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Research Paper

SOME RESULTS ON THE SUM-ANNIHILATING ESSENTIAL SUBMODULE GRAPH

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Abstract. Consider a commutative ring R with a non-zero identity $1 \neq 0$, and let M be a non-zero unitary module over R. In this document, our goal is to present the sum-annihilating essential submodule graph $\mathbb{AE}^0_R(M)$ and its subgraph $\mathbb{AE}^1_R(M)$ of a module M over a commutative ring R which is described in the following way: The vertex set of graph $\mathbb{AE}^0_R(M)$ (resp., $\mathbb{AE}^1_R(M)$) is the collection of all (resp., non-zero proper) annihilating submodules of M and two separate annihilating submodules N and K are connected anytime $N+K$ is essential in M. We study and investigate the basic properties of graphs $\mathbb{AE}^i_R(M)$ $(i = 0, 1)$ and will present some related results. Additionally, we explore how the properties of graphs interact with the algebraic structures they represent.

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1. Introduction

In the present paper, M is a non-zero unital module over a commutative ring R with non-zero identity element. In the case of a ring R, the collection of whole ideals in R is represented by $\mathbb{I}(R)$ and also $\mathbb{I}^*(R) = \mathbb{I}(R) \setminus \{0, R\}$ is the collection of entire non-zero proper (non-trivial) ideals in R. In addition, the collection of whole submodules of M is represented by the symbol $\mathbb{S}(M)$ and $\mathbb{S}^*(M) = \mathbb{S}(M) \setminus \{0, M\}$ is the collection of entire non-zero proper submodules of M. In addition, $\mathrm{J}(R)$ will represente the Jacobson radical of R, and it is the intersection of collection of maximal ideals in R and also it is the sum of all superfluous ideals in R. If R does not have superfluous ideals, then we put $\mathrm{J}(R) = 0$. If N is a submodule of M, then the residual of N by M will represent by $(N :_R M)$. This refers to the collection of elements r in R such that when multiplied by M, the result is contained within N i.e., $rM \subseteq N$. For any subset Y of R, $\mathrm{ann}_M(Y)$ represents as the collection of elements m in M where m multiplied by a equals 0 for every $a \in Y$. In particular, for an element x in R, $\mathrm{ann}_M(x) = \{m \in M : xm = 0\}$ is named an *annihilator submodule* of M. Also, $\mathrm{ann}_R(M) = (0 :_R M)$ represents the annihilator of M. An element x in R is named a *zero-divisor* on M whenever there exists a non-zero element m in M such that $xm = 0$, i.e., $\mathrm{ann}_M(x) \neq 0$. By $\mathrm{Z}_R(M)$ (briefly, $\mathrm{Z}(M)$), we express the collection of entire zero-divisors of R on M, i.e., $\mathrm{Z}(M) = \{r \in R : \mathrm{ann}_M(r) \neq 0\}$. When R is considered as an R-module, then we use $\mathrm{Z}(R)$ as a substitute for $\mathrm{Z}_R(R)$. A non-empty subset S of R is named multiplicatively closed subset (briefly, m.c.s.) exactly when $0 \in S$, $1 \notin S$ and $xy \in S$ for all $x, y \in S$. For instance, $S = R - \mathrm{Z}(M)$ is a m.c.s. of R. For further information, we direct the reader to [5, 13, 14, 21].

A ring R has property (A), whenever each finitely generated ideal I contained in $\mathrm{Z}(R)$ has a non-zero annihilator, i.e., $\mathrm{ann}_R(I) \neq 0$, see [11, 12]. In [10], the author investigated rings with property (A) and he named them McCoy. A Noetherian ring is an instance of a McCoy ring. A McCoy module is an R-module M such that for each finitely generated ideal I of R where I is contained in $\mathrm{Z}(M)$, $\mathrm{ann}_M(I) \neq 0$. An R-module M is named *super coprimal* when for each finite subset X in $\mathrm{Z}(M)$, $\mathrm{ann}_M(X) \neq 0$.

