Algebraic Structures and Their Applications Vol. 12 No. 1 (2025) pp 65-76.

Research Paper

ON THE SUBSPACE DISTANCE OF THE SUBSPACE CODES

SEYEDEH HAWRA SADROLHOFFAZ AND REZA KAHKESHANI*

ABSTRACT. Let $\mathcal{P}_q(n)$ be the set of all subspaces in the vector space $\mathbb{F}_q^n$. There is a subspace distance $d_S(U,V)$ between any two subspaces U and V. A subspace code is also a subset of $\mathcal{P}_q(n)$. It is known that $d_S(U,V) \geq d_H(\nu(\pi U), \nu(\pi V))$, where $\pi \in S_n$, $\nu(U)$ denotes the pivot vector of $E(U)$ and $E(U)$ is the reduced row echelon form of the generator matrix of U. In this paper, we show that if $E(U)$ and $E(V)$ have at most one non-zero entry in each rows and each columns then the equality holds. Moreover, we introduce the sets $\mathcal{G}_{U,V} = \{\pi \in S_n \mid d_S(U,V) = d_H(\nu(\pi U), \nu(\pi V))\}$ for any $U, V \in \mathcal{P}_q(n)$ and examine them in the spaces $\mathcal{P}_2(4)$, $\mathcal{P}_2(5)$, $\mathcal{P}_2(6)$ and $\mathcal{P}_3(4)$. It is shown that the groups 1, $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, S_3, S_4 and 1, $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, S_3, D_8, $S_3 \times \mathbb{Z}_2$, S_4, S_5 appears between these sets in $\mathcal{P}_2(4)$ and $\mathcal{P}_2(5)$, respectively. Moreover, the groups 1, $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, S_3, D_8, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $S_3 \times \mathbb{Z}_2$, $D_8 \times \mathbb{Z}_2$, S_4, $S_3 \times S_3$, $S_4 \times \mathbb{Z}_2$, $(S_3 \times S_3)$:2, S_5, S_6 and 1, $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, S_3, D_8, S_4 appears between these sets in $\mathcal{P}_2(6)$ and $\mathcal{P}_3(4)$, respectively.

DOI: 10.22034/as.2024.20208.1646

MSC(2010): Primary: 94B60, Secondary: 05A05, 05B20.

Keywords: Pivot vector, Subspace code, Subspace distance.

Received: 16 June 2023, Accepted: 01 September 2024.

*Corresponding author

1. Introduction

Let Σ be a non-empty set and $n \geq 1$ be an integer. Any subset $\mathcal{C}$ of Σ^n is called a block code of length n over the alphabet Σ. The elements of $\mathcal{C}$ are called codewords. The Hamming distance between two words $\mathbf{w} = w_1 w_2 \cdots w_n$ and $\mathbf{w}' = w'_1 w'_2 \cdots w'_n$ in Σ^n is

$$d_H(\mathbf{w}, \mathbf{w}') = \#\{i \mid 1 \leq i \leq n,\ w_i \neq w'_i\},$$

and the minimum (Hamming) distance of $\mathcal{C}$ is given by

$$d_H(\mathcal{C}) = \min\{d_H(\mathbf{c}, \mathbf{c}') \mid \mathbf{c}, \mathbf{c}' \in \mathcal{C},\ \mathbf{c} \neq \mathbf{c}'\}.$$

When the alphabet is the finite field $\mathbb{F}_q$ of order q, the code $\mathcal{C}$ is called linear if it is closed under addition and scalar multiplication. The code $\mathcal{C}$ is binary if $\Sigma = \mathbb{F}_2$. The (Hamming) weight of the word $\mathbf{w} = w_1 w_2 \cdots w_n$ is defined by $\mathrm{wt}(\mathbf{w}) = \#\{i \mid 1 \leq i \leq n,\ w_i \neq 0\}$. Clearly, $\mathrm{wt}(\mathbf{w}) = d_H(\mathbf{w}, 0)$. Also, set $\mathbf{w} * \mathbf{w}' = (w_1 w'_1)(w_2 w'_2) \cdots (w_n w'_n)$, where $\mathbf{w} = w_1 w_2 \cdots w_n$ and $\mathbf{w}' = w'_1 w'_2 \cdots w'_n$ are two words in $\mathbb{F}_q^n$. See [4, 5] for more details.

