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Research Paper

STUDYING SOME SEMISIMPLE MODULES VIA HYPERGRAPHS

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Abstract. Recent studies have shown that hypergraphs are useful in solving real-life problems. Hypergraphs have been successfully applied in various fields. Inspiring by the importance, we shall introduce a new hypergraph assigned to a given module. By the way, vertices of this hypergraph (which we call sum hypergraph) are all nontrivial submodules of a module P and a subset E of the vertices is a hyperedge in case the sum of each two elements of E is equal to P and E is maximal with respect to this condition. Some general properties of such hypergraphs are discussed. Semisimple modules with length 2 are characterized by their corresponding sum hypergraphs. It is shown that the sum hypergraph assigned to a finite module P is connected if and only if P is semisimple.

1. Introduction

According to [5, 6], a hypergraph denoted by $\mathcal{H}$ consists of a group of vertices or nodes, represented as $V = \{v_1, \ldots, v_n\}$, and a collection of hyperedges denoted by $E = \{e_j \mid 1 \leq j \leq$

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$m\}$. Each hyperedge is a set of vertices that is not empty and the union of all hyperedges forms the set of vertices V. This implies that, in a hypergraph, a hyperedge connects two or more vertices.

Hypergraphs are an extension of graphs where edges can connect more than two vertices. While in a graph, vertices represent the elements of a set, the hyperedges represent subsets of any cardinality (not just 2 as in graphs), or, even more generally, arbitrary statements about arbitrary subsets. Overall, hypergraphs have a wide range of applications and can be used to represent complex structures and relationships between nodes. Over the past decade, studies have shown that hypergraphs are useful in solving real-life problems. Hypergraphs have been successfully applied in various fields such as network modeling, data structures, system modeling for engineering, cellular mobile communication systems, image processing, machine learning, data mining, soft sets, social networks, chemistry, and more. Some examples of studies include [20], [12], [15], [8, 9, 10, 11], [13], [3], [16], [7, 14, 22] and [23].

From a theoretical point of view, hypergraphs allow to generalize certain theorems on graphs, even to replace several theorems on graphs by a single theorem of hypergraphs. For instance, the Berge's weak perfect graph conjecture, which says that a graph is perfect if and only if its complement is perfect, was proved thanks to the concept of normal hypergraph. From a practical point of view, they are now increasingly preferred to graphs. It is known that complete graphs play an important rule in graph theory. Via introducing co-intersection hypergraph in [18] and also intersection hypergraph in [19], the authors characterized some special modules and find the number of complete subgraphs of co-intersection graph and intersection graph defined on modules.

Note throughout the text, R denotes an associative ring with identity $1 \neq 0$ and all modules will be assumed as unitary right R-modules. A submodule T of a module P is said to be *essential (large)*, provided $T \cap F = 0$ implies $F = 0$, where F is a submodule of P. Changing intersection to sum and zero submodule to the whole module, a submodule Q of P is said to be *superfluous (small)* provided $Q + L = P$ where $L \leq P$ implies $L = P$. Note that if every submodule of the module P is a large submodule, then P is called a uniform module. An analogue for a uniform module is a hollow module. The module P is called hollow, if every proper submodule is small in P. A hollow module P with a largest submodule (the sum of all proper submodules of P) is a local module.

The sum of all simple submodules of a module P is said to be the Socle of P, denoted by $Soc(P)$. If P has no large submodules, then $Soc(P) = P$. In this case, we say P is semisimple. The radical of P, denoted by $Rad(P)$ is the sum of all small submodules of P, that is equivalent to the intersection of all maximal submodules of P (see [26]).

In this manuscript, we intend to introduce a novel hypergraph on a given module. By the way, we introduce sum-hypergraph on a given module P where the vertices are all nontrivial submodules of P and a subset E of vertices of this hypergraph forms a hyperedge provided the sum of each two elements of E is equal to P and E is maximal with respect to this condition. Motivating by [18, 19], here we are interested in finding maximal cliques in the corresponding graph (the sum graph defined on P, where vertices are all nontrivial submodules of P and two vertices ar adjacent in case their sum is equal to P). A semisimple module with length 2 can be specificized by its sum-hypergraph.

2. Sum hypergraph assigned to a module

In [18] and [19], the authors introduced two new hypergraphs defined on a given module namely co-intersection hypergraph and intersection hypergraph, respectively. Via those hypergraphs, some (semisimple) modules have been characterized. Looking through in [18] and [19], encourage us to introduce a new aspect of those definitions via replacing "intersection" with "sum". By the way, the key definition will be presented.

Definition 2.1. Assume that P is an R-module. A hypergraph can be defined on P as follows: the vertices are all nontrivial submodules of P and a set E (such that $\mid E \mid\geq 2$) of nontrivial submodules of P forms a hyperedge provided the sum of each two distinct elements of E is equal to P and E is maximal with respect to this property. We denote such hypergraph via $\mathcal{SUM}_R(P)$. Note that any hyperedge of $\mathcal{SUM}_R(P)$ includes at least two elements.

