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Research Paper

SPECIAL REGULAR CLEAN RINGS

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ABSTRACT. In this paper we introduce the concepts of special regular clean elements and regular clean decomposition in a ring R. These concepts lead us to the notion of special regular clean ring. We prove that for a special regular clean element $a = e + r \in R$ and unit $u \in R$ then au is a special regular clean if u is an inner inverse of e. We establish that an abelian ring R is a special regular clean ring if and only if the twisted power series ring $R[[x, \sigma]]$ is a special regular clean ring. We also study various characterizations of special clean and special regular clean rings.

1. Introduction

Throughout the discussion R means ring with unity unless otherwise specified. An element $a \in R$ is called clean if a can be written as a sum of unit and idempotent. The notion of clean elements were first introduced in late 70s by W. K. Nicholson [16] . Since then many authors have studied various concepts related to clean elements. In 2011 N. Ashrafi and E.

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Nasibi [3] defined regular clean rings and investigated many properties of this ring. Later in 2013 they have shown that the concept of clean ring and regular clean ring coincides for abelian rings [4]. Moreover, the authors have characterized regular rings using the concept of Twisted power series rings. In 2020 Khurana et al.[12] defined special clean rings and gave a many useful results with a different direction. Meanwhile several authors have studied the properties of clean rings and gave many results for the readers which are available in [5], [10], [1], [17], [15], [6] etc. It is interesting to note that a special regular clean ring is not special clean ring. This motivates us to study this structure of ring. $idem(R)$ denotes the set of all idempotent elements in R. $U(R)$ denotes the group of all units or invertible elements in R. $M_n(R)$ denotes the ring of all $n \times n$ square matrices over R. $UT_n(R)$ denotes the $n \times n$ upper triangular matrix ring over R. E_{ij} denotes the $n \times n$ matrix with all 0 except the $(i, j)^{th}$ place is 1. We note that $aE_{ij}bE_{pq} = abE_{iq}$ if $j = p$ and $aE_{ij}bE_{pq} = 0$ otherwise.

2. Preliminaries

In this section, we present the basic results and notations needed for our work. We also present the definition of special regular clean decomposition and special regular clean elements in a ring.

In 1936, von Neumann defined an element $a \in R$ is a regular (sometimes Von Neumann regular) if $a = ara$ for some $r \in R$. A ring R is called a regular if each of its elements are regular. The set $I(a) := \{r \in R : a = ara\}$ is called the "inner inverses of a" according to [14]. $reg(R)$ denotes the set of all the regular elements in R. Many properties of regular elements and regular rings has been studied in [9]. An element $a \in R$ is called a regular clean $(r-clean)$ if $a = r + e$ for some $r \in reg(R)$ and $e \in idem(R)$ according to [3]. $r - clean(R)$ denotes the set of all regular clean elements in R. A ring R is called a regular clean if $R = r - clean(R)$. Clearly regular rings and clean rings are regular clean. But converses are not true, for example we take $\mathbb{Z}_4$ which is a regular clean ring but not regular. For a regular clean ring which is not clean we consider Example 1 in [11].

Definition 2.1. Let R be a ring. An element $a \in R$ is called special regular clean or in short $sp - rclean$ if $a = e + r$ for some idempotent e and regular element r such that $aR \cap eR = 0$.

In working we will often refer the equation $a = e + r$ with $aR \cap eR = 0$ as a special regular clean decomposition.

If in the above definition $r \in U(R)$ then a is regarded as special clean element in a ring R in the sense of [12]. So, $sp - cn(R) \subseteq sp - rcn(R)$. But special regular clean decomposition may not be a special clean decomposition. Also a special regular clean element may not be special clean. For we take the following examples.

Example 2.2. Let $R = M_{2\times 2}(\mathbb{Z})$. Then we notice the following decomposition

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Here, $-E_{11} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \in reg(R)$. As $(-E_{11})(-E11)(-E_{11}) = -E_{11}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \in idem(R)$. Also $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} R \cap \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} R = \{\begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} : a, b \in R\} \cap \{\begin{pmatrix} a & b \\ a & b \end{pmatrix} : a, b \in R\} = 0$. Therefore, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ is a special regular clean decomposition. But as $-E_{11}$ is not unit, the decomposition is not special clean.

The above example doesn't conclude that a special regular clean element is not special clean. Because if we write

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}.$$

Then we get the element is a special clean. For an element which is a special regular clean but not a special clean we take the following example.