Let $\mathbb{F}_q^n$ be the vector space of dimension $n \geq 0$ over $\mathbb{F}_q$. We denote by $\mathcal{P}_q(n)$ the set of all subspaces in $\mathbb{F}_q^n$. Each subset of $\mathcal{P}_q(n)$ is called a subspace code. See [2, 6] for some applications of subspace codes in random linear network coding, cryptography and distributed storage. The subspace distance between two subspaces $U, V \in \mathcal{P}_q(n)$ is defined by

$$\begin{aligned} d_S(U, V) &= \dim(U + V) - \dim(U \cap V) \\ &= \dim(U) + \dim(V) - 2\dim(U \cap V). \end{aligned}$$

If $\mathcal{C}$ is a subspace code then the minimum subspace distance of $\mathcal{C}$ is

$$d_S(\mathcal{C}) = \min\{d_S(U, V) \mid U, V \in \mathcal{C},\ U \neq V\}.$$

An $k \times n$ matrix G over $\mathbb{F}_q$ is called a generator matrix of the k-dimensional subspace $U \in \mathcal{P}_q(n)$ if the rows of G form a basis of U, i.e., $U = \langle G \rangle$. Applying the elementary row operations on G, we can obtain the generator matrix $E(G)$ in reduced row echelon form such that $U = \langle E(G) \rangle$. Since such a matrix $E(G)$ is unique for any given subspace U, we also denote it by $E(U)$. Denote by $\nu(U)$ the characteristic vector of the pivot columns in $E(U)$. This binary vector is called a pivot vector. Clearly, $\nu(U) \in \mathbb{F}_2^n$ and $\nu(U)$ depends on the ordering of the positions. Note that if U is a k-dimensional subspace of $\mathbb{F}_q^n$ then $\mathrm{wt}(\nu(U)) = k$. For example, if

$$U = \left\langle \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \right\rangle \subseteq \mathbb{F}_3^5,$$

then

$$E(U) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

and $\nu(U) = (11010) \in \mathbb{F}_2^5$. See [3] for more information. Moreover, see [7, 8] for some recent research on the subspace codes.

Let U and V be two subspaces in $\mathcal{P}_q(n)$ with the pivot vectors $\nu(U)$ and $\nu(V)$, respectively. It is known that $d_S(U,V) \geq d_H(\nu(U), \nu(V))$ [3]. Let S_n be the symmetric group on $\{1, \ldots, n\}$. For a matrix M of size $k \times n$ over $\mathbb{F}_q$ and a permutation $\pi \in S_n$, let πM denotes the matrix arising from M by permuting its columns according to π. If U is a subspace of $\mathcal{P}_q(n)$ then πU is defined to be $\langle \pi E(U) \rangle$. It is easy to see that $\langle E(U \cap V) \rangle = \langle E(U) \rangle \cap \langle E(V) \rangle$, $\langle \pi E(U \cap V) \rangle = \langle \pi E(U) \rangle \cap \langle \pi E(V) \rangle$ and $d_S(U,V) = d_S(\pi U, \pi V)$ for all $U, V \in \mathcal{P}_q(n)$ and $\pi \in S_n$. Moreover, $d_S(U,V) \geq d_H(\nu(\pi U), \nu(\pi V))$. In this paper, we show that if $E(U)$ and $E(V)$ are two matrices with at most one non-zero entry in each rows and each columns then $d_S(U,V)$ and $d_H(\nu(\pi U), \nu(\pi V))$ are equal to each other. Moreover, we define the subset $\mathcal{G}_{U,V} = \{\pi \in S_n \mid d_S(U,V) = d_H(\nu(\pi U), \nu(\pi V))\}$ of S_n. This set consists of permutations that achieve equality in the mentioned inequality. Generally, this set is not a group. We examine these sets for any unordered pair $\{U,V\}$ in $\mathcal{P}_2(4)$, $\mathcal{P}_2(5)$, $\mathcal{P}_2(6)$ and $\mathcal{P}_3(4)$. We prove that the groups 1, $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, S_3, S_4 and 1, $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, S_3, D_8, $S_3 \times \mathbb{Z}_2$, S_4, S_5 appears between these sets in $\mathcal{P}_2(4)$ and $\mathcal{P}_2(5)$, respectively. Furthermore, the groups 1, $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, S_3, D_8, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $S_3 \times \mathbb{Z}_2$, $D_8 \times \mathbb{Z}_2$, S_4, $S_3 \times S_3$, $S_4 \times \mathbb{Z}_2$, $(S_3 \times S_3){:}2$, S_5, S_6 and 1, $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, S_3, D_8, S_4 appears between the mentioned sets in $\mathcal{P}_2(6)$ and $\mathcal{P}_3(4)$, respectively.