Throughout this manuscript, we consider modules with at least two nontrivial submodules.

Proposition 2.2. *Let P be an R-module. Then the hypergraph $\mathcal{SUM}_R(P)$ is null if and only if P is hollow.*

Proof. Let $\mathcal{SUM}_R(P)$ be null and N be an arbitrary nontrivial submodule of P. As the hypergraph has no hyperedges, so for each nontrivial submodule K of P, we can say $N+K \neq P$. Hence $N \ll P$. The converse follows from the fact that the sum of each two nontrivial submodules of P is not equal to P. $\square$

From Proposition 2.2, the sum hypergraph of the $\mathbb{Z}$-modules $\mathbb{Z}_{p^n}$ $(n \geq 2)$ and $\mathbb{Z}_{p^\infty}$ are null.

We next characterize modules P for which $\mathcal{SUM}_R(P)$ has just one hyperedge containing all vertices.

Theorem 2.3. *The following statements are equivalent for a module P:*

(1) *$\mathcal{SUM}_R(P)$ has just exactly one hyperedge containing all vertices;*

(2) *The (direct) sum of each two nontrivial submodules of P is equal to P;*

(3) *The module P is semisimple with length 2.*

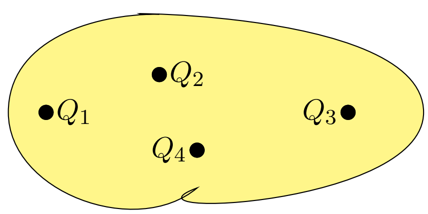
Proof. (1) $\Rightarrow$ (2) It is straightforward.

(2) $\Rightarrow$ (3) Consider two arbitrary submodules N and K of P. If $N \cap K \neq 0$, then by assumption $N = N + (N \cap K) = P$ which contradicts $N \neq P$. So $P = N \oplus K$. Hence P is semisimple. It is clear that the length of P is two.

(3) $\Rightarrow$ (1) To show that $\mathcal{SUM}_R(P)$ has just one hyperedge containing all nontrivial submodules, we may show that the sum of each two nontrivial submodules of P is equal to P. Suppose that N and K are two nontrivial submodules of P. If $N + K \neq P$, then we have a chain of submodules $0 \subset N \subseteq N + K \subset P$ which has length 3, a contradiction. So $N + K = P$. □

Following Theorem 2.3, the sum hypergraph of the $\mathbb{Z}$-module $\mathbb{Z}_p \oplus \mathbb{Z}_p$ has just one hyperedge including all $p+1$ nontrivial submodules.

Example 2.4. Consider the semisimple $\mathbb{Z}$-module $P = \mathbb{Z}_3 \oplus \mathbb{Z}_3$. Then $Q_1 = \{(0,0),(0,1),(0,2)\}$, $Q_2 = \{(0,0),(1,0),(2,0)\}$, $Q_3 = \{(0,0),(1,1),(2,2)\}$ and $Q_4 = \{(0,0),(1,2),(2,1)\}$ are all nontrivial submodules of P. Being P semisimple with length 2, implies $V = \{Q_1, Q_2, Q_3, Q_4\}$, $E = \{\{Q_1, Q_2, Q_3, Q_4\}\}$ and the hypergraph $\mathcal{SUM}_{\mathbb{Z}}(P)$ has the form (Theorem 2.3):



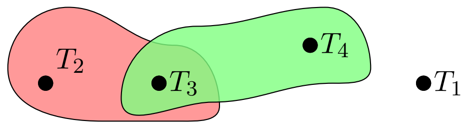
Proposition 2.5. *Let P be an R-module. Then the sum hypergraph of P is connected if and only if $Rad(P) = 0$.*

Proof. Suppose that $\mathcal{SUM}_R(P)$ is connected. If N is a nonzero small submodule of P, then N can not be included in an hyperedge of $\mathcal{SUM}_R(P)$. So N is isolated which is a contradiction. So $N = 0$. It follows that $Rad(P) = 0$. For the converse, suppose that $Rad(P) = 0$ and N, K be two nontrivial submodules of P. As $Rad(P) = 0$, there exist two nontrivial submodules L and T of P such that $N + L = P$ and $K + T = P$. Now consider, the submodule $L + T$ of P. If $L + T = P$, then $N - W_1 - L - W_2 - T - W_3 - K$ is a path from N to K. Otherwise, $L + T \neq P$. In this case $N - U_1 - K + T - U_2 - K$ will be a path from N to K. By the way, $\mathcal{SUM}_R(P)$ is connected. □

The following is an immediate corollary due to the fact that for a finite module P, being semisimple is equivalent to $Rad(P) = 0$.

