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Research Paper

THE STRONGLY IRREDUCIBLE DIMENSION OF RINGS VS. THE DERIVED DIMENSION OF THE SPACE OF STRONGLY IRREDUCIBLE IDEALS WITH V-TOPOLOGY

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ABSTRACT. An ideal I of a ring R is called strongly irreducible ideal (SI-ideal, for short), whenever the inclusion $J \cap K \subseteq I$, implies that $J \subseteq I$ or $K \subseteq I$. Let $X = \mathrm{SSpec}(R)$ be the set of all strongly irreducible ideals of a ring R. Then X with certain topology has derived dimension if and only if R has strongly irreducible dimension. Moreover, the two dimensions differ by at most 1.

1. Introduction

Karamzadeh in [4] has proved a useful result connecting two different concepts from topological spaces and general rings which is toward the unity in mathematics. This in particular shows that given any non-limit ordinal $\alpha = \beta + 1$ we may use the set of $X = \mathrm{Spec}(R)$ with a certain quasi-compact topology, where R is a Noetherian domain with Krull-dimension β

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(note, such a domain always exists, see [4, Note added in proof]), to provide a quasi-compact space with given a non-limit ordinal as its derived dimension which is important in topology (note, derived dimension of a topological space is also called Cantor-Bendixon dimension, and it is well-known that this dimension for quasi-compact spaces is always non-limit ordinal, see [6, P. 176]. Motivated by this fact in [4], in what follows we are going to set the stage for proving a similar connecting result. A proper ideal I of a ring R is said to be strongly irreducible (briefly, SI-ideal) if for each pair of ideals J and K of R, $J \cap K \subseteq I$ implies that either $J \subseteq I$ or $K \subseteq I$. A strongly irreducible ring R (briefly, SI-ring) is a ring in which 0 is strongly irreducible. A minimal strongly irreducible ideal in a ring R is any strongly irreducible ideal of R such that it does not properly contain any other strongly irreducible ideal. It is easy to see that every SI-ideal contains a minimal strongly irreducible ideal, for more details see [1]. If I is an ideal in a ring R, the set of all minimal strongly irreducible ideals Q containing I is denoted by $s\mathrm{iMin}(I)$. In what follows we recall the classical Krull-dimension of a ring R. Let $X = \mathrm{Spec}(R)$ be the set of all prime ideals in R and $\mathrm{Spec}_0(R)$ denotes the set of all maximal ideals of R. For an ordinal $\alpha > 0$ denote $\mathrm{Spec}_\alpha(R)$ to be the set of all prime ideals P in R such that whenever a prime ideal Q properly contains P, then Q belongs to $\mathrm{Spec}_\beta(R)$ for some $\beta < \alpha$. The smallest ordinal α such that $\mathrm{Spec}_\alpha(R) = X$ is called classical Krull-dimension of R, denoted by $\mathrm{cl.K.dim}(R)$. It is well-known that $\mathrm{cl.K.dim}(R)$ exists if and only if R has acc on prime ideals, for more details, see [2, 4]. Motivated by the above definition we introduce a new dimension for a ring R using SI-ideals instead of prime ideals and denote by $\mathrm{si.dim}(R)$. In this article, we extend the results of [4], in the same vein. For example we show that the existence of $\mathrm{si.dim}(R)$ is equivalent having acc on SI-ideals. Also similarly we study the derived dimension of a topological space defined on $X = \mathrm{SSpec}(R)$ (The set of all SI-ideals). In particular we show that this derived dimension exists if and only if $\mathrm{si.dim}(R)$ exists and they differ by at most one. In this article all rings are associative with $1 \neq \circ$.

2. Strongly irreducible dimension

Definition 2.1. Let $\mathrm{SSpec}(R)$ be the set of all SI-ideals of a ring R and $\mathrm{SSpec}_0(R)$ denote the set of all maximal ideals of R. Let $\alpha > 0$ be an ordinal and define $\mathrm{SSpec}_\alpha(R)$ to be the set of all ideals Q of R such that for any SI-ideal Q' where $Q \subset Q'$, then Q' belongs to $\mathrm{SSpec}_\beta(R)$ for some $\beta < \alpha$. Then the smallest ordinal α for which $\mathrm{SSpec}(R) = \mathrm{SSpec}_\alpha(R)$ is called *strongly irreducible dimension of* R, denoted by $\mathrm{si.dim}(R)$.

The following result provides an equivalent condition for the existence of $\mathrm{si.dim}(R)$ in a ring R.