2. On the subspace distance and pivot vectors

In this section, a sufficient condition for the equality between subspace distance of two subspaces and the hamming distance of their pivot vectors is given. In fact, we have the following theorem:

Theorem 2.1. *Let U and V be two subspaces in $\mathcal{P}_q(n)$. If $E(U)$ and $E(V)$ are two matrices with at most one non-zero entry in each rows and each columns then $d_S(U,V) = d_H(\nu(\pi U), \nu(\pi V))$ for all $\pi \in S_n$.*

Proof. Suppose that U and V are two subspaces of dimensions k and l, respectively. Set $E(U) = (\alpha_{ij})_{k \times n}$, $E(V) = (\beta_{hj})_{l \times n}$, $\nu(U) = (a_1, a_2, \ldots, a_n)$ and $\nu(V) = (b_1, b_2, \ldots, b_n)$. Since $E(U)$ and $E(V)$ have at most one non-zero entry in each rows and each columns,

$$a_j = \begin{cases} 1, & \text{if } \exists 1 \leq i \leq k \ \alpha_{ij} \neq 0; \\ 0, & \text{otherwise;} \end{cases}$$

and

$$b_j = \begin{cases} 1, & \text{if } \exists 1 \leq h \leq l \ \beta_{hj} \neq 0; \\ 0, & \text{otherwise;} \end{cases}$$

for any $1 \leq j \leq n$. Hence,

$$\begin{aligned} \operatorname{wt}(\nu(U) * \nu(V)) &= \operatorname{wt}(a_1 b_1, a_2 b_2, \ldots, a_n b_n) \\ &= \#\{1 \leq j \leq n \mid a_j = b_j = 1\} \\ &= \#\{1 \leq j \leq n \mid \exists 1 \leq i \leq k \ \exists 1 \leq h \leq l \ \alpha_{ij} \neq 0, \beta_{hj} \neq 0\} \\ &= \dim(\langle E(U) \rangle \cap \langle E(V) \rangle) \\ &= \dim(U \cap V). \end{aligned} \tag{1}$$

Now, let π be an arbitary permutation in S_n and set $\pi U = \pi E(U) = (\gamma_{ij})_{k \times n}$ and $\pi V = \pi E(V) = (\delta_{hj})_{l \times n}$. Clearly, $\gamma_{ij} = \alpha_{i\pi^{-1}(j)}$ and $\delta_{hj} = \beta_{h\pi^{-1}(j)}$ for all $1 \leq i \leq k$, $1 \leq h \leq l$ and $1 \leq j \leq n$. We can say that $\gamma_{i\pi(j)} \neq 0$ and $\delta_{h\pi(j)} \neq 0$ if and only if $\alpha_{ij} \neq 0$ and $\beta_{hj} \neq 0$. It follows from the assumption that $\pi E(U)$ and $\pi E(V)$ are also matrices that they have at most one non-zero entry in each rows and each columns. By (1), we have

$$\begin{aligned} \operatorname{wt}(\nu(\pi U) * \nu(\pi V)) &= \#\{1 \leq j \leq n \mid \exists 1 \leq i \leq k \ \exists 1 \leq h \leq l \ \gamma_{ij} \neq 0, \delta_{hj} \neq 0\} \\ &= \#\{1 \leq j \leq n \mid \exists 1 \leq i \leq k \ \exists 1 \leq h \leq l \ \alpha_{i\pi^{-1}(j)} \neq 0, \beta_{h\pi^{-1}(j)} \neq 0\} \\ &= \#\{1 \leq \pi(j) \leq n \mid \exists 1 \leq i \leq k \ \exists 1 \leq h \leq l \ \alpha_{ij} \neq 0, \beta_{hj} \neq 0\} \\ &= \operatorname{wt}(\nu(U) * \nu(V)) \\ &= \dim(U \cap V). \end{aligned} \tag{2}$$

So, (2) implies that

$$\begin{aligned} d_H(\nu(\pi U), \nu(\pi V)) &= \operatorname{wt}(\nu(\pi U)) + \operatorname{wt}(\nu(\pi V)) - 2 \operatorname{wt}(\nu(\pi U) * \nu(\pi V)) \\ &= \dim(U) + \dim(V) - 2 \dim(U \cap V) \\ &= d_S(U, V). \end{aligned}$$

Now, the proof is complete. □

The mentioned condition in Theorem 2.1 is sufficient but it is not necessary. To see this, consider the subspaces

$$U = \left\langle \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \right\rangle, \quad \text{and} \quad V = \left\langle \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \right\rangle.$$

So, we have

$$E(U) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad E(V) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This implies that $\nu(U) = (11100)$ and $\nu(V) = (10011)$. By a computer promgram in Maple software, $d_H(\nu(\pi U), \nu(\pi V)) = 4$ for all $\pi \in S_5$. On the other hand, by definition, $U \cap V = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle$ and $d_S(U, V) = 3 + 3 - 2 \cdot 1 = 4$. So, the equality is hold but $E(U)$ does not satisfy in the condition of Theorem 2.1. Now, we present an example where equality does not hold.