A *prime submodule* P of M is a proper submodule such that for $r \in R$ and $m \in M$, in the event that $rm \in P$ gives the result that either $r \in (P :_R M)$ or $m \in P$. The collection of all prime submodules of M is denoted by $\mathrm{Spec}(M)$. If P is a prime submodule, then $\mathfrak{p} := (P :_R M)$ is a prime ideal of R and P is named the $\mathfrak{p}$-prime submodule of M, see [16]. Equivalently, for the ideal I of R and m in M, whenever $Im \subseteq P$, then either $I \subseteq \mathrm{ann}_R(M/P)$ or $m \in P$. Note that when Q is a maximal submodule of M, then $Q \in \mathrm{Spec}(M)$ and also $\mathfrak{m} = (Q : M) \in \mathrm{Max}(R)$ such that $\mathrm{Max}(R)$ is the set of all maximal ideals of R. In this case, we state that Q is an $\mathfrak{m}$-*maximal submodule* of M, see [15, p. 61]. The set of all minimal (resp., maximal) submodules of M is denoted by $\mathrm{Min}(M)$ (resp., $\mathrm{Max}(M)$). An R-module M

is named *prime* whenever for each non-zero submodule X of M, $\mathrm{ann}(X) = \mathrm{ann}(M)$. Also, M is a *multiplication module* whenever for each submodule N of M there exists an ideal I of R where $N = IM$. In addition, in this case, $N = (N :_R M)M$, refer to [7, 9].

Dually, M is referred to as a *comultiplication module* whenever for each submodule N of M, there exists an ideal I of R such that N is equal to the set of elements in M that are annihilated by I, i.e., $N = (0 :_M I)$, see [1]. For instance, $M = \mathbb{Z}_{2^\infty}$ as a $\mathbb{Z}$-module is comultiplication because every proper submodule of M is as $(0 :_M 2^k\mathbb{Z})$ for $k = 0, 1, \ldots$. Obviously, M is comultiplication exactly when for every submodule N of M, we have the relation $\mathrm{ann}_M(\mathrm{ann}_R(N)) = N$. The ideal I of R where $N = (0 :_M I)$ is unique when M is comultiplication and in addition, it has the double annihilator condition (briefly, DAC) that is, $\mathrm{ann}_R(\mathrm{ann}_M(I)) = I$ for each ideal I of R. Such modules are named *strong comultiplication modules.* For a positive integer n and a prime number p the $\mathbb{Z}$-modules $\mathbb{Z}_{p^\infty}$ and $\mathbb{Z}_n$ are comultiplication whereas they are not strong comultiplication, refer to [2]. By [19, Theorem 1.1], when R is *completely primary*, then every ideal of R is the annihilator of some subset of R exactly when R has a unique minimal ideal. In simple terms, a ring R is considered a *fully elemental annihilator ring* if, for every ideal I of R, there exists an element x in R such that I is equal to the set of all elements that annihilate x in R, i.e., $I = \mathrm{ann}_R(x)$. This is true exactly when R is a direct sum of completely primary principal ideal rings.

A lot of research have been done to associate graphs with algebraic structures such as rings or modules, the reader refers to [3, 4, 6, 8, 17, 18]. An ideal A of R is named an *annihilating ideal*, whenever $\mathrm{ann}_R(A) \neq 0$. It follows that there exists a non-zero ideal B of R such that $AB = 0$. The collection of all ideals with non-zero annihilators is denoted by $\mathbb{A}(R)$.

Recently in [17], the author introduced the annihilators comaximal graph of $G^*(M)$. In addition, in [18], the authors studied the comaximal colon ideal graph of $C^*(M)$.

Motivated by [3, 4, 6, 8, 17, 18], we introduce the sum-annihilating essential submodule graph $\mathbb{AE}^0_R(M)$ and its subgraph $\mathbb{AE}^1_R(M)$ as follows: The vertex set of graph $\mathbb{AE}^0_R(M)$ (resp., $\mathbb{AE}^1_R(M)$) is the collection of all (resp., non-zero proper) annihilating submodules of M. Two separate vertices $N = \mathrm{ann}_M(I)$ and $K = \mathrm{ann}_M(J)$ are connected whenever $N + K$ is essential in M. In particular, if we consider $M = R$ as an R-module, then the annihilating submodules of M are the same as the annihilating ideals of R. Additionally, two vertices $I = \mathrm{ann}_R(A)$ and $J = \mathrm{ann}_R(B)$ such that $A, B \in \mathbb{I}(R)$ are adjacent in $\mathbb{AE}^0_R(R)$ whenever $I + J$ is essential in R. In the case of, $M = R$, $\mathbb{AE}^1_R(R)$ is the subgraph of $\mathcal{E}_R$ generated by the collection of all non-trivial annihilating ideals of R. In particular, if $M = R$ is a comultiplication R-module, then $\mathbb{AE}^1_R(R)$ and $\mathcal{E}_R$ are the same. This article aims to explore certain characteristics of $\mathbb{AE}^i_R(M)$ for $i = 0, 1$.

