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Research Paper

MEET-NONESSENTIAL GRAPH OF AN ARTINIAN LATTICE

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Abstract. Let L be a lattice with 1. The meet-nonessential graph $\mathbb{MG}(L)$ of L is a graph whose vertices are all nonessential filters of L and two distinct filters F and G are adjacent if and only if $F \wedge G$ is a nonessential filter of L. The basic properties and possible structures of the graph $\mathbb{MG}(L)$ are investigated. The clique number, domination number and independence number of $\mathbb{MG}(L)$ and their relations to algebraic properties of L are explored.

1. Introduction

The study of algebraic structures, using the properties of graph theory, tends to an exciting research topic in last decade. Associating a graph with an algebraic structure allows us to obtain characterizations and representations of special classes of algebraic structures in terms of graphs and vice versa (see for example [1, 2, 3, 10, 9, 12]).

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Let L be a distributive lattice with 1. The purpose of this paper is to investigate a graph associated to a lattice L called the meet-nonessential graph of L. This will result in characterization of lattices in terms of some specific properties of those graphs. The meet-nonessential graph of L is a simple graph $\mathbb{MG}(L)$ whose vertices are all nonessential filters and two distinct vertices are adjacent if and only if the meet of the corresponding filters is not an essential filter of L. The small intersection graph $\Gamma_S(R)$ of a commutative ring R is a graph whose vertices are all non-small proper ideals of R and two distinct ideals I and J are adjacent if and only if $I \cap J$ is not small in R was introduced and investigated in [9]. The concept of the nonessential sum graph of a commutative Artinian ring R was introduced and studied in [3]. The sum-essential graph $\Gamma_M(R)$ of a left R-module M is a graph whose vertices are all nontrivial submodules of M and two distinct submodules are adjacent if and only if their sum is an essential submodule of M was introduced and investigated in [12].

Here is a brief outline of the article. Among many results in this paper, the first, Preliminaries section contains elementary observations needed later on. In Section 3, we show in Theorem 3.9 that $\mathbb{MG}(L)$ is connected if and only if $|\mathcal{S}(L)| \neq 2$. Also, if $\mathbb{MG}(L)$ is a connected graph, then $\text{diam}(\mathbb{MG}(L)) \leq 2$ and $\text{gr}(\mathbb{MG}(L)) = 3$ provided $\mathbb{MG}(L)$ contains a cycle (Thorem 3.10). For a lattice L, it is shown that $\mathbb{MG}(L)$ cannot be a complete r-partite graph (Theorem 3.12) and $\mathbb{MG}(L)$ has no cut vertex (Theorem 3.11). Moreover, $\mathbb{MG}(L)$ cannot be a complete graph (Theorem 3.13). Also it is proved that if $\mathbb{MG}(L)$ contains a vertex with degree 1, then $|\mathcal{S}(L)| = 2$ (Theorem 3.14). We also prove in Theorem 3.18 that every vertex of $\mathbb{MG}(L)$ is of finite degree if and only if the graph has only finitely many vertices. In Section 3, the clique number, domination number and independence number of $\mathbb{MG}(L)$ and their relations to algebraic properties of L are explored.

2. Preliminaries

Let G be a simple graph with vertex set $\mathcal{V}(G)$ and edge set $\mathcal{E}(G)$. For every vertex $v \in \mathcal{V}(G)$, the degree of v, denoted by $\deg_G(v)$, is defined the cardinality of the set of all vertices which are adjacent to v. A graph G is said to be connected if there exists a path between any two distinct vertices, G is a complete graph if every pair of distinct vertices of G are adjacent and K_n will stand for a complete graph with n vertices. The graph G is k-regular, if $\deg_G(v) = k < \infty$ for every $v \in \mathcal{V}(G)$. Let u and v be elements of $\mathcal{V}(G)$. We say that u is a universal vertex of G if u is adjacent to all other vertices of G and write $u \backsim v$ if u and v are adjacent. The distance $d(u, v)$ is the length of the shortest path from u to v if such path exists, otherwise, $d(a, b) = \infty$. The diameter of G is $\text{diam}(G) = \sup\{d(a, b) : a, b \in \mathcal{V}(G)\}$. The girth of a graph G, denoted by $\text{gr}(G)$, is the length of a shortest cycle in G. If G has no cycles, then $\text{gr}(G) = \infty$. A subset $S \subseteq \mathcal{V}(G)$ is an independent set if the subgraph induced by S is totally disconnected. The independence number $\alpha(G)$ is the maximum size of an independent set in G. A clique of a graph is its maximal complete subgraph and the number of vertices in the largest clique of graph G, denoted by $\omega(G)$, is called the clique number of G. For a positive integer k, a k-partite graph is a graph whose vertices can be partitioned into k nonempty independent sets. The complete bipartite graph with part sizes m and n is denoted by $K_{m,n}$. We will sometimes call $K_{1,n}$ a star graph. The (open) neighborhood $N(v)$ of a vertex v of $\mathcal{V}(G)$ is the set of vertices which are adjacent to v. For each $S \subseteq \mathcal{V}(G)$, $N(S) = \cup_{v\in S} N(v)$ and $N[S] = N(S) \cup S$ set of vertices S in G is a dominating set, if $N[S] = \mathcal{V}(G)$. The domination number, $\gamma(G)$, of G is the minimum cardinality of a dominating set of G. Note that a graph whose vertices set is empty is a null graph and a graph whose edge set is empty is an empty graph. A vertex x of a connected graph G is a cut vertex of G if there are vertices y and z of G such that x is in every path from y to z (and $x \neq y$, $x \neq z$). Equivalently, for a connected graph G, x is a cut vertex of G if $G \setminus \{x\}$ is not connected [13].