Example 2.2. Consider the subspaces

$$U = \left\langle \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \right\rangle, \quad \text{and} \quad V = \left\langle \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \right\rangle.$$

So,

$$E(U) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad \text{and} \quad E(V) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Clearly, $U \cap V = \left\langle \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \right\rangle$ and $\nu(U) = \nu(V) = (1110)$. Thus, $d_H(\nu(U), \nu(V)) = 0$ and $d_S(U, V) = dim(U) + dim(V) - 2dim(U \cap V) = 2$. This implies that $d_s(U, V) > d_H(\nu(U), \nu(V))$.

3. On the spaces $\mathcal{P}_q(n)$ for some n and q

Consider again the set $\mathcal{P}_q(n)$ consisting of all subspaces of $\mathbb{F}_q^n$. For any U and V in $\mathcal{P}_q(n)$, we define

$$\mathcal{G}_{U,V} = \{\pi \in S_n \mid d_S(U, V) = d_H(\nu(\pi U), \nu(\pi V))\}.$$

The subset $\mathcal{G}_{U,V} \subseteq S_n$ is not necessarily a group. For example, if

$$U = \left\langle \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\rangle, \quad \text{and} \quad V = \left\langle \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\rangle$$

then

$$\mathcal{G}_{U,V} = \{(), (12), (34), (23), (13), (24), (123), (132), (234), (243), (12)(34), (1342)\}$$

and clearly, this set is not a group. Also, consider the subspaces U and V of example 2.2. Since $1 \notin \mathcal{G}_{U,V}$, the set $\mathcal{G}_{U,V}$ is not a group. Now, a consequence of Theorem 2.1 is as follows:

Corollary 3.1. *If U and V satisfy in the assumptions of Theorem 1 then $\mathcal{G}_{U,V} \cong S_n$.*

In this section, we examine these sets by a computer program in Maple software, where $(n, q) = (4, 2), (5, 2), (6, 2), (4, 3)$. By [1, 3],

$$|\mathcal{P}_q(n)| = \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{k=1}^{n} \prod_{i=0}^{k-1} \frac{q^{n-i} - 1}{q^{k-i} - 1},$$

and hence, we have $\binom{|\mathcal{P}_2(4)|}{2} = 2145$, $\binom{|\mathcal{P}_2(5)|}{2} = 69378$, $\binom{|\mathcal{P}_2(6)|}{2} = 3986076$ and $\binom{|\mathcal{P}_3(4)|}{2} = 22155$ such sets in the cases $(n, q) = (4, 2)$, $(n, q) = (5, 2)$, $(n, q) = (6, 2)$ and $(n, q) = (4, 3)$, respectively. Among these sets, there are 602, 8030, 140912 and 2765 subgroups, respectively. The information we obtain about all such subgroups are listed in Tables 1-6. The subgroups and theirs numbers are written under the columns '$\mathcal{G}_{U,V}$' and '#', respectively. The headings '$\cong$' indicates the shape of the group $\mathcal{G}_{U,V}$. The last column gives an example for each case. Therefore, we obtain the following theorems:

Theorem 3.2. *Consider the words in the space $\mathcal{P}_2(4)$. There are 602 groups among the 2145 sets of $\mathcal{G}_{U,V}$ and these groups are isomorphic to 1, $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, S_3 and S_4.*

Theorem 3.3. *Consider the words in the space $\mathcal{P}_2(5)$. There are 8030 groups among the 69378 sets of $\mathcal{G}_{U,V}$ and these groups are isomorphic to 1, $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, S_3, D_8, $S_3 \times \mathbb{Z}_2$, S_4 and S_5.*

Theorem 3.4. *Consider the words in the space $\mathcal{P}_2(6)$. There are 140912 groups among the 3986076 sets of $\mathcal{G}_{U,V}$ and these groups are isomorphic to 1, $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, S_3, D_8, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $S_3 \times \mathbb{Z}_2$, $D_8 \times \mathbb{Z}_2$, S_4, $S_3 \times S_3$, $S_4 \times \mathbb{Z}_2$, $(S_3 \times S_3){:}2$, S_5 and S_6.*

Theorem 3.5. *Consider the words in the space $\mathcal{P}_3(4)$. There are 2765 groups among the 22155 sets of $\mathcal{G}_{U,V}$ and these groups are isomorphic to 1, $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, S_3, D_8 and S_4.*

Table 1. The groups $\mathcal{G}_{U,V}$, where $n = 4$ and $q = 2$.