Example 2.3. Let K be a field with $char \neq 2$, $A = K[[x]]$ and $\widetilde{A}$ be the field of fractions of A. Then we consider R_1 [[11], Example 1] defined as $R_1 = \{r \in End(A_K) : \exists q \in \widetilde{A} \text{ and } n \in \mathbb{Z}^+ \text{ with } r(a) = qa\ \forall a \in (x^n)\}$. By [[5], Proposition 10], R_1 is not clean. Consider $s \in R_1$ such that s is not clean. Again by [[11], Example 1], R_1 is a regular ring. So, there exist $y \in R_1$ such that $sys = s$. Now consider $R = M_{2\times 2}(\mathbb{Z}) \times R_1$. Then we notice the decomposition

$$(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, s) = (\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, 0) + (\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, s).$$

Here, $(-E_{11}, s) = (\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, s) = (-E_{11}, s)(-E_{11}, y)(-E_{11}, s) \in reg(R)$ and $(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, 0) \in idem(R)$. Then it can be easily shown that $(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, s)R \cap (\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, 0)R = 0$. Therefore, $(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, s)$ is a special regular clean element. But we have $A \times B$ is clean if and only if A and B are clean. So as s is not clean, $(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, s)$ is not clean. Consequently, it is not special clean.

If we remove the condition $aR \cap eR = 0$ then we get the regular clean decomposition in the sense of [3]. So $sp - rclean(R) \subseteq r - clean(R)$. But again the other inclusion is not true. We take the following example.

Example 2.4. We consider the ring $R = \mathbb{Z}_4$ which is $r - clean$. So, 2 is regular clean element. We claim that 2 is not special regular clean. For this we must show that there doesn't exist any $e \in idem(R)$ and $r \in reg(R)$ with $2 = e + r$ such that $2R \cap eR = 0$. Here, $reg(R) = \{0, 1, 3\}$ and $idem(R) = \{0, 1\}$. Then only possible decomposition for 2 is $2 = 1 + 1$. But $2R \cap 1R = 2\mathbb{Z}_4 \cap 1\mathbb{Z}_4 = \{0, 2\} \neq 0$.

So, from the above examples it is convenience to study this class of ring separately. An idempotent $e \in R$ is called left semicentral if $(1 - e)Re = 0$ and right semicentral if $eR(1 - e) = 0$. A ring R is called abelian if all its idempotents are either left semicentral or right semicentral. In particular in an abelian ring all idempotents commute with every element of the ring (central).

Lemma 2.5. *([7],Proposition 3.5) Let R be an abelian ring, $a \in R$ be a clean element in R and $e \in idem(R)$. Then the following holds.*

(1) *ae is clean.*
(2) *If $-a$ is clean then $a + e$ is also clean.*

Lemma 2.6. *([3], Theorem 10) A direct product ring $R = \prod_{i \in I} R_i$ is a regular clean ring if and only if each $\{R_i\}$ is a regular clean.*

It is well known that the structure of the rings can be determined by idempotents through direct sum decompositions called Peirce decomposition [2]. So, if $e \in R$ is an idempotent then we have $R = eRe \oplus eR(1 - e) \oplus (1 - e)Re \oplus (1 - e)R(1 - e)$. An element $a \in R$ is said to be special clean if $a = e + u$ for some idempotent e and unit u such that $aR \cap eR = 0$. $cn(R)$ denotes the set of all clean elements in R and $sp - cn(R)$ denotes the set of all special clean elements in R. A non trivial ring is called a division ring if every nonzero element is unit.

$R[[x]]$ denotes the power series ring where elements of R commute with the indeterminate x. Now if we remove the commutativity of R and x and define the operation as $xa = \sigma(a)x$ for all $a \in R$ and some $\sigma \in End(R)$(Endomorphism ring of R), we get a new structure of ring called twisted power series ring. This ring is denoted by $R[[x, \sigma]]$. A map $\delta : R \to R$ is said to be derivation if δ is additive and $\delta(ab) = a\delta(b) + \delta(a)b$. Now we define the operation on $R[[x]]$ as $xa = ax + \delta(a)$ for all $a \in R$ we get a new structure of ring called differential power series ring. The map δ is called inner derivation on R if there exists $r \in R$ such that $\delta(a) = ra - ar$ for every $a \in R$. For further information in this area is available in [[8],[13]].

3. Main results

In this section we present our main results.