By a lattice we mean a poset $(L, \leq)$ in which every couple elements x, y has a g.l.b. (called the meet of x and y, and written $x \wedge y$) and a l.u.b. (called the join of x and y, and written $x \vee y$). A lattice L is complete when each of its subsets X has a l.u.b. and a g.l.b. in L. Setting $X = L$, we see that any nonvoid complete lattice contains a least element 0 and greatest element 1 (in this case, we say that L is a lattice with 0 and 1). A lattice L is called a distributive lattice if $(a\vee b)\wedge c = (a\wedge c)\vee(b\wedge c)$ for all a, b, c in L (equivalently, L is distributive if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all a, b, c in L). A non-empty subset F of a lattice L is called a filter, if for $a \in F$, $b \in L$, $a \leq b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if L is a lattice with 1, then $1 \in F$ and $\{1\}$ is a filter of L). If A is a subset of L, then the filter generated by A, denoted by $T(A)$, is the intersection of all filters that is containing A. A subfilter G of a filter F of L is called essential in F (written $G \trianglelefteq F$) if $G \cap H \neq \{1\}$ for any subfilter $H \neq \{1\}$ of F. For terminology and notation not defined here, the reader is referred to [4].

Lemma 2.1. *Let L be a lattice* [4, 7, 8, 10].

(1) A non-empty subset F of L is a filter of L if and only if $x \vee z \in F$ and $x \wedge y \in F$ for all $x, y \in F$, $z \in L$. Moreover, since $x = x \vee (x \wedge y)$, $y = y \vee (x \wedge y)$ and F is a filter, $x \wedge y \in F$ gives $x, y \in F$ for all $x, y \in L$.

(2) If F_1, F_2 are filters of L and $a \in L$, then $F_1 \vee F_2 = \{a_1 \vee a_2 : a_1 \in F_1, a_2 \in F_2\}$ and $a \vee F_1 = \{a \vee a_1 : a_1 \in F_1\}$ are filters of L and $F_1 \cap F_2 = F_1 \vee F_2 \subseteq F_1, F_2$.

(3) If L is distributive, F, G are filters of L, and $x \in L$, then $(G :_L F) = \{x \in L : x \vee F \subseteq G\}$, $(F :_L T(\{x\}) = (F :_L x) = \{a \in L : a \vee x \in F\}$ and $(\{1\} :_L x) = (1 :_L x) = \{a \in L : a \vee x = 1\}$ are filters of L.

(4) If L is distributive and F_1, F_2 are filters of L, then $F_1 \wedge F_2 = \{a \wedge b : a \in F_1, b \in F_2\}$ is a filter of L, $F_1, F_2 \subseteq F_1 \wedge F_2$ (for if $x \in F_1$, then $x = x \wedge 1 \in F_1 \wedge F_2$) and if $F_1 \subseteq F_2$, then $F_1 \wedge F_2 = F_2$.

Lemma 2.2. *Let L be a lattice* [6].

(1) Let A be an arbitrary non-empty subset of L. Then $T(A) = \{x \in L : a_1 \wedge a_2 \wedge \cdots \wedge a_n \leq x$ for some $a_i \in A$ $(1 \leq i \leq n)\}$. Moreover, if F is a filter and A is a subset of L with $A \subseteq F$, then $T(A) \subseteq F$ and $T(F) = F$.

(2) If F and G are filters of L, then $T(G \cup F) = F \wedge G$;

(3) (modular law) If F, G and H are filters of L with $F \subseteq G$, then $G \cap (F \wedge H) = F \wedge (G \cap H)$.

Let U be a subfilter of a filter F of L. If subfilter V of F is maximal with respect to $U \cap V = \{1\}$, then we say that V is a complement of U. Using the maximal principle we readily see that if U is a subfilter of F, then the set of those subfilters of F whose intersection with U is $\{1\}$ contains a maximal element V. Thus every subfilter U of F has a complement. As a direct application of the Lemma 2.2 and [6] Lemma 2.15, we obtain the following lemma:

Lemma 2.3. *Let A, B, C and D be filters of L.*

(1) If $A \trianglelefteq B$ and $C \trianglelefteq D$, then $A \wedge C \trianglelefteq B \wedge D$;

(2) If $B \cap D = \{1\}$, then $A \trianglelefteq B$ and $C \trianglelefteq D$ if and only if $A \wedge C \trianglelefteq B \wedge D$.

(3) If B is a complement of A in L, then $A \wedge B \trianglelefteq L$.

A lattice L is called semisimple, if for each proper filter F of L, there exists a filter G of L such that $L = F \wedge G$ and $F \cap G = \{1\}$). In this case, we say that F is a direct meet of L, and we write $L = F \odot G$. A filter F of L is called a semisimple filter, if every subfilter of F is a direct meet. A simple filter is a filter that has no filters besides the $\{1\}$ and itself.

Let $\Lambda = \{F_i : i \in I\}$ be a set of filters of L. Then it is easy to see that $\bigwedge_{i \in I} F_i = \{\bigwedge_{i \in I'} f_i : f_i \in F_i, I' \subset I, I'$ is finite$\}$ is a filter of L (if $\Lambda = \emptyset$, then we set $\bigwedge_{i \in I} F_i = \{1\}$). $L = \bigodot_{i \in I} F_i$ is said to be a direct decomposition of L into the meet of the filters $\{F_i : i \in I\}$ if (1) $L = \bigwedge_{i \in I} F_i$ and (2) $\{F_i : i \in I\}$ is independent i.e for each $j \in I$, $F_j \cap \bigwedge_{j \neq i \in I} F_i = \{1\}$. For each filter F of L, $\mathrm{Soc}(F) = \bigwedge_{i \in \Lambda} F_i$, where $\{F_i\}_{i \in \Lambda}$ is the set of all simple filters of L contained in F.

3. Basic properties of $\mathbb{MG}(L)$

Throughout this paper, we shall assume unless otherwise stated, that L is a distributive lattice with 1. In this section, we collect basic properties concerning the graph $\mathbb{MG}(L)$. A filter $F \neq \{1\}$ of L is called L-second if for each $a \in L$, either $a \vee F = \{1\}$ or $a \vee F = F$. By [8] Proposition 2.1, F is L-second if and only if the only subfilters of F are $\{1\}$ and F itself (i.e. F is simple) and in this case, $|F| = 2$. The set of all simple filters of L is denoted by $\mathcal{S}(L)$. The next lemma plays a key role in the sequel.