$\mathcal{G}_{U,V}$	#	$\cong$	example
$\langle () \rangle$	2	1	$\begin{pmatrix} 0&1&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&1&0&0 \\ 0&0&1&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$
$\langle (12) \rangle$	3	$\mathbb{Z}_2$	$\begin{pmatrix} 0&0&1&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&1&0 \\ 0&1&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix}$
$\langle (23) \rangle$	1		$\begin{pmatrix} 1&1&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&1 \\ 0&0&1&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$
$\langle (34) \rangle$	3		$\begin{pmatrix} 0&1&1&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&1&0&0 \\ 0&0&1&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$
$\langle (12), (34) \rangle$	4	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{pmatrix} 1&0&1&1 \\ 0&1&1&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix}$
$\langle (12), (123) \rangle$	6	S_3	$\begin{pmatrix} 1&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&1 \\ 0&1&0&1 \\ 0&0&1&1 \\ 0&0&0&0 \end{pmatrix}$
$\langle (24), (234) \rangle$	6		$\begin{pmatrix} 1&1&1&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$
$\langle (12), (1234) \rangle$	577	S_4	$\begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$

4. conclusion

In this paper, we show that if $E(U)$ and $E(V)$ have at most one non-zero entry in each rows and each columns then inequality $d_S(U,V) \geq d_H(\nu(\pi U), \nu(\pi V))$ becomes equality. Moreover, we introduce the sets $\mathcal{G}_{U,V} = \{\pi \in S_n \mid d_S(U,V) = d_H(\nu(\pi U), \nu(\pi V))\}$ for any $U, V \in \mathcal{P}_q(n)$ and examine them for some n and q. Interested authors can work on finding a necessary condition for this inequality to become an equality. Also, interested authors can find the application of these results in coding theory.

5. Acknowledgments

The authors are grateful to the referees for their valuable comments. This work is partially supported by the University of Kashan under grant number 1073211.

TABLE 2. The groups $\mathcal{G}_{U,V}$, where $n = 5$ and $q = 2$.

$\mathcal{G}_{U,V}$	#	$\cong$	example
$\langle () \rangle$	4	1	$\begin{pmatrix} 0&1&1&0&0 \\ 0&0&0&1&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&1&0&0&0 \\ 0&0&1&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}$
$\langle (12) \rangle$	12	$\mathbb{Z}_2$	$\begin{pmatrix} 1&0&1&0&0 \\ 0&1&1&0&0 \\ 0&0&0&1&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}$
$\langle (23) \rangle$	2		$\begin{pmatrix} 1&1&1&0&1 \\ 0&0&0&1&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&1&0 \\ 0&0&1&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}$
$\langle (34) \rangle$	2		$\begin{pmatrix} 1&1&0&0&0 \\ 0&0&1&0&1 \\ 0&0&0&1&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&1&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}$
$\langle (45) \rangle$	12		$\begin{pmatrix} 1&1&0&1&1 \\ 0&0&1&1&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&1&0&0 \\ 0&0&0&1&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}$
$\langle (12), (34) \rangle$	3	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{pmatrix} 1&1&0&0&1 \\ 0&0&1&0&1 \\ 0&0&0&1&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&1&1&0 \\ 0&1&1&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}$
$\langle (12), (45) \rangle$	16		$\begin{pmatrix} 1&1&0&1&1 \\ 0&0&1&1&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&1&0&0 \\ 0&1&1&0&0 \\ 0&0&0&1&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}$
$\langle (23), (45) \rangle$	3		$\begin{pmatrix} 0&1&0&1&1 \\ 0&0&1&1&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&1&1&0&0 \\ 0&0&0&1&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}$
$\langle (12), (123) \rangle$	13	S_3	$\begin{pmatrix} 1&0&1&0&1 \\ 0&1&1&0&1 \\ 0&0&0&1&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&1&0 \\ 0&1&0&1&0 \\ 0&0&1&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \end{pmatrix}$
$\langle (23), (234) \rangle$	1		$\begin{pmatrix} 0&1&0&0&1 \\ 0&0&1&0&1 \\ 0&0&0&1&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&1&1&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}$
$\langle (34), (345) \rangle$	13		$\begin{pmatrix} 0&1&1&1&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \end{pmatrix}$
$\langle (12), (1425) \rangle$	1	D_8	$\begin{pmatrix} 1&1&0&1&1 \\ 0&0&1&1&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&1&0&1 \\ 0&1&1&0&1 \\ 0&0&0&1&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}$
$\langle (12), (123), (45) \rangle$	12	$S_3 \times \mathbb{Z}_2$	$\begin{pmatrix} 1&0&0&1&1 \\ 0&1&0&1&1 \\ 0&0&1&1&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&1&0&0 \\ 0&1&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \end{pmatrix}$
$\langle (34), (345), (12) \rangle$	12		$\begin{pmatrix} 1&0&1&1&1 \\ 0&1&1&1&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \end{pmatrix}$
$\langle (12), (1234) \rangle$	26	S_4	$\begin{pmatrix} 1&0&0&1&0 \\ 0&1&0&1&0 \\ 0&0&1&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&1 \\ 0&1&0&0&1 \\ 0&0&1&0&1 \\ 0&0&0&1&1 \\ 0&0&0&0&0 \end{pmatrix}$
$\langle (23), (2345) \rangle$	26		$\begin{pmatrix} 0&1&0&1&1 \\ 0&0&1&1&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&1&1&1&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}$
$\langle (12), (12345) \rangle$	7872	S_5	$\begin{pmatrix} 1&0&0&0&1 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}$