Theorem 2.2. *A ring R has acc on* SI-*ideals if and only if* $\mathrm{si.dim}(R)$ *exists.*

Proof. First suppose that R has acc on SI-ideals and define the sets $\text{SSpec}_\alpha(R)$ of SI-ideals as in the previous definition. Since for each α, $\text{card}(\text{SSpec}_\alpha(R)) \leq 2^{\text{card}(R)}$, the chain $\text{SSpec}_0(R) \subseteq \text{SSpec}_1(R) \subseteq \cdots$ cannot be properly increasing forever. Hence, there exists an ordinal γ such that $\text{SSpec}_\gamma(R) = \text{SSpec}_{\gamma+1}(R)$. If $\text{si.dim}(R)$ does not exist, then using acc on SI-ideals, there is an SI-ideal Q which is maximal with respect to the property $Q \notin \text{SSpec}_\alpha(R)$. Hence all SI-ideals properly containing Q lie in $\text{SSpec}_\alpha(R)$. Therefore, we infer that $Q \in \text{SSpec}_{\alpha+1}(R) = \text{SSpec}_\alpha(R)$, which is a contradiction.

For the converse, we show that every nonempty set of SSpec(R) has a maximal element. So, let $\text{si.dim}(R)$ exists and S be a nonempty set of $\text{SSpec}(R)$. Therefore, there is an ordinal $\alpha \geq 0$ such that $\text{SSpec}_\alpha(R) = \text{SSpec}(R)$, and we may assume that β is the smallest ordinal such that $S \cap \text{SSpec}_\beta(R)$ is not empty. Let $Q \in S \cap \text{SSpec}_\beta(R)$ and assume there is an element Q' in S which contains Q properly. Then $Q' \in \text{SSpec}_\gamma(R)$ for some $\gamma < \beta$ and hence $Q' \in \text{SSpec}_\gamma(R) \cap S$ for some $\gamma < \beta$, and this contradicts the fact that β is minimal . Thus Q is maximal in S, i.e., R has acc on SI-ideals. □

Remark 2.3. Clearly that every prime ideal of a ring R is an SI-ideal. Thus, if R is a ring with no acc on prime ideals (i.e., $\text{cl.K.dim}(R)$ does not exist, see [2, Ex. 14A(b)] and [2, Proposition 14.1]), then from theorm 2.2, we conclud that $\text{si.dim}(R)$ also does not exist.

Example 2.4. Let $\mathbb{Z}$ be the ring of integers. It is clear that every nonzero strongly irreducible ideal in $\mathbb{Z}$ is of the form $p^i\mathbb{Z}$, where p is a prime number and i is a positive integer. Therefore for any nonnagative integer n, $\text{SSpec}_n(\mathbb{Z})$ consists of all ideals of the form $p^i\mathbb{Z}$ where $1 \leq i \leq n+1$. So $\text{si.dim}(\mathbb{Z}) = \omega$, where ω is the first infinite ordinal. The same result holds for any commutative principal ideal domain that is not a field.

Example 2.5. Let R be a *strongly regular ring* (i.e., $a = a^2b$ for any $a \in R$ and some $b \in R$). Then $\text{cl.K.dim}(R) = \text{si.dim}(R) = 0$

Example 2.6. If R is a polynomial ring in an infinite number of indeterminates over a field k, then R does not have acc for prime ideals. Therfore according to Remark 2.3, $\text{si.dim}(R)$ is not exist.

Remark 2.7. We observe that if Q is an SI-ideal in R containing an ideal I, then Q/I is an SI-ideal of R/I. For the converse we also observe that if R is an arithmetical ring (note, we recall that a ring R is called arithmetical ring whenever the lattice of all ideals of R is distributive, i.e., for any three ideals I, J and K in R, $I + (J \cap K) = (I + J) \cap (I + K)$, or equivalently, $I \cap (J + K) = (I \cap J) + (I \cap K)$) and Q/I is an SI-ideal of R/I, then Q is an SI-ideal in R. To this end, let $J \cap K \subseteq Q$ then $(J \cap K) + I = (J + I) \cap (K + I) \subseteq Q$ and consequently

$(J+I)/I \cap (K+I)/I \subseteq Q/I$. Since Q/I is SI-ideeal, we infer that $(J+I)/I \subseteq Q/I$ or $(K+I)/I \subseteq Q/I$. Hence $J \subseteq Q$ or $K \subseteq Q$, i.e., Q is an SI-ideal.