The *diameter* of a graph G, represented as $\operatorname{diam}(G)$, is the maximum distance between each two vertices in G. The *girth* of a graph G, represented as $\operatorname{gr}(G)$, is the length of the shortest cycle in G when it contains a cycle, otherwise the girth of G is considered infinite. In a graph, a *clique* is the largest fully connected subgraph, and the number of vertices in the largest clique of graph G, represented as $\omega(G)$, is referred to as the clique number of G.

2. The sum-annihilating essential submodule graph

In this section, we present the sum-annihilating essential graph $\mathbb{AE}^0_R(M)$ and its subgraph $\mathbb{AE}^1_R(M)$ which are simple undirected graphs, with vertices set

$$\mathrm{V}(\mathbb{AE}^0_R(M)) = \{N \in \mathbb{S}(M) \mid N = \operatorname{ann}_M(I), \text{for some}\, I \in \mathbb{I}(R)\},$$

and

$$\mathrm{V}(\mathbb{AE}^1_R(M)) = \{N \in \mathbb{S}^*(M) \mid N = \operatorname{ann}_M(I), \text{for some}\, I \in \mathbb{I}^*(R)\}.$$

Two separate vertices N and K in $\mathbb{AE}^i_R(M)$ $(i = 0, 1)$ are connected only when $N + K$ is essential in M.

We start by introducing the following definition.

Definition 2.1. Let us have a non-zero module M over a ring R. A submodule N of M is considered an *annihilating submodule* of M if there is a (non-zero proper) ideal I of R such that N equals the annihilator of M with respect to I, i.e., $N = \operatorname{ann}_M(I) = (0 :_M I)$.

Clearly, $\operatorname{ann}_M(0) = M$, $\operatorname{ann}_M(R) = 0_M$ are trivial annihilating submodules of M. Particularly, if R is a principal ideal domain, then for each $a \in \mathrm{Z}(M)$, $\operatorname{ann}_M(a) = \operatorname{ann}_M(Ra) \neq 0$, is an annihilating submodule of M.

Definition 2.2. Consider M as an R-module.

(i) The *sum-annihilating essential submodule graph* of M, represented as $\mathbb{AE}^0_R(M)$ is an undirected graph with the vertex collection of entire annihilating submodules of M and two different vertices $N = \operatorname{ann}_M(I)$ and $K = \operatorname{ann}_M(J)$ are connected in $\mathbb{AE}^0_R(M)$, whenever $N + K$ is an essential submodule of M.

(ii) The *strong sum-annihilating essential submodule graph* of M, denoted by $\mathbb{AE}^1_R(M)$ is a simple undirected graph with the vertex set of all non-trivial annihilating submodules of M and two distinct vertices N and K are adjacent in $\mathbb{AE}^1_R(M)$, whenever $N + K$ is an essential submodule of M.

Clearly, $\mathbb{AE}^0_R(M)$ is a star graph that has universal vertex $M = \operatorname{ann}_M(0)$, because for each annihilating submodule N of M, the sum of N and M equals M is essential in M. Moreover, $\mathbb{AE}^0_R(M)$ is not an empty graph, since $0 - M$ is an edge. Also, $\mathbb{AE}^1_R(M)$ is a subgraph of $\mathbb{AE}^0_R(M)$ where does not take zero submodule and M to be vertices of $\mathbb{AE}^1_R(M)$. If there is no confusion regarding the ring we will write $\mathbb{AE}^i(M)$ instead of $\mathbb{AE}^i_R(M)$ for $i = 0, 1$. In particular, when we view R as an R-module, we use $\mathbb{AE}^i(R)$ instead of $\mathbb{AE}^i_R(R)$ for $i = 0, 1$. We present the degrees of vertex N in $\mathbb{AE}^0(M)$ and $\mathbb{AE}^1(M)$, respectively by $\deg_0(N)$ and $\deg_1(N)$.