TABLE 3. The groups $\mathcal{G}_{U,V}$, where $n = 6$ and $q = 2$.

$\mathcal{G}_{U,V}$	#	$\cong$	example
$\langle () \rangle$	16	1	$\begin{pmatrix} 1&1&0&0&0&0 \\ 0&0&1&1&0&0 \\ 0&0&0&0&1&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&1&0&0&0 \\ 0&0&0&1&1&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}$
$\langle (12) \rangle$	40		$\begin{pmatrix} 1&0&1&0&0&0 \\ 0&1&1&0&0&0 \\ 0&0&0&1&1&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&1&0&0&0&0 \\ 0&0&1&1&0&0 \\ 0&0&0&0&1&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}$
$\langle (23) \rangle$	8		$\begin{pmatrix} 1&1&1&0&0&0 \\ 0&0&0&1&1&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&1&0&0 \\ 0&0&1&1&0&0 \\ 0&0&0&0&1&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}$
$\langle (34) \rangle$	4	$\mathbb{Z}_2$	$\begin{pmatrix} 1&1&0&0&0&0 \\ 0&0&1&0&1&0 \\ 0&0&0&1&1&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&1&1&0&0 \\ 0&0&0&0&1&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}$
$\langle (45) \rangle$	8		$\begin{pmatrix} 0&1&1&0&0&0 \\ 0&0&0&1&0&1 \\ 0&0&0&0&1&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&1&0&0&0&0 \\ 0&0&1&1&1&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}$
$\langle (56) \rangle$	40		$\begin{pmatrix} 1&1&0&0&0&0 \\ 0&0&1&1&0&0 \\ 0&0&0&0&1&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&1&0&0&0 \\ 0&0&0&1&1&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}$
$\langle (12),(34) \rangle$	12		$\begin{pmatrix} 0&0&1&0&1&0 \\ 0&0&0&1&1&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&1&1&0&1 \\ 0&1&1&1&0&1 \\ 0&0&0&0&1&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}$
$\langle (12),(45) \rangle$	20		$\begin{pmatrix} 1&1&0&0&0&0 \\ 0&0&1&1&1&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&1&0&0&0 \\ 0&1&1&0&0&0 \\ 0&0&0&1&0&1 \\ 0&0&0&0&1&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}$
$\langle (12),(56) \rangle$	132	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{pmatrix} 1&0&1&0&0&1 \\ 0&1&1&0&0&0 \\ 0&0&0&1&1&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&1&0&0&0&1 \\ 0&0&1&1&0&0 \\ 0&0&0&0&1&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}$
$\langle (23),(45) \rangle$	2		$\begin{pmatrix} 1&1&1&0&0&0 \\ 0&0&0&1&0&1 \\ 0&0&0&0&1&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&1&1&0 \\ 0&0&1&1&1&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}$
$\langle (23),(56) \rangle$	20		$\begin{pmatrix} 0&1&0&1&0&0 \\ 0&0&1&1&0&0 \\ 0&0&0&0&1&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&1&1&0&0&0 \\ 0&0&0&1&1&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}$
$\langle (34),(56) \rangle$	12		$\begin{pmatrix} 1&1&0&0&0&0 \\ 0&0&1&0&1&1 \\ 0&0&0&1&1&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&1&1&0&0 \\ 0&0&0&0&1&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}$
$\langle (12),(123) \rangle$	104		$\begin{pmatrix} 1&0&1&0&0&0 \\ 0&1&1&0&0&0 \\ 0&0&0&1&1&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&1&0&0 \\ 0&1&0&1&0&0 \\ 0&0&1&1&0&1 \\ 0&0&0&0&1&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}$
$\langle (23),(234) \rangle$	2	S_3	$\begin{pmatrix} 1&1&1&1&0&0 \\ 0&0&0&0&1&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&1&0 \\ 0&0&1&0&1&0 \\ 0&0&0&1&1&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}$
$\langle (34),(345) \rangle$	2		$\begin{pmatrix} 0&1&1&1&1&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&1&0&0&0&0 \\ 0&0&1&0&0&1 \\ 0&0&0&1&0&1 \\ 0&0&0&0&1&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}$
$\langle (45),(456) \rangle$	104		$\begin{pmatrix} 0&1&1&0&0&0 \\ 0&0&0&1&0&1 \\ 0&0&0&0&1&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&1&0&0&0&0 \\ 0&0&1&1&1&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}$