Lemma 2.8. *Let R be an arithmetical ring and $\alpha \geq 0$ be an ordinal. Then:*

(1) *If I is an ideal contained in the* SI*-ideal Q, then $Q \in \mathrm{SSpec}_{\alpha}(R)$ if and only if $Q/I \in \mathrm{SSpec}_{\alpha}(R/I)$.*

(2) $\mathrm{si.dim}(R) = \alpha$ *implies* $\mathrm{si.dim}(R/I) \leq \alpha$ *for every ideal I of R.*

(3) *If R is an* SI*-ring with* $\mathrm{si.dim}(R) = \alpha$ *and $Q \neq 0$ is an* SI*-ideal, then* $\mathrm{si.dim}(R/Q) < \alpha$.

Proof. (1) The statement is obvious for $\alpha = 0$. Let $\alpha \geq 0$ and assume (1) holds for all $\beta < \alpha$. From the definition of $\mathrm{SSpec}_{\alpha}(R)$ and the induction hypothesis we get: $Q \in \mathrm{SSpec}_{\alpha}(R)$ if and only if $Q \subset Q^{'} \in \mathrm{SSpec}(R)$ implies $Q^{'} \in \mathrm{SSpec}_{\beta}(R)$ for some $\beta < \alpha$. And we also note that $Q/I \subset Q^{'}/I \in \mathrm{SSpec}(R/I)$ implies $Q^{'}/I \in \mathrm{SSpec}_{\beta}(R/I)$ for some $\beta < \alpha$, if and only if $Q/I \in \mathrm{SSpec}_{\alpha}(R/I)$.
(2) If $Q/I \in \mathrm{SSpec}(R/I)$, then $I \subseteq Q \in \mathrm{SSpec}(R) = \mathrm{SSpec}_{\alpha}(R)$. By (1), $Q/I \in \mathrm{SSpec}_{\alpha}(R/I)$ and hence $\mathrm{si.dim}(R/I) \leq \alpha$.
(3) Since R is an SI-ring and $0 \subset Q$, clearly $Q \in SSec_{\beta}(R)$ for some $\beta < \alpha$. Now, if $Q^{'}/Q \in \mathrm{SSpec}(R/Q)$ then $Q^{'} \in \mathrm{SSpec}_{\gamma}(R)$ for some $\gamma < \beta$ and hence $Q^{'}/Q \in \mathrm{SSpec}_{\gamma}(R/Q)$ by (1). Thus $\mathrm{SSpec}(R/Q) \subseteq \mathrm{SSpec}_{\gamma}(R/Q)$ which implies that $\mathrm{si.dim}(R/Q) < \alpha$. □

The following corollary is now immediate.

Corollary 2.9. *Let R be an arithmetical ring and* $\mathrm{si.dim}(R)$ *exists. If P and Q are* SI*-ideals with $P \subset Q$, then* $\mathrm{si.dim}(R/Q) < \mathrm{si.dim}(R/P)$.

In what follows we present several more results concerning SI-ideal in arithmetical rings.

Lemma 2.10. *Let R be an arithmetical ring with* $\mathrm{si.dim}(R) = \alpha$. *If $\beta \geq 0$ is any ordinal strictly less than α, then there is an SI-ideal Q such that* $\mathrm{si.dim}(R/Q) = \beta$. *If R is a right or left Neotherain, then there is a minimal SI-ideal Q such that* $\mathrm{si.dim}(R/Q) = \alpha$.

Proof. First, we observe that for an SI-ideal Q, we have si.dim(R/Q) $= \beta$ if and only if β is the smallest ordinal such that $Q \in \mathrm{SSpec}_{\beta}(R)$. If there is no SI-ideal Q such that $\mathrm{si.dim}(R/Q) = \beta$, then we must have $\mathrm{SSpec}_{\beta}(R) = \mathrm{SSpec}_{\beta+1}(R)$, which implies that $\mathrm{SSpec}_{\beta}(R) = \mathrm{SSpec}_{\gamma}(R)$ for any $\beta < \gamma$. Therefore, we infer that $\mathrm{si.dim}(R) \leq \beta$, which is a contradiction. Here, we observe that this argument especially shows that $\mathrm{si.dim}(R)$ is the supremum of the ordinals $\mathrm{si.dim}(R/Q)$, where Q ranges over the set of SI-ideals, and it is clear that we may restrict the set of SI-ideals to the set of minimal SI-ideals, for, every SI-ideals contains a minimal SI-ideal. If R is right or left Noetherian, then there are only finitely many minimal SI-ideals, therefore there is a minimal SI-ideal, Q say, such that $\mathrm{si.dim}(R/Q) = \alpha$. □