Note 2.3. *Let M be a non-zero R-module.*

(i) *For each $a \in R$, $\mathrm{ann}_M(a) = \mathrm{ann}_M(Ra) \neq 0$ is a vertex in $\mathbb{AE}^1_R(M)$ if and only if $a \in \mathrm{Z}(M) \setminus \{0\}$.*

(ii) *In general, $\mathrm{Z}(M)$ may not be an ideal of R for an R-module M. For instance, consider $M = \mathbb{Z}_2 \times \mathbb{Z}_3$ as a $\mathbb{Z}$-module. Then one can check that $\mathrm{Z}(M) = 2\mathbb{Z} \cup 3\mathbb{Z}$. Of course, in this case, since $\mathbb{Z}$ is a PID, M is McCoy thus, for each finitely generated ideal $I \subseteq \mathrm{Z}(M)$, $\mathrm{ann}_M(I) \neq 0$ is a vertex of $\mathbb{AE}^0_R(M)$.*

Example 2.4. (i) For a simple R-module M, $\mathbb{AE}^0_R(M)$ is of the form $0 - M$, and $\mathbb{AE}^1_R(M)$ is the null graph.

(i) Let us propose $M = \mathbb{Z}$ as an $\mathbb{Z}$-module. Then $(0 :_{\mathbb{Z}} k\mathbb{Z}) = 0$ for each ideal $k\mathbb{Z}$ in $\mathbb{Z}$ with $0 \neq k \in \mathbb{N}$ and for $k = 0$, $(0 :_{\mathbb{Z}} 0) = \mathbb{Z}$. Therefore, $\mathbb{Z}$ has no non-trivial annihilating submodule as a $\mathbb{Z}$-module. Thus, $\mathbb{AE}^0_{\mathbb{Z}}(\mathbb{Z})$ has only two vertices 0 and $\mathbb{Z}$ and only an edge $0 - \mathbb{Z}$. Also, $\mathbb{AE}^1_{\mathbb{Z}}(\mathbb{Z})$ is a null graph.

(ii) Consider $M = \mathbb{Z}_6$ as an $\mathbb{Z}$-module. Then $\langle \bar{2} \rangle = \mathrm{ann}_{\mathbb{Z}_6}(3\mathbb{Z})$, and $\langle \bar{3} \rangle = \mathrm{ann}_{\mathbb{Z}_6}(2\mathbb{Z})$ are non-trivial annihilating submodules of $\mathbb{Z}_6$. Clearly, $\langle \bar{2} \rangle + \langle \bar{3} \rangle = \mathbb{Z}_6$ which is an essential submodule of $\mathbb{Z}_6$. Therefore, $\mathbb{AE}^1_{\mathbb{Z}}(\mathbb{Z}_6)$ is the graph with only an edge $\langle \bar{2} \rangle - \langle \bar{3} \rangle$, see Figure 1.

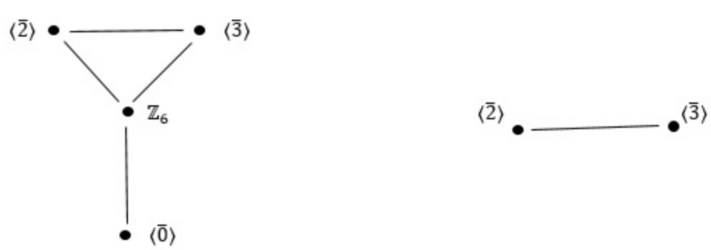


FIGURE 1. $\mathbb{AE}^0_{\mathbb{Z}}(\mathbb{Z}_6)$ $\mathbb{AE}^1_{\mathbb{Z}}(\mathbb{Z}_6)$.