Proposition 3.1. *Let R be a special regular clean ring and $e \in idem(R)$. If e is left semicentral then the corner ring eRe is a special regular clean ring.*

Proof. Let R be a special regular clean ring. We consider $e\alpha e \in eRe$ with $\alpha \in R$. Then there exist $f \in idem(R)$ and $r \in reg(R)$ with $\alpha = f + r$ such that $\alpha R \cap fR = 0$. As, $r \in reg(R)$ there exist $b \in R$ such that $rbr = r$. Now multiplying α by e both left and right side we get $e\alpha e = efe + ere$. Here $(ere)(b)(ere) = er(ebe)re = erb(ere) = e(rbr)e = ere \in reg(eRe)$ and $(efe)(efe) = efe(fe) = effe = efe \in idem(eRe)$. Also $e\alpha eR \cap efeR \subseteq \alpha eR \cap feR \subseteq \alpha R \cap fR = 0$. Therefore, eRe is a special regular clean ring. □

Proposition 3.2. *A direct product ring $R = R_1 \times R_2$ is a special regular clean if both R_1 and R_2 are special regular clean.*

Proof. From Lemma 2.6 we have if R_1, R_2 are regular clean then $R = R_1 \times R_2$ is a regular clean. Now we consider any element $(a, b) \in R$. As R_1 and R_2 are special regular clean we have $a = e_1 + r_1 \in R_1$ and $b = e_2 + r_2 \in R_2$ are regular clean decomposition for some $e_1 \in idem(R_1), e_2 \in idem(R_2), r_1 \in reg(R_1), r_2 \in reg(R_2)$ such that $aR_1 \cap e_1R_1 = 0$ and $bR_2 \cap e_2R_2 = 0$. Then from fact that if for any non empty sets $A \cap B = 0$ and $C \cap D = 0 \implies (A \times C) \cap (B \times D) = 0$, we have $(aR_1 \times bR_2) \cap (e_1R_1 \times e_2R_2) = 0 \implies (a, b)(R_1 \times R_2) \cap (e_1, e_2)(R_1 \times R_2) = 0$. Therefore, $(a, b) = (e_1, e_2) + (r_1, r_2)$ such that $(a, b)R \cap (e_1, e_2)R = 0$. Hence R is a special regular clean ring. □

Theorem 3.3. *Let $e \in R$ be both left semicentral and right semicentral. Then R is special regular clean if eRe and fRf are both special regular clean. Where $f = 1-e$, the complimentary idempotent.*

Proof. We can write $R = eRe \oplus eRf \oplus fRe \oplus fRf$. Therefore, $R \cong \begin{pmatrix} eRe & eRf \\ fRe & fRf \end{pmatrix}$. In this case as $e \in idem(R)$ is both left and right semicentral, so $eRf = fRe = 0$ and hence we get the $R \cong \begin{pmatrix} eRe & 0 \\ 0 & fRf \end{pmatrix}$. Now it is enough to prove the diagonal representation of R is special regular clean. We consider $A \in R$ such that $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ for $a \in eRe$ and $b \in fRf$. As, eRe and fRf are special regular clean $a = e_1 + r_1$ and $b = e_2 + r_2$ are special regular clean decomposition. We consider $r_1 = r_1b_1r_1$ and $r_2 = r_2b_2r_2$ for some $b_1, b_2 \in R$. Therefore,

$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} e_1 + r_1 & 0 \\ 0 & e_2 + r_2 \end{pmatrix} = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} + \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$. Here $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} \in idem(R)$ and $\begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} = \begin{pmatrix} r_1 b_1 r_1 & 0 \\ 0 & r_2 b_2 r_2 \end{pmatrix} = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \in reg(R)$.
Also $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} R \cap \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} R = \begin{pmatrix} aR \cap e_1 R & 0 \\ 0 & bR \cap e_2 R \end{pmatrix} = 0$. Therefore, A has a special regular clean decomposition and consequently R is a special regular clean ring. □

Proposition 3.4. *Let R be a division ring. Then R is a special clean ring if and only if R is a special regular clean ring.*

Proof. (=>) It is clear by the fact that if $a \in R$ has a special clean decomposition then it is also a special regular clean decomposition.
(<=) Let $x \in R$ and $x = e + r$ such that $e^2 = e$ and $r = rbr$ for some $b \in R$. If $r = 0$ then $x = e = (2e-1)+(1-e)$ is a special clean by [[12],Example 2.7(B)]. If $r \neq 0$,we have $rbr - r = 0 \implies (rb-1)r = 0 = r(br-1)$. As R is a division ring we get $rb = 1 = br \implies r \in U(R)$. So, R is a special clean ring. □

Remark 3.5. Let $R = M_2(\mathbb{Z})$ and $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \in sp - rclean(R)$.
Then left multiplying A by i and keeping the idempotent same we have $iA = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ -1+i & 0 \end{pmatrix}$. This is again a special regular clean decomposition in the bigger ring $R' = M_2(\mathbb{Z}[i])$.
Because, $\begin{pmatrix} -1 & 0 \\ -1+i & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ -1+i & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1+i & 0 \end{pmatrix} \in reg(R')$ and clearly $\begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} R' \cap \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} R' = 0$. So, from a special regular clean element in a ring R we can construct another special regular clean element in some bigger ring.