Table 4. The groups $\mathcal{G}_{U,V}$, where $n=6$ and $q=2$ (Continued).

$\mathcal{G}_{U,V}$	#	$\cong$	example
$\langle(56),(1625)\rangle$	3	D_8	$\begin{pmatrix}1&0&1&0&1&0\\0&1&1&0&0&1\\0&0&0&1&1&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},\begin{pmatrix}1&1&0&0&0&0\\0&0&1&1&0&0\\0&0&0&0&1&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$
$\langle(12),(34),(56)\rangle$	16	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$	$\begin{pmatrix}0&0&1&0&1&1\\0&0&0&1&1&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},\begin{pmatrix}1&0&1&1&0&1\\0&1&1&1&0&1\\0&0&0&0&1&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$
$\langle(12),(123),(45)\rangle$	13	$S_3\times\mathbb{Z}_2$	$\begin{pmatrix}0&1&1&0&0&0\\0&0&0&1&0&1\\0&0&0&0&1&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},\begin{pmatrix}1&0&0&1&1&0\\0&1&0&1&1&0\\0&0&1&1&1&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$
$\langle(12),(123),(56)\rangle$	117		$\begin{pmatrix}0&0&0&1&1&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},\begin{pmatrix}1&0&0&1&0&0\\0&1&0&1&0&0\\0&0&1&1&0&1\\0&0&0&0&1&1\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$
$\langle(24),(243),(56)\rangle$	3		$\begin{pmatrix}0&1&0&0&1&1\\0&0&1&0&1&1\\0&0&0&1&1&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},\begin{pmatrix}1&1&1&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$
$\langle(34),(345),(12)\rangle$	3		$\begin{pmatrix}0&0&1&0&0&1\\0&0&0&1&0&1\\0&0&0&0&1&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},\begin{pmatrix}1&0&1&1&1&0\\0&1&1&1&1&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$
$\langle(45),(456),(12)\rangle$	117		$\begin{pmatrix}1&0&1&0&0&1\\0&1&1&0&0&0\\0&0&0&1&0&1\\0&0&0&0&1&1\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},\begin{pmatrix}1&1&0&0&0&1\\0&0&1&1&1&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$
$\langle(45),(456),(23)\rangle$	13		$\begin{pmatrix}0&1&0&1&1&1\\0&0&1&1&1&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},\begin{pmatrix}1&1&1&0&0&0\\0&0&0&1&1&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$
$\langle(12),(1526),(34)\rangle$	1	$D_8\times\mathbb{Z}_2$	$\begin{pmatrix}1&1&0&0&1&1\\0&0&1&0&1&1\\0&0&0&1&1&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},\begin{pmatrix}1&0&1&1&0&1\\0&1&1&1&0&1\\0&0&0&0&1&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$
$\langle(12),(1234)\rangle$	83	S_4	$\begin{pmatrix}0&0&0&0&1&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},\begin{pmatrix}1&0&0&0&1&0\\0&1&0&0&1&0\\0&0&1&0&1&0\\0&0&0&1&1&0\\0&0&0&0&0&1\\0&0&0&0&0&0\end{pmatrix}$
$\langle(23),(2345)\rangle$	1		$\begin{pmatrix}1&1&1&1&1&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&1&0&0&0&1\\0&0&1&0&0&1\\0&0&0&1&0&1\\0&0&0&0&1&1\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$
$\langle(34),(3456)\rangle$	83		$\begin{pmatrix}0&1&1&1&1&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},\begin{pmatrix}1&1&0&0&0&0\\0&0&1&0&1&0\\0&0&0&1&1&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$
$\langle(12),(123),(45),(456)\rangle$	35	$S_3\times S_3$	$\begin{pmatrix}0&0&0&1&1&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},\begin{pmatrix}1&0&0&1&1&1\\0&1&0&1&1&1\\0&0&1&1&1&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$
$\langle(14),(1423),(56)\rangle$	52	$S_4\times\mathbb{Z}_2$	$\begin{pmatrix}0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},\begin{pmatrix}1&0&0&0&1&1\\0&1&0&0&1&1\\0&0&1&0&1&1\\0&0&0&1&1&1\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$
$\langle(34),(3456),(12)\rangle$	52		$\begin{pmatrix}1&0&1&1&1&1\\0&1&1&1&1&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},\begin{pmatrix}0&0&1&0&0&1\\0&0&0&1&0&1\\0&0&0&0&1&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$

TABLE 5. The groups $\mathcal{G}_{U,V}$, where $n = 6$ and $q = 2$ (Continued).