Lemma 2.11. *Let R be an arithmetical ring with $\mathrm{si.dim}(R) \geq \alpha \geq 0$. If $\mathrm{si.dim}(R/I) < \alpha$ for every ideal $I \neq 0$, then R is an SI-ring with $\mathrm{si.dim}(R) = \alpha$.*

Proof. We show that $\mathrm{SSpec}_\alpha(R) = \mathrm{SSpec}(R)$. Let P and Q be two SI-ideals with $P \subset Q$. Since $\mathrm{si.dim}(R/P) = \beta < \alpha$ and Q/P is an SI-ideal of R/P, by Lemma 2.8(1), we get $Q \in \mathrm{SSpec}_\beta(R)$. Thus $P \in \mathrm{SSpec}_\alpha(R)$ for all $P \in \mathrm{SSpec}(R)$, and this yields $\mathrm{si.dim}(R) \leq \alpha$, i.e., $\mathrm{si.dim}(R) = \alpha$.

Now suppose that R is not an SI-ring, therefore there are nonzero SI-ideals I and J such that $I \cap J = 0$. Let β be the maximum of $\mathrm{si.dim}(R/I)$ and $\mathrm{si.dim}(R/J)$. It is sufficient to prove that $\mathrm{SSpec}_\beta(R) = \mathrm{SSpec}(R)$, which is a contradiction because we show that $\mathrm{si.dim}(R) = \alpha$. To this end, suppose that P is an SI-ideal of R. Since $I \cap J = 0 \subseteq P$, we may assume that $I \subseteq P$. Hence (P/I) is an SI-ideal of (R/I) (note that R is an arithmetical ring). Now by Lemma 2.8 (2), $\mathrm{si.dim}(R/I)/(P/I) \leq \mathrm{si.dim}(R/I)$, so $\mathrm{si.dim}(R/P) \leq \beta < \alpha$, i.e., $(P/I) \in \mathrm{SSpec}_\beta(R/I)$ and consequently $P \in \mathrm{SSpec}_\beta(R)$. □

Lemma 2.12. *Let R be an arithmetical ring with acc on two-sided ideals. The following are equivalent:*

(1) *R is an SI-ring.*
(2) *For every SI-ideal $Q \neq 0$, $\mathrm{si.dim}(R/Q) < \mathrm{si.dim}(R)$.*
(3) *For every ideal $I \neq 0$, $\mathrm{si.dim}(R/I) < \mathrm{si.dim}(R)$.*

Proof. (1) implies (2) by Lemma 2.8 (3). (3) implies (1) by the previous lemma. Finally, in what follows we prove (2) implies (3). Suppose that (2) holds and let I be an ideal which is maximal with respect to $\mathrm{si.dim}(R/I) = \mathrm{si.dim}(R)$. Now, suppose that K/I is a nonzero ideal of R/I, then by the maximality of I, we have $\mathrm{si.dim}(R/I)/(K/I) = \mathrm{si.dim}(R/K) < \mathrm{si.dim}(R)$. Thus by the previous lemma, (R/I) is an SI-ring. So I is an SI-ideal and by (2), we get $I = 0$. □

The following corollary is evident.

Corollary 2.13. *Let R be a right or left Noetherian arithmetical ring and $\mathrm{si.dim}(R)$ exists. If Q is an SI-ideal of R and I is an ideal of R with $Q \subset I$, then $\mathrm{si.dim}(R/I) < \mathrm{si.dim}(R/Q)$.*

Proposition 2.14. *Let R be a ring with acc on SI-ideals. Then R has acc on ideals I of the form $I_k = \bigcap_{Q \in F_k} Q$, where F_k is a finite set of noncomparable SI-ideals.*