Lemma 3.1. *Let L be an Artinian lattice. Then:*

(1) If F is a filter of L with $F \neq \{1\}$, then F contains only a finite number of simple filters. In particular, $\mathcal{S}(L)$ is a finite set;

(2) $\mathrm{Soc}(L) \trianglelefteq L$ *and* $\mathrm{Soc}(L)$ *contains only finitely many subfilters.*

Proof. Clearly, $\mathcal{S}(F) \neq \emptyset$ since L is Artinian. Indeed (1) is a direct consequence of [8], Theorem 2.2 (i) and (2) is a consequence of (1). □

The proof of the following Lemma (Lemma 3.2 (1)) can be found in [11] (with some different proof and notions), but we give the details for convenience.

Lemma 3.2. *(1) If $\mathbb{F}(L)$ is the set of all filters of L, then*

$$\mathrm{Soc}(L) = \cap\{F \in \mathbb{F}(L) : F \textit{ is essential in } L\};$$

(2) If $G \in \mathbb{F}(L)$, then $G \trianglelefteq L$ if and only if $\mathrm{Soc}(L) \subseteq G$*;*

(3) If H is a nontrivial subfilter of $\mathrm{Soc}(L)$*, then H is not essential in L.*

Proof. (1) Let $\mathrm{Soc}(L) = \bigwedge_{i\in I} S_i$, where $\{S_i\}_{i\in I}$ is the set of all simple filters of L. Set $\mathcal{K} = \cap\{F \in \mathbb{F}(L) : F$ is essential in $L\}$. Let S be a simple filter of L. If $G \trianglelefteq L$, then $S \cap G \neq \{1\}$, so $S \subseteq G$. Thus $\mathrm{Soc}(L)$ is contained in every essential filter of L; so $\mathrm{Soc}(L) \subseteq \mathcal{K}$. We claim that $\mathcal{K}$ is semisimple. Let G be a filter of L such that $G \subseteq \mathcal{K}$. If $G \trianglelefteq L$, then $\mathcal{K} \subseteq G$; hence $G = \mathcal{K}$. So we may assume that G is not essential in L. Let G' be a complement of G in L; so $G \wedge G' \trianglelefteq L$ by Lemma 2.3. It follows that $G \subseteq \mathcal{K} \subseteq G \wedge G'$, and by modularity $\mathcal{K} = \mathcal{K} \cap (G \wedge G') = G \wedge (\mathcal{K} \cap G')$ which implies that $\mathcal{K}$ is semisimple; thus $\mathcal{K} \subseteq \mathrm{Soc}(L)$ and so we have equality.

(2) One side is clear by (1). To prove the other side, assume to the contrary, that G is not essential in L. Then there exists a filter H of L such that $G \cap H = \{1\}$. By Lemma 2.1, there is a simple filter S of L such that $S \subseteq H$. So we have $S \cap G \subseteq H \cap G = \{1\}$ which implies that $S \nsubseteq G$, a contradiction. Thus $G \trianglelefteq L$.

(3) This is straightforward. □

Lemma 3.3. *Assume that $\mathcal{S}(L) = \{S_i\}_{i\in\Lambda}$ and let I be a nonempty proper finite subset of Λ, where $|\Lambda| > 1$. Then $\bigwedge_{i\in I} S_i$ is a nonessential filter of L.*

Proof. Suppose to the contrary, that $\bigwedge_{i\in I} S_i \trianglelefteq L$. Since each $S_j \neq \{1\}$, so $(\bigwedge_{i\in I} S_i) \cap S_j \neq \{1\}$ for $j \notin I$ which implies that $S_j \subseteq \bigwedge_{i\in I} S_i$. If $1 \neq x \in S_j$, then $x = \wedge_{i\in I} s_i$ for some $s_i \in S_i$ $(i \in I)$. Then there is an element $t \in I$ such that $s_t \neq 1$, as $x \neq 1$. Now S_j is a filter gives $s_t \in S_j \cap S_t = \{1\}$ by Lemma 2.1, a contradiction. This completes the proof. □

Lemma 3.4. *If S is a simple filter of L and F, G are two filters such that $S \subseteq F \wedge G$, then either $S \subseteq F$ or $S \subseteq G$.*

Proof. If $1 \neq s \in S$, then $s = a \wedge b$ for some $a \in F$ and $b \in G$. Now S is a filter gives $a, b \in S$ by Lemma 2.1 (so either $a \neq 1$ or $b \neq 1$). Without loss of generality, we can assume that $a \neq 1$. It follows that $F \cap S \neq \{1\}$ which gives $S \subseteq F$. □

Henceforth we will assume that all considered lattices L are Artinian. We recall that $\mathcal{S}(L) \neq \emptyset$ and L contains only a finite number of simple filters by Lemma 3.1.

Proposition 3.5. $\mathbb{MG}(L)$ *is a null graph if and only if L has exactly one simple filter.*

Proof. One side is clear. To prove the other side, suppose that L has exactly one simple filter S (so $\mathrm{Soc}(L) = S \trianglelefteq L$ by Lemma 2.1). Let G be a nontrivial filter of L. If H is a non-trivial filter of L, then Lemma 3.1 shows that $S \subseteq H \cap G$; so G is essential in L. Thus every nontivial filter of L is essential in L; hence $\mathbb{MG}(L)$ is a null graph. □

Example 3.6. (1) Let $L = \{0, a, b, c, 1\}$ be a lattice with $0 \leq a \leq c \leq 1$, $0 \leq b \leq c \leq 1$, $a \vee b = c$ and $a \wedge b = 0$. An inspection will show that the nontrivial filters of L are $S_1 = \{1, a, c\}$, $S_2 = \{1, b, c\}$ and $S_3 = \{1, c\}$ with S_3 is a simple filter of L and S_1, S_2 are essential in L. Thus $\mathbb{MG}(L)$ is a null graph by Proposition 3.5.