The following two results explain some methods to construct special regular clean elements in a ring from known elements of the same type.

Theorem 3.6. *Let R be a ring and $\alpha = e + r \in sp - rclean(R)$. If $u \in I(e)$ then $\alpha u, u\alpha \in sp - rclean(R)$.*

Proof. Let $a = e + r$ such that $e^2 = e$ and $rbr = r$ for some $b \in R$ such that $aR \cap eR = 0$. We consider the following decomposition $au = eu + ru$. Then since $u \in I(e)$, $eueu = eu \in idem(R)$. Also $(ru)(u^{-1}b)(ru) = (rbr)u = ru \in reg(R)$. Finally we see $auR \cap euR = aR \cap eR = 0$. Therefore, $au \in sp - rclean(R)$. Similar proof for $ua \in sp - rclean(R)$. □

Theorem 3.7. *Let R be a ring and $a \in sp - rclean(R)$ such that $a = gu$ is an unit regular factorization for some $g \in idem(R)$ and $u \in U(R)$. Then $b = eu^{-1} \in sp - rclean(R)$.*

Proof. Let $a = e + r$ such that $e^2 = e$ and $ryr = r$ for some $y \in R$ such that $aR \cap eR = 0$. Replacing $a = gu$ we get $gu = e + r \implies g = eu^{-1} + ru^{-1} \implies eu^{-1} = g + r(-u^{-1}) = b$. Now we see $r(-u^{-1})(-uy)r(-u^{-1}) = (ryr)(-u^{-1}) = r(-u^{-1}) \in reg(R)$. Also, $bR \cap gR = eu^{-1}R \cap au^{-1}R = eR \cap aR = 0$. Therefore, $b \in sp - rclean(R)$. □

Proposition 3.8. *Let R be an abelian ring. Then R is special regular clean if and only if R is special clean.*

Proof. (<=) Clear.
(=>) Let R be a special regular clean ring and $a \in R$. Consider the decomposition $a = e + r$ where $e \in idem(R)$ and $rbr = r$ for some $b \in R$ such that $aR \cap eR = 0$. Now we consider $e = rb \in idem(R)$. As, R is an abelian ring, we see $(re + (1-e))(be + (1-e)) = rebe + 1 - e = rbe + 1 - e = 1$. Again $(be + (1-e))(re + 1 - e) = bere + 1 - e = bre + 1 - e = (br)rb + 1 - e = rbrb + 1 - e = 1$. So $u = re + (1-e) \in U(R)$ and $eu = ere = er = rbr = r$. Therefore ,$u = re + f = ere + f = eue + f = eu + f$ is unit $\implies -(eu + f)$ is unit, where $f = 1 - e$. Now we have a clean decomposition $-r = f + (-(eu + f))$. By Lemma 2.5 $r + e'$ is clean $\implies$ x is clean and consequently x is special clean. □

Let $f(x) \in R[[x, \sigma]]$ be a regular element. Then there exist $g(x) \in R[[x, \sigma]]$ such that $f(x)g(x)f(x) = f(x)$. Then degree of the polynomial $f(x)g(x)f(x)$ is obviously greater than the degree of $f(x)$ unless $g(x)f(x) = id$ or, $f(x)g(x) = id$. But it is well known that $U(R) = U(R[[x, \sigma]])$ which implies $f(x)$ is a unit in $R[[x, \sigma]]$. Therefore, $reg(R[[x, \sigma]]) = U(R)$. Similarly it is easy to verify that $idem(R) = idem(R[[x, \sigma]])$.

With this observation we get the following theorem.

