))$ for any $n \geq 1$ and $E^{(k)}$ is an equimultiple module (Corollary 6.2, [1]). □

References

[1] A. L. B. Correia and S. Zarzuela, *On the asymptotic properties of the Rees power of a module*, J. Pure Appl. Algebra, **207** No. 2 (2006) 373-385.

[2] R. C. Cowsik, *Symbolic powers and numbers of defining equations*, Lecture Notes in Pure and Appl. Math., **91** (1987) 13-14.

[3] S. Goto, M. Herrmann, and K. Nishida, *On the structure of Noetherian symbolic Rees algebras*, Manuscripta Math., **67** No. 1 (1990) 197-225.

[4] S. Goto, K. Nishida, and K. I. Watanabe, *Non-Cohen-Macaulay symbolic blowups for space monomial curves and counterexamples to Cowsik's question*, Proc. Amer. Math. Soc., **120** No. 2 (1994) 383-392.

[5] C. Huneke, *On the finite generation of symbolic blow-ups*, Math. Z., **179** No. 4 (1982) 465-472.

[6] C. Huneke, *Hilbert functions and symbolic powers*, Michigan Math. J., **34** No. 2 (1987) 293-318.

[7] D. G. Northcott and D. Rees, *Reductions of ideals in local rings*, Proc. Cam. Philos. Soc., **50** No. 2 (1954) 145-158.

[8] D. Rees, *On a problem of Zariski*, Illinois J. Math., **2** (1958) 145-149.

[9] P. Roberts, *A prime ideal in a polynomial ring whose symbolic blow-up is not Noetherian*, Proc. Amer. Math. Soc., **94** No. 4 (1985) 589-592.

[10] A. K. Singh, *Multi-symbolic Rees algebras and strong F-regularity*, Math. Z., **235** No. 2 (2000) 335-344.

[11] P. Singh and S. D. Kumar, *Reductions in the Rees algebra of modules*, ALGEBR. REPRESENT. TH., **17** (2014) 1785-1795.

[12] I. Swanson and C. Huneke, *Integral Closure of Ideals, Rings, and Modules*, Cambridge University Press, Cambridge, 2006.

Priti Singh

Department of Mathematics, Patna Science College,

Patna University, India.

pritisingh.mnnit@gmail.com

Sharvan Kumar

Department of Mathematics,

Patna University, India.
skgu2810@gmail.com

Ajim Uddin Ansari
Department of Mathematics, CMP Degree College,
University of Allahabad, India.
ajimmatau@gmail.com

Shiv Datt Kumar
Department of Mathematics,
MNNIT Allahabad, India.
sdt@mnnit.ac.in