(2) Assume that R is a discrete valuation ring with unique maximal ideal $P = Rp$ and let $E = E(R/P)$, the R-injective hull of R/P. For each positive integer n, set $A_n = (0 :_E P^n)$. Then by [5] Lemma 2.6, every non-zero proper submodule of E is equal to A_m for some m with a strictly increasing sequence of submodules $A_1 \subset A_2 \subset \cdots \subset A_n \subset A_{n+1} \subset \cdots$. The collection of submodules of E form a complete lattice which is a chain under set inclusion which we shall denote by $L(E)$ with respect to the following definitions: $A_n \vee A_m = A_n + A_m$ and $A_n \wedge A_m = A_n \cap A_m$ for all submodules A_n and A_m of E. Then by [8] Example 2.3 (b), we have

(i) Every proper filter of $L(E)$ is of the form $[A_n, E] = \{X \in L(E) : A_n \subseteq X \subseteq E\}$ for some n. For each positive integer n, set $F_n = [A_n, E]$. Then $F_1 \supsetneqq F_2 \supsetneqq \cdots F_n \supsetneqq F_{n+1} \cdots$ gives L is not Artinian.

(ii) $\mathcal{S}(L(E)) = \emptyset$ and $F_n \trianglelefteq L(E)$ for each $n \in \mathbb{N}$; so $\mathcal{V}$ $(\mathbb{MG}(L) = \emptyset$. Thus $\mathbb{MG}(L)$ is a null graph by Proposition 3.5.

Theorem 3.7. $\mathbb{MG}(L)$ *is an empty graph if and only if L has exactly two simple filters which are the only nonessential filters of L.*

Proof. Let $\mathbb{MG}(L)$ be an empty graph. If $|\mathcal{S}(L)| = 1$, then $\mathbb{MG}(L)$ is a null graph by Proposition 3.5 which is impossible. Suppose that $|\mathcal{S}(L)| \geq 3$ and let S_1, S_2 and S_3 be simple filters of L. Then S_1 and S_2 are adjacent in $\mathbb{MG}(L)$ by Lemma 3.3 which is a contradiction. So we may assume that $\mathcal{S}(L) = \{S_1, S_2\}$ with $S_1 \neq S_2$ (so S_1 and S_2 are nonessential filters of L). If $G \neq \{1\}$ is a nonessential filter of L with $G \neq S_1, S_2$, then either $S_1 \subseteq G$ or $S_2 \subseteq G$ by Lemma 3.1. Without loss of generality, we can assume that $S_1 \subseteq G$. This gives $G = G \wedge S_1$ is not essential in L; hence S_1 and G adjacent in $\mathbb{MG}(L)$ which is impossible. Thus S_1 and S_2 are the only non-essental filters of L. To prove the other side, we consider L has exactely two simple filters which are the only nonessential filters of L. Thus $S_1 \wedge S_2 = \mathrm{Soc}(L) \trianglelefteq L$ by Lemma 3.1. Hence $\mathbb{MG}(L)$ is an empty graph. □

Example 3.8. Let $L = \{0, a, b, c, d, 1\}$ be a lattice with $0 \leq d \leq c \leq a \leq 1$, $0 \leq d \leq c \leq b \leq 1$, $a \vee b = 1$ and $a \wedge b = c$. An inspection will show that the nontrivial filters of L are $S_1 = \{1, a\}$, $S_2 = \{1, b\}$, $S_3 = \{1, a, b, c\}$ and $S_4 = \{1, a, b, c, d\}$ with S_1, S_2 are the only nonessential simple filter of L and S_3, S_4 are essential in L. Thus $\mathbb{MG}(L)$ is an empty graph by Theorem 3.7.

Theorem 3.9. *For the lattice L, the following conditions are equivalent:*

(1) $\mathbb{MG}(L)$ *is not connected;*

(2) $|\mathcal{S}(L)| = 2$*;*

(3) There exist two disjoint complete subgraphs H_1, H_2 *of* $\mathbb{MG}(L)$ *such that* $\mathbb{MG}(L) = H_1 \cup H_2$.

Proof. (1) ⇒ (2) Assume that H_1 and H_2 are two components of $\mathbb{MG}(L)$ and let F, G be filters of L such that $F \in \mathcal{V}(H_1)$ and $G \in \mathcal{V}(H_2)$ (so F and G are not essential in L). There are simple filters S_1 and S_2 such that $S_1 \subseteq F$ and $S_2 \subseteq G$ by Lemma 3.1. If $S_1 = S_2$, then $F \backsim S_1 \backsim G$ is a path in $\mathbb{MG}(L)$, a contradiction. So we may assume that $S_1 \neq S_2$. If $|\mathcal{S}(L)| \geq 3$, then $S_1 \wedge S_2$ is not essential in L by Lemma 3.3 which gives $F \backsim S_1 \backsim S_2 \backsim G$ is a path in $\mathbb{MG}(L)$, a contradiction. Thus $|\mathcal{S}(L)| = 2$.

(2) ⇒ (3) Let $|\mathcal{S}(L)| = 2$. Then $\mathrm{Soc}(L) = S_1 \wedge S_2$, where S_1, S_2 are simple filters of L. Let $H_1 = \{F \in \mathbb{F}(L) : S_1 \subseteq F \text{ and } F \text{ is not essential in } L\}$ and

$$H_2 = \{F \in \mathbb{F}(L) : S_2 \subseteq F \text{ and } F \text{ is not essential in } L\}.$$

Let $F, G \in \mathcal{V}(H_1)$. If F and G are not adjacent in $\mathbb{MG}(L)$, then $G \wedge F \trianglelefteq L$ which gives $S_2 \subseteq \mathrm{Soc}(L) \subseteq G \wedge F$ by Lemma 3.2. So either $S_2 \subseteq F$ or $S_2 \subseteq G$ by Lemma 3.4, a contradiction because in that case either F is essential or G is essential. Thus H_1 is a complete subgraph of $\mathbb{MG}(L)$. Similarly, H_2 is a complete subgraph of $\mathbb{MG}(L)$. It remains to show that there is no path between H_1 and H_2. Assume to the contrary, that there exist $F \in \mathcal{V}(H_1)$ and $G \in \mathcal{V}(H_2)$ such that F and G are adjacent in $\mathbb{MG}(L)$ (note that each vertex in $\mathbb{MG}(L)$

is contained in $\mathcal{V}(H_1)$ or $\mathcal{V}(H_2)$). Since $\mathrm{Soc}(L) = S_1 \wedge S_2 \subseteq F \wedge G$, we have $F \wedge G$ is essential in L by Lemma 3.2 which is impossible. This completes the proof.