Proof. Let $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ be an infinite ascending chain of ideals, each of which is of the form $I_k = \bigcap_{P \in F_k} P$, where F_k is a finite set of noncomparable SI-ideals. If it happens that $F_{r_1} = F_{r_2} = \cdots = F_{r_n} = \cdots$, where $r_1 < r_2, \cdots < r_n < \cdots$ is an infinite sequence, then $I_{r_1} = I_{r_2} = \cdots = I_{r_n} = \cdots$ and we are through. Now we may assume $F_{n+1} - F_n \neq \emptyset$, for all n and complete the proof by obtaining a contradiction. We note that $F_i \cap F_r \subseteq F_{i-1} \cap F_r$ for all r and $r \leq i-1$, for if not, then there exists $Q_i \in F_i \cap F_r$ such that $Q_i \notin F_{i-1}$. Hence there exists $Q_{i-1} \in F_{i-1}$ such that $Q_{i-1} \subset Q_i$ and since $r \leq i-1$, there exists $Q_r \in F_r$ such that $Q_r \subseteq Q_{i-1} \subset Q_i$. But Q_r and Q_i are both in F_r and can not be comparable. This shows that without loss of generality we can assume that $F_{i-1} \cap F_r = F_i \cap F_r$, for all r and $r \leq i-1$. Now let $Q_m \in F_m - F_{m-1}$, for any integer $m > 0$. Then $Q_m \notin \bigcup_{i=1}^{m-1} F_i$, for otherwise $Q_m \in F_r$, for some $r \leq m-1$ and $F_m \cap F_r = F_{m-1} \cap F_r$ implies that $Q_m \in F_{m-1}$, which is impossible. Hence there exists $Q_{m-1} \in F_{m-1}$ such that $Q_{m-1} \subset Q_m$ and $Q_{m-1} \notin \bigcup_{i=1}^{m-2} F_i$, for otherwise $Q_{m-1} \in F_{m-1} \cap F_r$ for some $r \leq m-2$ implies that $Q_{m-1} \in F_m$, which is impossible. Repeating this process we get a chain $Q_1 \subset Q_2 \subset \cdots Q_n$ of SI-ideals such taht each Q_i belongs to F_i. Now put $F_1^n = \{Q_1 \in F_1\colon$ there exists a chain $Q_1 \subset Q_2 \subset \cdots Q_n$, where $Q_i \in F_i, i = 1, \cdots, n\}$. We have already shown that $F_1^n \neq \emptyset$ for all n. Moreover, F_1^n is finite and $F_1^n \subseteq F_1^m$, for $m \leq n$. Therefore the chain $F_1^1 \supseteq F_1^2 \supseteq \cdots \supseteq F_1^n \supseteq \cdots$ stationary and we can choose $Q_1^{'} \in \bigcap_{n=1}^{\infty} F_1^n$. Now for each $n \geq 2$, let $F_2^n = \{Q_2 \in F_2 :$ there exists a chain $Q_1^{'} \subset Q_2 \subset \cdots Q_n$ where $Q_i \in F_i, i = 2, \cdots, n\}$. It is clear that $F_2^n \neq \emptyset$ for all n. Now we can choose $Q_2^{'} \in \bigcap_{n=2}^{\infty} F_2^n$. Hence proceeding inductively we get a chain $Q_1^{'} \subset Q_2^{'} \subset \cdots \subset Q_n^{'} \subset \cdots$ of SI-ideals, which is the desired contradiction. □

First let us recall that the Z-topology on the set of prime ideals in noncommutative rings as in [4]. If A is an ideal of R we let $V(A)$ denote the subset of $\mathrm{Spec}(R)$ consisting of those prime ideals that contain A. Now just by replacing the prime ideal in the previous definition by SI-ideals we get a topology on the SI-ideals, see also [1].

It is shown that in [4] that the set of $X - V(A)$ satisfy the axioms for open sets in this topological space and we call it the *SIZ- topology* on $X = \mathrm{SSpec}(R)$. Put $B = \{V(A) : A$ is a ideal of R $\}$, then clearly B can be take as a base for open sets on $\mathrm{SSpec}(R)$. This topology is also called V-topology. The name V-topology is first introduced by Karamzadeh in [4].

The following result is in [4], without proof. Next we give a proof for the sake of the reader.

Proposition 2.15. *Let* $X = \mathrm{SSpec}(R)$ *be with the* SIZ-*topology, then the following statements are equivalent:*

(1) *X has acc on open subsets.*
(2) *Every subset of X is quasi-compact.*
(3) *X has acc on intersections of* SI-*ideals.*