Theorem 3.9. *Let R be an abelian ring. Then the following statements are equivalent.*

(1) *R is a special regular clean ring.*
(2) *The twisted power series ring $\mathcal{T} = R[[x, \sigma]]$ of R is a special regular clean ring for every $\sigma \in End(R)$.*

Proof. (=>) Let R be a special regular clean and $f(x) = a_0 + a_1x + a_2x^2 + \cdots \in R[[x, \sigma]]$. By Proposition 3.8 we find that R is special clean. So, $a_0 = e_0 + u_0$ is a special clean decomposition for some $e_0 \in idem(R)$ and $u_0 \in U(R)$. Now writing $f(x) = e_0 + (u_0 + a_1x + a_2x^2 + ...)$ we get a clean decomposition which also turns out to be regular clean as $reg(\mathcal{T}) = U(R)$ and $idem(\mathcal{T}) = idem(R)$. Finally, for $f(x)\mathcal{T} \cap e_0\mathcal{T} = 0$, if $f(x) = w$ is unit or idempotent, we consider $f(x) = 0 + w$. If not then clearly the intersection of left ideal generated by $f(x)$ and e_0 is 0. Therefore, $\mathcal{T}$ is special regular clean. □

Corollary 3.10. *Let R be an abelian ring. Then R is special regular clean if and only if the power series ring $R[[x]]$ is special regular clean.*

Proof. If we consider $\sigma = id \in End(R)$ in Theorem 3.9 then we get the result. □

Theorem 3.11. *Let R be an abelian ring and δ be an inner derivation on R. Then the differential polynomial ring $R[[x, \delta]]$ is special regular clean if R is special regular clean.*

Proof. As δ is an inner derivation, there exist $r \in R$ such that $\delta(a) = ra - ar$ for every $a \in R$. Now we see $(x - r)a = xa - ra = ax + \delta(a) - ra = ax + ra - ar - ra = a(x - r)$ for every $a \in R$. By taking $y = x - r$ we get y commutes with R. Hence, $R[[x, \delta]] \cong R[[y]]$. Therefore, from Corollary 3.10 we get $R[[y]]$ is a special regular clean ring. □

Theorem 3.12. *Let the general upper triangular matrix ring of the form $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ where A and B are any ring and M be left$-A$ right$-B$ bimodule. Then A and B are special regular clean if R is special regular clean.*

Proof. Let R be special regular clean and $T = \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} + \begin{pmatrix} r_1 & m \\ 0 & r_2 \end{pmatrix} \in sp-rclean(R)$. We need to show $a = e_1 + r_1 \in A, b = e_2 + r_2 \in B$ are special regular clean. As, $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} \in idem(R)$ we have $e_1 \in idem(A)$ and $e_2 \in idem(B)$ and there exist a $\begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} \in R$ such that $\begin{pmatrix} r_1 & m \\ 0 & r_2 \end{pmatrix} \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} \begin{pmatrix} r_1 & m \\ 0 & r_2 \end{pmatrix} = \begin{pmatrix} r_1 & m \\ 0 & r_2 \end{pmatrix} \implies \begin{pmatrix} r_1y_1r_1 & r_1y_1m + my_2r_2 \\ 0 & r_2y_2r_2 \end{pmatrix} = \begin{pmatrix} r_1 & m \\ 0 & r_2 \end{pmatrix}$. Therefore, $r_1 \in reg(A)$ and $r_2 \in reg(B)$. Now $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} R \cap \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} R = 0 \implies aA \cap e_1A = 0$ and $bB \cap e_2B = 0$. Therefore, A and B are special regular clean. □

Corollary 3.13. *If the ring R of all upper(respectively lower) triangular square matrices over a ring S is special regular clean then S is special regular clean.*

Proof. The proof follows as in Theorem 3.12. □

Theorem 3.14. *Consider R as a module over itself. R is special regular clean ring if and only if $R = rR \oplus eR$, for some $r \in reg(R)$ and $e \in idem(R)$.*

Proof. Let R be a regular special clean ring. Then $a = e + r$ for some $r \in reg(R)$ and $e \in idem(R)$ such that $aR \cap eR = 0$. Replacing $a = e + r$ we have $(e + r)R \cap eR = 0 \implies (eR + rR) \cap eR = 0$. By modular law we have, $rR \cap (eR + eR) = 0 \implies rR \cap eR = 0$. As $a \in R$ is arbitary we have $R = rR \oplus eR$ for some $r \in reg(R)$ and $e \in idem(R)$. Similarly, we can prove the converse part. □

4. Conclusion

We have defined the concept of special regular clean ring using the definition of special regular clean elements. We have established several characteristics of this ring. In future we can study these rings satisfying finiteness conditions which will open up a new area of research. Moreover, this idea can be further extended to unit regular clean elements in a ring.

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