The implication (3) $\Rightarrow$ (1) is clear. □

Note that the condition "L is an Artinian lattice" is necessary in Theorem 3.9 by Example 3.6 (2).

Theorem 3.10. *For the lattice L, the following statements hold:*

(1) If $\mathbb{MG}(L)$ is a connected graph, then $\mathrm{diam}(\mathbb{MG}(L)) \leq 2$.

(2) If $\mathbb{MG}(L)$ contains a cycle, then $\mathrm{gr}(\mathbb{MG}(L)) = 3$.

Proof. (1) By Theorem 3.9, $\mathbb{MG}(L)$ is a connected graph. Let F and G be nonessential filters of L such that $G \wedge F \trianglelefteq L$. Then there exist simple filters S_1 and S_2 such that $S_1 \subseteq F$ and $S_2 \subseteq G$ by Lemma 3.1. If $F \wedge S_2$ is not essential in L, then $F \backsim S_2 \backsim G$ is a path in $\mathbb{MG}(L)$ with $\mathrm{d}(F, G) = 2$. Similarly, if $G \wedge S_1$ is not essential in L, then $\mathrm{d}(G, F) = 2$. So we may assume that $F \wedge S_2 \trianglelefteq L$ and $G \wedge S_1 \trianglelefteq L$. As $\mathbb{MG}(L)$ is connected, $|\mathcal{S}(L)| \geq 3$ by Theorem 2.9. Let S_3 be a simple filter of L such that $S_1 \neq S_3$ and $S_2 \neq S_3$. Since $G \wedge F \trianglelefteq L$, we have $S_3 \subseteq \mathrm{Soc}(L) \subseteq G \wedge F$ by Lemma 3.2 which gives either $S_3 \subseteq F$ or $S_3 \subseteq G$. We can assume that $S_3 \subseteq F$. Then $F = F \wedge S_3$ is nonessential in L. We claim that $S_3 \wedge G$ is nonessential in L. If $S_3 \wedge G \trianglelefteq L$, then $S_1 \subseteq \mathrm{Soc}(L) \subseteq S_3 \wedge G$ gives $S_1 \subseteq G$; hence $S_1 \wedge G = G$ is nonessential in L, a contradiction. Thus $F \backsim S_3 \backsim G$ is a path in $\mathbb{MG}(L)$ with $\mathrm{d}(G, F) = 2$.

(2) If $|\mathcal{S}(L)| = 2$ and $\mathbb{MG}(L)$ contains a cycle, then $\mathrm{gr}(\mathbb{MG}(L)) = 3$ by Theorem 3.9. So we may assume that $|\mathcal{S}(L)| \geq 3$. Let S_1, S_2 and S_3 be three distinct simple filters of L. Then by Lemma 3.3, $S_1 \wedge S_2$, $S_2 \wedge S_3$ and $S_3 \wedge S_1$ are nonessential in L; so $S_1 \backsim S_2 \backsim S_3 \backsim S_1$ is a cycle in $\mathbb{MG}(L)$ which implies that $\mathrm{gr}(\mathbb{MG}(L)) = 3$. □

Theorem 3.11. *If $\mathbb{MG}(L)$ is a connected graph, then $\mathbb{MG}(L)$ has no cut vertex.*

Proof. Assume to the contrary, that $\mathbb{MG}(L)$ has a cut vertex S (so $\mathbb{MG}(L) \setminus \{S\}$ is not connected). Then there are vertices G and H such that S lies on every path from H to G. Thus $G \backsim S \backsim H$ is a path between G and H by Theorem 3.10 (1). It follows that $G \wedge S$ is not essential in L, $G \wedge H \trianglelefteq L$ and $S \wedge H$ is not essential in L. Let $K \subsetneqq S$ for any filter K of L. By Lemma 2.3, S is not essential in L gives K is not essential in L. As $G \wedge K \subseteq G \wedge S$, we get that $G \wedge K$ is not essential in L. Similarly, $H \wedge K$ is not essential in L. So $G \backsim K \backsim H$ is a path in $\mathbb{MG}(L) \setminus S$ which is impossible. Thus S is a simple filter of L. We claim that there is a simple filter $S_i \neq S$ of L such that $S_i \nsubseteq G$. Otherwise, $\bigwedge_{S \neq S_i} S_i \subseteq G$ which gives $\mathrm{Soc}(L) = S \wedge \bigwedge_{S \neq S_i} S_i \subseteq S \wedge G$, a contradictin to the fact that $S \wedge G$ is not essential. Similarly, there is a simple filter $S_i \neq S$ of L such that $S_i \nsubseteq H$. Since $G \wedge H \trianglelefteq L$, we have

$S_i \subseteq \text{Soc}(L) \subseteq G \wedge H$ for each $S_i \in \mathcal{S}(L)$ which gives either $S_i \subseteq G$ or $S_i \subseteq H$. So for each $S_i \in \mathcal{S}(L)$, we have either $S_i \subseteq G$ or $S_i \subseteq H$. As $\mathbb{MG}(L)$ is a connected graph, Theorem 3.8 gives $|\mathcal{S}(L)| \geq 3$. Let S_i and S_j be simple filters of L such that $S_i \neq S$, $S_j \neq S$, $S_i \nsubseteq G$ and $S_j \nsubseteq H$. It follows that $S_i \subseteq H$ and $S_j \subseteq G$. Thus $G \backsim S_j \backsim S_i \backsim H$ is a path in $\mathbb{MG}(L)$ which is a contradiction. So $\mathbb{MG}(L)$ has no cut vertex. □