Proof. (1) $\Rightarrow$ (2) Let $A \subseteq \bigcup_{\lambda\in\Lambda} O_\lambda$, where each O_λ is an open subset of X and A is a subset of X. Since X has acc on open subsets, then $T = \{\bigcup_{i=1}^{n} O_{\lambda_i} : \lambda_i \in \Lambda, n \in \mathbb{N}\}$ has maximal element, say $\bigcup_{i=1}^{n} O_{\lambda_i}$. It is clear that $A \subseteq \bigcup_{i=1}^{n} O_{\lambda_i}$.
(2) $\Rightarrow$ (3) Let $I_1 \subseteq I_2 \subseteq \cdots$ be an infinite ascending chain of ideals, where each I_k is an intersection of a family of SI-ideals. Then

$$X - V(I_1) \subseteq X - V(I_2) \subseteq \cdots.$$

Let $A = \bigcup_{k\in\mathbb{N}}(X - V(I_k))$, then By assumption, there exists $n \in \mathbb{N}$ such that $A = X - V(I_n)$. It follows that $V(I_n) = V(I_{n+k})$, and consequently $I_n = I_{n+k}$ for all $k \in \mathbb{N}$.
(3) $\Rightarrow$ (1) Since every open set of X is of the form $X - V(I)$, where I is an ideal of R, so to prove 3, it suffices to prove that X has the dcc on closed sets. Hence let $V(I_1) \supseteq V(I_2) \supseteq \cdots$ be an infinite descending chain of closed subset of X. It is clear that $\bigcap V(I_1) \subseteq \bigcap V(I_2) \subseteq \cdots$ and by hypothesis, there exists $n \in N$ such that $\bigcap V(I_n) = \bigcap V(I_{n+k})$, for all $k \in \mathbb{N}$. Since for every ideal I of R, $V(\bigcap V(I)) = V(I)$, we infer that $V(I_n) = V(I_{n+k})$ for all $k \in \mathbb{N}$, and the proof is complete. □

In what follows we prove some needed results which are counterparts of similar results in [4]

Corollary 2.16. *If R is a ring with* si.dim(R) *and has only finitely many minimal* SI-*ideals over any ideal, then every* SI-*ideal is minimal over some finitely generated subideal.*

Proof. If I is an ideal in R, let $P(I)$ denote the intersection of SI-ideal containing I. It is sufficient to show that $P(I) = P(\langle x_1, x_2, \cdots, x_n\rangle)$, where $\langle x_1, x_2, \cdots, x_n\rangle$ is the ideal generated by $x_1, x_2, \cdots, x_n \in I$. It is clear that $V(I) = \bigcap_{x\in I} V(\langle x\rangle)$ and $X - V(I) = \bigcup_{x\in I}(X - V(\langle x\rangle))$. Now by Proposition 2.14, R has acc on intersections of SI-ideals. Therefore Proposition 2.15 shows that every subset and in particular $X - V(I)$ is quasi-compact. Thus there are some element $x_1, x_2, \cdots, x_n \in I$ such that $X - V(I) = \bigcup_{i=1}^{n}(X - V(\langle x_i\rangle))$, hence $V(I) = \bigcap_{i=1}^{n} V(\langle x_i\rangle)$ implies that $P(I) = P(\langle x_1, x_2, \cdots, x_n\rangle)$. □

We cite the following proposition from [3], which is counterpart of the prime avoidance lemma for SI-ideals. We give the proof for the sake of the reader.

Proposition 2.17. *Let $I, Q_1, Q_2, \cdots, Q_n$, $n \geq 2$, be ideals of a ring R and $I \subseteq \bigcup_{i=1}^{n} Q_i$. If at most two of the Q_i's are not SI-ideal, then $I \subseteq Q_i$ for some Q_i.*

Proof. For $n = 2$, the assertion holds, even if Q_1 and Q_2 are not SI-ideal, which is a classical result in ring theory. Now assume $n \geq 3$. In this case, without loss of generality we may assume that Q_1 is an SI-ideal and $Q_i \nsubseteq Q_j$ for $i \neq j$. Also by induction we may assume that

$I \nsubseteq \bigcup_{i=2}^{n} Q_i$. Hence there is $x \in I$ such that $x \notin \bigcup_{i=2}^{n} Q_i$. We show that $I \subseteq Q_1$ and we are done. Let us put $J = \bigcap_{i=2}^{n} Q_i$ and note that for each $y \in I \cap J$ we have $x + y \notin Q_i$ for all $i \geq 2$. Therefore $x + y \in Q_1$ which means $y \in Q_1$ and so $I \cap J \subseteq Q_1$. Since Q_1 is an SI-ideal and $J \nsubseteq Q_1$, we infer that $I \subseteq Q_1$. □

The next proposition is the counterpart of Proposition 2 in [4]. But before that, we express the concept of the rank of an SI-ideal. Let Q, $Q_1, \cdots$, Q_n be distinc SI-ideals and $Q = Q_0 \supset Q_1 \supset \cdots \supset Q_n$, then we say that this chain is of length n. Now, we say that Q has rank n which is abbriveted by $\text{rank}(Q) = n$, if there exists a chain of length n descending from Q, but no longer chain. If for any positive integer n, there exists a descending chain from Q of length n, then we say that Q has rank ∞. We note that a minimal SI-ideal has rank 0.