Corollary 2.6. *Let P be a finite module. Then $\mathcal{SUM}_R(P)$ is connected if and only if P is semisimple.*

Example 2.7. Suppose that $P = \mathbb{Z}_{p^2q}$ as an $\mathbb{Z}$-module where p, q are distinct prime numbers and $p < q$. The list of all nontrivial submodules of B is $T_1 =< pq >$, $T_2 =< p^2 >$, $T_3 =< q >$ and $T_4 =< p >$. By the way, $\mathcal{SUM}_R(P)$ has two hyperedges $L_1 = \{T_2, T_3\}$ and $L_2 = \{T_3, T_4\}$. Note that $T_1 \ll P$, so that T_1 is an isolated vertex in $\mathcal{SUM}_{\mathbb{Z}}(P)$.



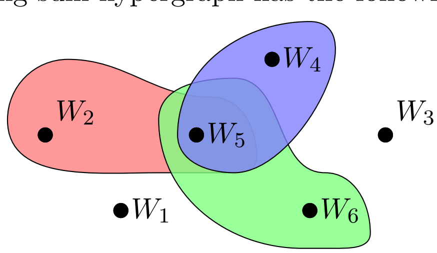
Note that the diameter of a hypergraph $\mathcal{H}$ is the maximum distance between any pair of vertices in H. Now, we are in a position to find the diameter of $\mathcal{SUM}_R(P)$ when it is a connected hypergraph. By definition, if $\mathcal{SUM}_R(P)$ is not connected, then diam$(\mathcal{SUM}_R(P) = \infty$.

Proposition 2.8. *Let P be an R-module. If $\mathcal{SUM}_R(P)$ is a connected hypergraph, then diam$(\mathcal{SIH}_R(P)) \leq 3$.*

Proof. Assume that $\mathcal{SUM}_R(P)$ is connected. By Proposition 2.5, we have $Rad(P) = 0$. Let N_1 and N_2 be two non-adjacent vertices of $\mathcal{SUM}_R(P)$.

As $Rad(P) = 0$, we conclude that $N_1 + K = P$ and $N_2 + T = P$ for some nontrivial submodules K and T of P. If $K + T = P$, then $N_1 - W_1 - K - W_2 - T - W_3 - N_2$ is a path. So $d(N_1, N_2) = 3$. Otherwise, $K + T \neq P$. So, $N_1 - U_1 - K + T - U_2 - N_2$ is a path. Hence $d(N_1, N_2) = 2$. Therefore, $diam(\mathcal{SUM}_R(P)) \leq 3$. □

Example 2.9. Let $P = \mathbb{Z}_{p^3q}$ as an $\mathbb{Z}$-module. All nontrivial submodules are $W_1 =< p^2q >$, $W_2 =< p^3 >$, $W_3 =< pq >$, $W_4 =< p^2 >$, $W_5 =< q >$ and $W_6 =< p >$. Then $\mathcal{SUM}_R(P)$ has three hyperedges $E_1 = \{W_2, W_5\}$, $E_2 = \{W_4, W_5\}$ and $E_3 = \{W_5, W_6\}$. As W_5, W_6 are maximal submodules of P, then $Rad(P) = W_5 \cap W_6 = W_3$ which is a small submodule of P. Note that W_1 as a submodule of W_3 is also small. So W_1 and W_3 are isolated vertices in $\mathcal{SUM}_{\mathbb{Z}}(P)$. The corresponding sum hypergraph has the following figure:



Recall that by $l_R(P)$, we mean the length of the R-module P. In other words, we say P has length $n \in \mathbb{N}$, provided n is the length of the largest chain of submodules of P. If no such largest chain exists, then $l_R(P) = \infty$. Note also that by "finitely many submodules" we mean, infinite number of submodules.

Lemma 2.10. *Let P be an R-module. Assume that $\Delta(\mathcal{SUM}_R(P)) < \infty$ and $\delta(\mathcal{SUM}_R(P)) \geq 1$. Then $l_R(P) \leq \Delta(\mathcal{SUM}_R(P)) + 1$ and every nontrivial submodule of P has finitely many submodules.*

Proof. Suppose that $L_1 \supset L_2 \supset ...$ is a descending chain of nontrivial submodules of P. Since $\delta(\mathcal{SUM}_R(P)) \geq 1$, we have $N + L_{\Delta(\mathcal{SIH}_R(B))+1} = P$ for some proper submodule N of P. From this, we get that $N + L_i = P$, for each $1 \leq i \leq \Delta(\mathcal{SUM}_R(P)) + 1$ and so $deg(N) \geq \Delta(\mathcal{SUM}_R(P)) + 1$, a contradiction. Hence every descending chain of nontrivial submodules of P has maximum length $\Delta(\mathcal{SUM}_R(P)) + 1$.