Theorem 3.12. *For a positive integer r, $\mathbb{MG}(L)$ is not a complete r-partite graph.*

Proof. Assume to the contrary, that $\mathbb{MG}(L)$ is a complete r-partite graph with parts $V_1, \cdots, V_r$. Since two distinct simple filters are always adjacent by Lemma 3.3, so each V_i contains at most one simple filter of L. Therefore by Pigeon hole principle we have $|\mathcal{S}(L)| \leq r$. We claim that $|\mathcal{S}(L)| = r$. Let $\mathcal{S}(L) = \{S_1, \cdots, S_k\}$, where $k < r$. If $S_i \in V_i$ for $1 \leq i \leq k$, then V_{k+1} contains no simple filter. As the number of simple filters is finite, $\bigwedge_{j \neq i} S_j$ is not essential in L by Lemma 3.3. Since $(\bigwedge_{j \neq i} S_j) \wedge S_i = \text{Soc}(L) \trianglelefteq L$ by Lemma 3.2, so $\bigwedge_{j \neq i} S_j$ and S_i are not adjacent. Thus $\bigwedge_{j \neq i} S_j \in V_i$, as $S_i \in V_i$. Assume that G is a vertex in V_{s+1} and let $S_m \subseteq G$ for some simple filter S_m of L. So G is adjacent to all elements of V_m. It follows that G is adjacent to $\bigwedge_{j \neq m} S_j$ which is impossible, as $\text{Soc}(L) \subseteq G \wedge (\bigwedge_{j \neq m} S_j)$ and $\text{Soc}(L) \trianglelefteq L$. Hence $k = r$. Consider the filter $H = \bigwedge_{i=3}^{r} S_i$ (so H is not essential in L by Lemma 3.3). Since $H \wedge S_1 = \bigwedge_{i \neq 2} S_i$ is not essential in L, we obtain that H and S_1 are adjacent. Similarly, H and S_2 are adjacent. So $H \notin V_1$ and $H \notin V_2$. It is clear that $H \wedge S_i = H$ is not essential in L for each $3 \leq i \leq r$. Hence H is adjacent to all simple filters S_i of L; so $H \in V_i$ for each $1 \leq i \leq r$ which is impossible, as required. □

Theorem 3.13. *For the Lattice L, the following conditions hold:*

(1) $\mathbb{MG}(L)$ has no a universal vertex;

(2) $\mathbb{MG}(L)$ is not a complete graph.

Proof. (1) Set $\mathcal{S}(L) = \{S_1, \cdots, S_n\}$ by Lemma 3.1. Assume to the contrary, that $\mathbb{MG}(L)$ has a universal vertex G. Then there is a simple filter S_j such that $S_j \subseteq G$. By Lemma 3.3, $H = \bigwedge_{i \neq j} S_i$ is not essential in L (so H is a vertex of $\mathbb{MG}(L)$). Since G is a universal vertex, G and H are adjacent in $\mathbb{MG}(L)$; hence $H \wedge G$ is not essential in L. But $\text{Soc}(L) = S_j \wedge H \subseteq H \wedge G$ gives $H \wedge G \trianglelefteq L$ which is impossible. So there is no vertex in $\mathbb{MG}(L)$ which is adjacent to every other vertex.

(2) By an argument like that (1), $\mathbb{MG}(L)$ cannot be a complete graph. □

Theorem 3.14. $\mathbb{MG}(L)$ *contains a vertex with degree one if and only if* $\mathbb{MG}(L) = H_1 \cup H_2$, *where H_1, H_2 are two disjoint complete subgraphs of* $\mathbb{MG}(L)$ *and* $|\mathcal{V}\ (H_i)| = 2$ *for some* $i = 1, 2$.

Proof. Let G be a vertex of $\mathbb{MG}(L)$ with $\deg(G) = 1$. By Proposition 3.5, $|\mathcal{S}(L)| > 1$. Suppose that $|\mathcal{S}(L)| \geq 3$. By Lemma 3.3, for each simple filter S_i of L, S_i is adjacent to every other simple filter of L; so $\deg(S_i) \geq 2$. It follows that G is not a simple filter of L. Without loss of generality, let $S_1 \subseteq G$. Then G and S_1 are adjacent in $\mathbb{MG}(L)$. Since $\deg(G) = 1$, so the only vertex adjacent to G is S_1 and $S_k \nsubseteq G$ for $k \neq 1$; hence G and S_2 are not adjacent. Thus $S_2 \wedge G \trianglelefteq L$ which implies that $S_j \subseteq \mathrm{Soc}(L) \subseteq G \wedge S_2$ for $j \neq 1, 2$; hence $S_j \subseteq G$ for $j \neq 1, 2$, a contradiction. Therefore $|\mathcal{S}(L)| = 2$. Now by theorem 3.9, $\mathbb{MG}(L) = H_1 \cup H_2$, where H_1, H_2 are two disjoint complete subgraphs of $\mathbb{MG}(L)$. Without loss of generality, suppose $G \in H_1$. As H_1 is a complete subgraph and $\deg(G) = 1$, we get that $|\mathcal{V}\ (H_1)| = 2$. This completes the proof. □

Corollary 3.15. *For the lattice L, $\mathbb{MG}(L)$ is not a star graph.*

Proof. Assume to the contrary, that $\mathbb{MG}(L)$ is a star graph. Then $\mathbb{MG}(L)$ has a vertex with degree one. Thus $|\mathcal{S}(L)| = 2$ by Theorem 3.14; so $\mathbb{MG}(L)$ is not connected by Theorem 3.9 which is impossible. Therefore $\mathbb{MG}(L)$ cannot be a star graph □

Theorem 3.16. *If $\mathbb{MG}(L)$ is a k-regular graph, then $|\mathcal{V}\ (\mathbb{MG}(L))| = 2k + 2$.*