Proposition 2.18. *Let R be an arithmetical ring with* $\text{si.dim}(R) = n$ *and have only finitely many minimal* SI-*ideals over any ideal, then every* SI-*ideal is minimal over a subideal generated by less than or equal to n elements.*

Proof. Let $\text{si.dim}(R) = n$, Q is an SI-ideal and $Q_1, Q_2, \cdots, Q_t$ be all minimal SI-ideals. By Corollary 2.9, it is clear that for any SI-ideal Q, $\text{rank}(Q) \leq \text{si.dim}(R)$. We may proceed by induction on $k = \text{rank}(Q)$ and show that Q is minimal over a subideal which is generated by $\leq k$ elements. For $\text{rank}(Q) = 0$ is clear. Now suppose that it is true for $\text{rank}(Q) \leq k - 1$. And let $\text{rank}(Q) = k$ where $k > 0$. By Proposition 2.17, we infer that $Q \nsubseteq \bigcup_{i=1}^{t} Q_i$ because $k > 0$, so there exists $a_1 \in Q$ such that $a_1 \notin \bigcup_{i=1}^{t} Q_i$. Now let $\pi : R \to R/\langle a_1 \rangle$ with $\pi(x) = \bar{x}$ be the canonical projection and we observe that $\text{rank}(Q/\langle a_1 \rangle) \leq k - 1$. Therefore $Q/\langle a_1 \rangle$ is minimal over $\langle \bar{a}_2, \bar{a}_3, \cdots, \bar{a}_k \rangle$ by hypothesis induction, then it is clear that Q is minimal over $\langle a_1, a_2, \cdots, a_k \rangle$. □

3. Drived dimension of a topological space

Recall that in a topological space X an element $x \in X$ is called a limit point of a subset A of X if each open set of X contains at least one point of A distinct from x. The set of all limit points of A is denoted by A' and is called the *drived* set of A. A point $a \in A$ is called *isoleted* whenever $a \in A - A'$.

Without further ado we begin with the definition of the above dimension.

Definition 3.1. The α-derivative of a toplogical space X is defined by transfinite induction: $X_0 = X$, $X_{\alpha+1} = X'_\alpha$ and $X_\alpha = \bigcap_{\beta<\alpha} X_\beta$, for a limit ordinal α. If for an ordinal α we have $X_\alpha = \emptyset$ then X is called *scattered*. If X is scattered and α is the smallest ordinal such that $X_\alpha = \emptyset$, then α is called *derived dimension* of X and is denoted by $\text{d}(X) = \alpha$, for more information see [4].

The following lemma is well known, see[4].

Lemma 3.2. *Let X be a topoligical space, then the following are equivalent.*

(1) *Every non empty subset of X contains an isolated point.*
(2) *There is an $\alpha > 0$ such that $X_\alpha = \emptyset$*

The following is the counterpart of [4, lemma 4].

Lemma 3.3. *Let $X = \mathrm{SSpec}(R)$ be the space with the* V*-topology and $S \subseteq X$, then an element $Q \in S$ is an isolated point of S if and only if it is a maximal element of S.*

Proof. If $Q \in S$ is maximal element of S then $V(Q) \cap S = \{Q\}$ shows that P is an isolated point of S. Now suppose that Q is an isolated point of S then there exists an open subset G such that $P \in G$ and $G \cap S = \{Q\}$. But there exists $V(A)$ such that $Q \in V(A) \subseteq G$, then $V(A) \cap S = \{Q\}$. Now we claim that Q is a maximal in S. If $Q \subset Q^{'}$ and $Q^{'} \in S$ then $Q^{'} \in V(A)$ which is impossible. □

We need the next proposition which is also the counterpart of [4, Corollary 3].