$\mathcal{G}_{U,V}$	#	$\cong$	example
$\langle(142536), (13)\rangle$	1	$(S_3 \times S_3){:}2$	$\begin{pmatrix} 1&1&1&0&0&1\\ 0&0&0&1&0&1\\ 0&0&0&0&1&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&1&1&1\\ 0&1&0&1&1&1\\ 0&0&1&1&1&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}$
$\langle(12), (12345)\rangle$	158	S_5	$\begin{pmatrix} 1&1&0&0&0&0\\ 0&0&1&0&1&0\\ 0&0&0&1&1&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&1\\ 0&1&0&0&0&1\\ 0&0&1&0&0&1\\ 0&0&0&1&0&1\\ 0&0&0&0&1&1\\ 0&0&0&0&0&0 \end{pmatrix}$
$\langle(23), (23456)\rangle$	158		$\begin{pmatrix} 1&1&1&1&1&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&1&1&1&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}$
$\langle(12), (123456)\rangle$	139477	S_6	$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&1\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&1&1&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}$

TABLE 6. The groups $\mathcal{G}_{U,V}$, where $n = 4$ and $q = 3$.

$\mathcal{G}_{U,V}$	#	$\cong$	example
$\langle()\rangle$	24	1	$\begin{pmatrix} 1&2&0&2\\ 0&0&1&1\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&1&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}$
$\langle(12)\rangle$	56	$\mathbb{Z}_2$	$\begin{pmatrix} 1&0&1&0\\ 0&1&1&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&2&0&2\\ 0&0&1&1\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}$
$\langle(23)\rangle$	16		$\begin{pmatrix} 1&1&1&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&1\\ 0&0&1&2\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}$
$\langle(34)\rangle$	56		$\begin{pmatrix} 0&1&1&2\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&2&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix}$
$\langle(12), (34)\rangle$	136	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{pmatrix} 1&0&2&2\\ 0&1&2&1\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix}$
$\langle(12), (123)\rangle$	120	S_3	$\begin{pmatrix} 1&0&2&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&2\\ 0&1&0&1\\ 0&0&1&1\\ 0&0&0&0 \end{pmatrix}$
$\langle(24), (234)\rangle$	120		$\begin{pmatrix} 1&2&1&2\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix}$
$\langle(34), (1324)\rangle$	8	D_8	$\begin{pmatrix} 1&0&1&1\\ 0&1&2&2\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&2&0&1\\ 0&0&1&2\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}$
$\langle(12), (1234)\rangle$	2229	S_4	$\begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&2\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}$

REFERENCES

[1] P. J. Cameron, *Combinatorics: Topics, Techniques, Algorithms*, Cambridge University Press, 1994.

[2] W. C. Huffman, J. -L. Kim and P. Solé, *Concise Encyclopedia of Coding Theory*, Chapman and Hall/CRC, 2021.

[3] S. Kurz, *Construction and bounds for subspace codes*, (2023) arXiv:2112.11766v2.

[4] S. Ling and C. Xing, *Coding Theory, A First Course*, Cambridge University Press, 2004.

[5] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*, North-Holland, 1977.

[6] N. Raviv, *Subspace Codes and Distributed Storage Codes*, PhD thesis, Computer Science Department, Technion, 2017.

[7] H. Zhang and C. Tang, *Further constructions of large cyclic subspace codes via Sidon spaces*, Linear Algebra Appl. **661** (2023) 106-115.

[8] F. Zullo, *Multi-orbit cyclic subspace codes and linear sets*, Finite Fields Their Appl. **87** (2023) 102153.

Seyedeh Hawra Sadrolhoffaz

Department of Pure Mathematics, Faculty of Mathematical Sciences,

University of Kashan

Kashan, Iran.

sadr@grad.kashanu.ac.ir

Reza Kahkeshani

Department of Pure Mathematics, Faculty of Mathematical Sciences,

University of Kashan

Kashan, Iran.

kahkeshanireza@kashanu.ac.ir