Proof. At first we show that if F and G are vertices of $\mathbb{MG}(L)$ with $F \subseteq G$, then $\deg(G) \leq \deg(F)$. If K is a vertex adjacent to G, then $K \wedge G$ is not essential in L gives $K \wedge F$ is not essenial in L By Lemma 2.3; hence $\deg(G) \leq \deg(F)$. Let $\mathbb{MG}(L)$ be a k-regular graph. Then for each simple filter S_i of L, $\deg(S_i) = k$. Let $\mathcal{S}(L) = \{S_1, \cdots, S_n\}$, where $n \geq 3$. By Lemma 3.3, $H = \bigwedge_{i \neq 2} S_i$ is not essential in L. It is clearly that H is adjacent to S_1 but H is not adjacent to $S_1 \wedge S_2$ since $H \wedge (S_1 \wedge S_2 = \mathrm{Soc}(L) \trianglelefteq L$ by Lemma 3.1; hence $\deg(S_1 \wedge S_2) \gneqq \deg(S_1)$. It follows that $\deg(S_1 \wedge S_2) < k$ which is impossible. Thus $|\mathcal{S}(L)| \leq 2$. Since $\mathbb{MG}(L)$ is not a null graph, we have $|\mathcal{S}(L)| \neq 1$. Therefore $\mathcal{S}(L) = \{S_1, S_2\}$. Thus by Theorem 3.9, There exist two disjoint complete subgraphs H_1, H_2 of $\mathbb{MG}(L)$ such that $\mathbb{MG}(L) = H_1 \cup H_2$. We can assume that $S_1 \in H_1$ and $S_2 \in H_2$. Since $\deg(S_1) = k$, we have $|H_1| = k + 1$. Similarly, $|H_2| = k + 1$. Hence $|\mathcal{V}\ (\mathbb{MG}(L))| = 2k + 2$. □

We say that filters F and G of L are strongly disjoint if for any elements $1 \neq f \in F$ and $1 \neq g \in G$, $(1 :_L f) \neq (1 :_L g)$.

Theorem 3.17. *If $\mathbb{MG}(L)$ is not an empty graph and is a tree, then the following conditions are hold:*

(1) If F and G are elements of $\mathcal{V}\ (\mathbb{MG}(L))$ with $G \wedge F$ is not essential in L, then G and F are strongly disjoint.

(2) If F and G are elements of $\mathcal{V}$ $(\mathbb{MG}(L))$ with $G \wedge F$ is not essential in L, then one of F and G is a simple filter.

Proof. (1) Since $\mathbb{MG}(L)$ is a tree, so it is a triangle-free graph. Let F, G be elements of $\mathcal{V}$ $(\Gamma_P(L))$ such that $G \wedge F$ is not essential in L. At first we show that $G \cap F = \{1\}$ and if S is a subfilter of $G \wedge F$ with $S \neq \{1\}$, then $S \cap F \neq \{1\}$ or $S \cap G \neq \{1\}$. Assume that $K = F \cap G \neq \{1\}$ and let S' be a simple filter of L such that $S' \subseteq K$. Then F, S', G would form a triangle. This is impossible, so $K = \{1\}$. Let S be a subfilter of $F \wedge G$ with $S \neq \{1\}$ (so S is not essential in L). If $\{1\} \subsetneqq H \subsetneqq S$, the $H, S, F \wedge G$ would form a triangle, a contradiction. Thus S is a simple filter with $S \subseteq F \wedge G$ which implies that either $S \subseteq F$ or $S \subseteq G$ by Lemma 3.4. Assume to contrary, that there are elements $a \in F$ and $b \in G$ such that $(1 :_L a) = (1 :_L b)$. Then $\{1\} \neq T(\{a, b\}) \subseteq G \wedge F$ gives either $F \cap T(\{a, b\}) \neq \{1\}$ or $G \cap T(\{a, b\}) \neq \{1\}$. Without loss of generality, we can assume that $F \cap T(\{a, b\}) \neq \{1\}$. Then there exists $x \in F$ such that $a \wedge b \leq x$ which implies that $x = (x \vee b) \wedge (x \vee a) \in F$. Now F is a filter gives $x \vee b \in F \cap G = \{1\}$; so $x \in (1 :_L a) = (1 :_L b)$. It follows that $x = 1$ which is a contradiction. Thus F and G are strongly disjoint.

(2) Let $\{1\} \neq S_1 \subsetneqq G$ and $\{1\} \neq S_2 \subsetneqq F$. Since every tree is a bipartite graph, we have a cycle $S_1 \backsim G \backsim F \backsim S_2 \backsim S_1$ in a tree which is impossible. Thus one of F and G is a simple filter and so (2) holds. □

Theorem 3.18. *For the lattice L, the following conditions are equivalent:*

(1) Every vertex of $\mathbb{MG}(L)$ is of finite degree;

(2) The graph $\mathbb{MG}(L)$ is finite.

Proof. (1) $\Rightarrow$ (2) Let every vertex of $\mathbb{MG}(L)$ is of finite degree. By Lemma 3.1, $G \cap \mathrm{Soc}(L) \neq \{1\}$ for every nontrivial filter G of L and $\mathrm{Soc}(L)$ contains only finitely many subfilters. Assume that K is any non-trivial subfilter of $\mathrm{Soc}(L)$ and let $\mathcal{A}_K = \{G \in \mathbb{F}(L) : G \cap \mathrm{Soc}(L) = K\}$. At first we show that $\mathcal{V}(\mathbb{MG}(L)) = \bigcup_{K \subsetneqq \mathrm{Soc}(L)} \mathcal{A}_K$. Since the inclusion $\mathcal{V}(\mathbb{MG}(L)) \subseteq \bigcup_{K \subsetneqq \mathrm{Soc}(L)} \mathcal{A}_K$ is clear, we will prove the reverse inclusion. Suppose that $H \in \bigcup_{K \subsetneqq \mathrm{Soc}(L)} \mathcal{A}_K$. Then there is a proper subfilter G of $\mathrm{Soc}(L)$ such that $H \cap \mathrm{Soc}(L) = G$. Then G is not essential gives H is not essential; hence $H \in \mathcal{V}(\mathbb{MG}(L))$ and so we have equality. Now it is enough to show that $\mathcal{A}_K$ is a finite set for every proper subfilter K of $\mathrm{Soc}(L)$. Let S_K be a simple filter of L such that $S_K \subseteq K$. Let $U \in \mathcal{A}_K$ (so U is not essential in L). Then $S_K = S_K \cap K = S_K \cap (U \cap \mathrm{Soc}(L)) = S_K \cap U$ gives $S_K \subseteq U$; hence S_K is adjacent to any $U \in \mathcal{A}_K$. As S_K is of finite degree, we obtain that $\mathcal{A}_K$ is finite. This completes the proof. □