Proposition 3.4. *Let $X = \mathrm{SSpec}(R)$ be the space with the* V*-topology. Then $\mathrm{SSpec}_\alpha(R) = \bigcup_{\beta \leq \alpha} S_\beta$, where S_β is the set of isolated points of X_β.*

Proof. We proceed by induction on α. For $\alpha = 0$ it is clear. Let us assume that $\mathrm{SSpec}_\beta(R) = \bigcup_{\gamma \leq \beta} S_\gamma$ for all $\beta < \alpha$. Now let $Q \in \bigcup_{\beta \leq \alpha} S_\beta$. If $Q \in S_\alpha$, then Q is a maximal element of X_α and so $Q^{'} \in X$, $Q \subset Q^{'}$ implies that $Q^{'} \notin X_\alpha = X - \bigcup_{\beta < \alpha} S_\beta$. Hence we have $Q^{'} \in S_\beta$ for some $\beta < \alpha$. Thus $Q^{'} \in \bigcup_{\gamma \leq \beta} S_\gamma = \mathrm{SSpec}_\beta(R)$ which implies that $Q \in \mathrm{SSpec}_\alpha(R)$, and if $Q \notin S_\alpha$, then $Q \in S_\beta$ for some $\beta < \alpha$ which implies that $Q \in \bigcup_{\gamma \leq \beta} S_\gamma = \mathrm{SSpec}_\beta(R) \subseteq \mathrm{SSpec}_\alpha(R)$. Therefore, we have $\bigcup_{\beta \leq \alpha} S_\beta \subseteq \mathrm{SSpec}_\alpha(R)$.
Conversely, let $Q \in \mathrm{SSpec}_\alpha(R)$, then if $Q \notin \bigcup_{\beta < \alpha} S_\beta$, we show that $Q \in S_\alpha$. To this end, let $Q^{'} \in X$, $Q \subset Q^{'}$, then $Q^{'} \in \mathrm{SSpec}_\beta(R) = \bigcup_{\gamma \leq \beta} S_\gamma$ implies that $Q^{'} \notin X_\alpha = X - \bigcup_{\gamma < \alpha} S_\gamma$. But $Q \in X_\alpha$ shows that Q must be a maximal element of X_α, so by the previous lemma $Q \in S_\alpha$. Therefore we have $\mathrm{SSpec}_\alpha(R) \subseteq \bigcup_{\beta \leq \alpha} S_\beta$. □

Corollary 3.5. *Let $\mathrm{si.dim}(R) = \alpha$, then $X = \mathrm{SSpec}(R)$ with* V*-topology have derived dimension and $\mathrm{d}(X) \leq \alpha + 1$*

Proof. Let S be a non empty subset of X, then by Theorem 2.2, there exists a maximal element Q in S. We note that $V(Q) \cap S = \{Q\}$. This shows that Q is an isolated points of S with respect to V-topology. Hence by Lemma 3.2, there is an $\alpha > 0$ such that $X_\alpha = \emptyset$. Hence $d(X)$

exists and since according to the previous proposition $X_{\alpha+1} = X - \bigcup_{\beta\leq\alpha} S_\beta = \emptyset$, therefore we have $\mathrm{d}(X) \leq \alpha + 1$. □

The next result is our main theorem.

Theorem 3.6. *Let* $X = \mathrm{SSpec}(R)$ *be the space with the* V*-topology, then the derived dimension of* X *exists if and only if* $\mathrm{si.dim}(R)$ *exists and* $\mathrm{d}(X) = \mathrm{si.dim}(R)$ *if* $\mathrm{d}(X)$ *is a limit ordinal and* $\mathrm{d}(X) = \mathrm{si.dim}(R) + 1$ *if* $\mathrm{d}(X)$ *is not a limit ordinal.*

Proof. Since $\mathrm{SSpec}_\alpha(R) = \bigcup_{\beta\leq\alpha} S_\beta$ and $X_{\alpha+1} = X - \bigcup_{\beta\leq\alpha} S_\beta$, so the first part holds. For the last part, first consider $\mathrm{d}(X) = \alpha$, where α is a limit ordinal. In this case we have $X_\alpha = X - \bigcup_{\beta<\alpha} S_\beta = \emptyset$, therefore $X = \bigcup_{\beta<\alpha} S_\beta = \bigcup_{\beta\leq\alpha} S_\beta = \mathrm{SSpec}_\alpha(R)$. Hence si.dim(R) $\leq \alpha$, and since by Corollary 3.5, $\mathrm{d}(X) \leq \mathrm{si.dim}(R) + 1$, thus $\mathrm{si.dim}(R) = \alpha$. Now let $\mathrm{d}(X) = \alpha + 1$, then $X_{\alpha+1} = \emptyset$ which implies that $X = \bigcup_{\beta\leq\alpha} S_\beta = \mathrm{SSpec}_\alpha(R)$. Therefore $\mathrm{si.dim}(R) \leq \alpha$ and from $\mathrm{d}(X) \leq \mathrm{si.dim}(R) + 1$, we get $\mathrm{si.dim}(R) = \alpha$. □

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