Next, if $H_1 \subset H_2 \subset ...$ is an ascending chain of non-trivial submodules of P, then $T + H_1 = P$, for some non-trivial submodule T of P. From this, we get that $T + H_i = P$, for each $i \in \mathbb{N}$ and so $deg(T)$ will be infinite, a contradiction. It follows that any ascending chain of nontrivial submodules of P is finite. These two facts imply that $l_R(P) \leq \Delta(\mathcal{SUM}_R(P)) + 1$.

Let N be a nontrivial submodule of P. Then there exists a non-trivial submodule L of P such that $N + L = P$, because $\delta(\mathcal{SUM}_R(P)) \geq 1$. Therefore every submodule of N is adjacent to L. As $\Delta(\mathcal{SUM}_R(P))$ is finite, N has finitely many submodules. □

The girth of a hypergraph is the length of a shortest cycle if such exists. If there is no cycle, then the girth is defined to be infinite. The following theorem gives the girth of $\mathcal{SUM}_R(P)$ which is denoted by $gr(\mathcal{SUM}R(P))$.

Theorem 2.11. *If P is a module such that $\mathcal{SUM}_R(P)$ contains a cycle, then $gr(\mathcal{SUM}_R(P)) \leq 4$. Further if $\mathcal{SUM}_R(P)$ is connected, then $gr(\mathcal{SUM}_R(P)) = 3$.*

Proof. First we claim that $gr(\mathcal{SUM}_R(P)) \leq 4$ and we prove the same by the method contradiction. Suppose that $gr(\mathcal{SUM}_R(P)) = t \geq 5$. Let $N_1 - E_1 - N_2 - E_2 - \cdots - E_{t-1} - N_t - E_t - N_1$ be a cycle of the shortest length in $\mathcal{SUM}_R(P)$. Since N_1 and N_3 are not adjacent, $N_1 + N_3$ is not equal to P and similarly $N_2 + N_4$ is not equal to P and hence they are proper submodules of P.

If $N_1 = N_1 + N_3$, then $N_1 + N_4 = N_1 + N_3 + N_4 = P$. It follows that $N_1 - E_1 - N_2 - E_2 - N_3 - E_3 - N_4 - E_4 - N_1$ is a cycle, which is impossible. Hence $N_1 \neq N_1 + N_3$.

If $N_1 = N_2 + N_4$, then $N_1 = P$ again a contradiction. Therefore $N_1 \neq N_2 + N_4$.

If $N_2 = N_1 + N_3$, then $N_2 = P$ which is a contradiction. Thus $N_2 \neq N_1 + N_3$.

If $N_2 = N_2 + N_4$, then $N_2 + N_5 = P$. This implies that $N_2 - E_2 - N_3 - E_3 - N_4 - E_4 - N_5 - E_5 - N_2$ is a cycle, a contradiction. Hence $N_2 \neq N_2 + N_4$.

If $N_1 + N_3 = N_2 + N_4$, then it is easy to see that $N_1 + N_3 = P$, which is impossible.

From the above arguments, $\{N_1, N_2, N_1 + N_3, N_2 + N_4\}$ is a set of four distinct vertices in $\mathcal{SUM}_R(P)$ and $N_1 - U_1 - N_2 - U_2 - N_1 + N_3 - U_3 - N_2 + N_4 - U_4 - N_1$ is a cycle, a contradiction. Therefore $gr(\mathcal{SUM}_R(P)) = t \leq 4$.

Next, suppose that $\mathcal{SUM}_R(P)$ is connected and let $N_1-E_1-N_2-E_2-N_3-E_3-N_4-E_4-N_1$ be a cycle of length four in $\mathcal{SUM}_R(P)$. Since $\mathcal{SUM}_R(P)$ is connected, by Proposition 2.5, we have $Rad(P) = 0$

Let $N_1 - E_1 - N_2 - E_2 - N_3 - E_3 - N_4 - E_4 - N_1$ be a cycle of length four. If either $N_1 + N_3 = P$ or $N_2 + N_4 = P$, then $N_1 - E_1 - N_2 - E_2 - N_3 - E_3 - N_4 - E_4 - N_1$ contains a cycle of length three. Hence $gr(\mathcal{SUM}_R(P)) = 3$. If $N_1 + N_3 \neq P$ and $N_2 + N_4 \neq P$, let $H = (N_1 + N_3) \cap (N_2 + N_4)$. It is clear that $H < P$. Since $Rad(P) = 0$, we have $P = H + T$ for some $T \leq P$. Now, $(N_1 + N_3) - U_1 - T - U_2 - (N_2 + N_4) - U_3 - (N_1 + N_3)$ as a cycle in $\mathcal{SUM}_R(P)$. Therefore $gr(\mathcal{SUM}_R(P)) = 3$. □

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