4. Clique number, independence number and domination number

Let us begin this section with the following theorem:

Theorem 4.1. *For the lattice L, the following statements hold:*

(1) If $\mathbb{MG}(L)$ is a non-empty graph, then $\omega(\mathbb{MG}(L)) \geq |\mathcal{S}(L)|$;

(2) If $\mathbb{MG}(L)$ is an empty graph, then $\omega(\mathbb{MG}(L)) = 1$ if and only if $\mathcal{S}(L) = \{S_1, S_2\}$, where S_1, and S_2 are the only nonessential distinct simple filters of L;

(3) If $\omega(\mathbb{MG}(L))$ is finite, then $\omega(\mathbb{MG}(L)) \geq 2^{|\mathcal{S}(L)|-1} - 1$.

Proof. (1) Since any two simple distinct filters of L are adjacent by Lemma 3.3 , the subgraph of $\mathbb{MG}(L)$ with the vertex set of $\{S_i\}_{S_i \in \mathcal{S}(L)}$ is a complete subgraph of $\mathbb{MG}(L)$. Thus $\omega(\mathbb{MG}(L)) \geq |\mathcal{S}(L)|$.

(2) This is a direct consequence of Theorem 3.7.

(3) Let $\mathcal{S}(L) = \{S_1, \cdots, S_n\}$. Also for each $1 \leq i \leq n$, set

$$A_i = \{S_1, \cdots S_{i-1}, S_{i+1}, \cdots, S_n\}.$$

For each i $(1 \leq i \leq n)$, Let $P(A_i)$ be the power set of A_i and set $B_X = \bigwedge_{F_i \in X} F_i$ for each $\emptyset \neq X \in P(A_i)$. Then the subgraph of $\mathbb{MG}(L)$ with the vertex set $\{B_X\}_{X \in P(A_i) \setminus \{\emptyset\}}$ is a complete subgraph of $\mathbb{MG}(L)$ by Lemma 3.3. Clearly, $|\{B_X\}_{X \in P(A_i) \setminus \{\emptyset\}}| = 2^{|\mathcal{S}(L)|-1} - 1$. Hence $\omega(\mathbb{MG}(L)) \geq 2^{|\mathcal{S}(L)|-1} - 1$. □

Theorem 4.2. *For the lattice L, $\alpha(\mathbb{MG}(L)) = |\mathcal{S}(L)|$.*

Proof. Let $\mathcal{S}(L) = \{S_1, \cdots, S_n\}$. Since $\Omega = \{\bigwedge_{j=1, i \neq j}^{n} S_j\}_{i=1}^{n}$ is an independent set in $\mathbb{MG}(L)$, we have $n \leq \alpha(\mathbb{MG}(L))$ (note that if $C, D \in \Omega$, then $C \wedge D = \mathrm{Soc}(L) \trianglelefteq L$, and so C is not adjacent to D by Lmma 3.2). Assume that $\alpha(\mathbb{MG}(L)) = t$ and let $A = \{F_1, \cdots, F_t\}$ be a maximal independent set in $\mathbb{MG}(L)$. Then for each $F \in A$, F is not essential in L (so $\mathrm{Soc}(L) \nsubseteq F$ by Lemma 3.2); hence $S \nsubseteq G$ for some simple filter S of L. If $t > n$, then by Pigeon hole principal, there exist $1 \leq i, j \leq n$ and $S \in \mathcal{S}(L)$ such that $S \nsubseteq F_i$ and $S \nsubseteq F_j$; so $S \nsubseteq F_i \wedge F_j$ by Lemma 3.4. Since A is an independent set in $\mathbb{MG}(L)$, F_i and F_j are not adjacent, and so $F_i \wedge F_j \trianglelefteq L$. It follows that $S \subseteq \mathrm{Soc}(L) \subseteq F_i \wedge F_j$ which is impossible. If $\alpha(\mathbb{MG}(L)) = \infty$, then by a similar argument as above, we have a contradiction. This proves that $\alpha(\mathbb{MG}(L)) = |\mathcal{S}(L)|$. □

Theorem 4.3. *For the lattice L, $\gamma(\mathbb{MG}(L)) = 2$.*

Proof. Note that $|\mathcal{S}(L)| \geq 2$, as $\mathbb{MG}(L)$ is a non-null graph. Set $A = \{S_1, S_2\}$, where S_1, S_2 are distinct simple filters of L. Let G be a vertex of $\mathbb{MG}(L)$. If either $S_1 \subseteq G$ or $S_2 \subseteq G$, the $S_1 \wedge G = G$ or $S_2 \wedge G = G$ is non-essential in L. Then G is adjacent to S_1 or S_2. So we may assume that $S_1 \nsubseteq G$ and $S_2 \nsubseteq G$. Without loss of generality, we can assume that G is not adjacent to S_1 (so $G \wedge S_1 \trianglelefteq L$). Then $S_2 \subseteq \mathrm{Soc}(L) \subseteq S_1 \wedge G$ by Lemma 3.2. It follows that $S_2 \subseteq G$ by Lemma 3.4 which is impossible. Thus G is adjacent to S_1. Similarly, G is adjacent to S_2. Hence $\gamma(\mathbb{MG}(L)) \leq 2$. By Theorem 3.13, $\mathbb{MG}(L)$ has no a universal vertex; so $\gamma(\mathbb{MG}(L)) \neq 1$. Thus $\gamma(\mathbb{MG}(L)) = 2$. □

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