In general, take $M = \mathbb{Z}_{p_1 \cdots p_s}$ as a $\mathbb{Z}$-module such that all p_i's $(1 \leq i \leq s)$ are distinct prime numbers, then $\langle \bar{p_i} \rangle = \mathrm{ann}_M(p_1 \cdots p_{i-1} p_{i+1} \cdots p_s \mathbb{Z})$ is a non-trivial annihilating submodule of M for every $1 \leq i \leq s$ and $\langle \bar{p_i} \rangle + \langle \bar{p_j} \rangle = M$ for each $1 \leq i \neq j \leq s$. Hence, the subgraph of $\mathbb{AE}^1_{\mathbb{Z}}(M)$ generated by $\{\langle \bar{p_1} \rangle, \cdots, \langle \bar{p_s} \rangle\}$ is the maximal complete subgraph of $\mathbb{AE}^1_{\mathbb{Z}}(M)$. In fact, $\mathbb{AE}^1_{\mathbb{Z}}(M)$ has the maximal complete subgraph isomorphic to K_s. So, $\omega(\mathbb{AE}^0_{\mathbb{Z}}(M)) = s + 1$ and $\omega(\mathbb{AE}^1_{\mathbb{Z}}(M)) = s$, because the subgraph of $\mathbb{AE}^0_{\mathbb{Z}}(M)$ generated by $\{\langle \bar{p_1} \rangle, \cdots, \langle \bar{p_s} \rangle, M\}$ is the maximal complete subgraph. For example, we have $\omega(\mathbb{AE}^1_{\mathbb{Z}}(\mathbb{Z}_6)) = 2$ and $\omega(\mathbb{AE}^0_{\mathbb{Z}}(\mathbb{Z}_6)) = 3$.

(iii) Consider the uniserial $\mathbb{Z}$-module $M = \mathbb{Z}_{16}$. Then $\langle\bar{2}\rangle = \mathrm{ann}_{\mathbb{Z}_{16}}(8\mathbb{Z})$, $\langle\bar{4}\rangle = \mathrm{ann}_{\mathbb{Z}_{16}}(4\mathbb{Z})$, and $\langle\bar{8}\rangle = \mathrm{ann}_{\mathbb{Z}_{16}}(2\mathbb{Z})$ are all non-trivial annihilating submodules of M. One can check that the graphs $\mathbb{AE}^0_{\mathbb{Z}}(\mathbb{Z}_{16})$ and $\mathbb{AE}^1_{\mathbb{Z}}(\mathbb{Z}_{16})$ are as in Figure 2.

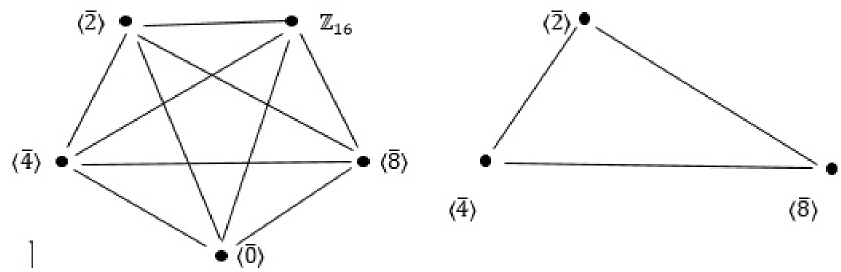


FIGURE 2. $\mathbb{AE}^0_{\mathbb{Z}}(\mathbb{Z}_{16})$ $\mathbb{AE}^1_{\mathbb{Z}}(\mathbb{Z}_{16})$.

Example 2.5. (i) Take $M = \mathbb{Z}_{12}$ as a $\mathbb{Z}$-module. Then $\langle\bar{2}\rangle = \mathrm{ann}_M(6\mathbb{Z})$, $\langle\bar{3}\rangle = \mathrm{ann}_M(4\mathbb{Z})$, $\langle\bar{4}\rangle = \mathrm{ann}_M(3\mathbb{Z})$ and $\langle\bar{6}\rangle = \mathrm{ann}_M(2\mathbb{Z})$. One can check that the graphs $\mathbb{AE}^i_{\mathbb{Z}}(\mathbb{Z}_{12})$ $(i = 0, 1)$ are as Figure 3.

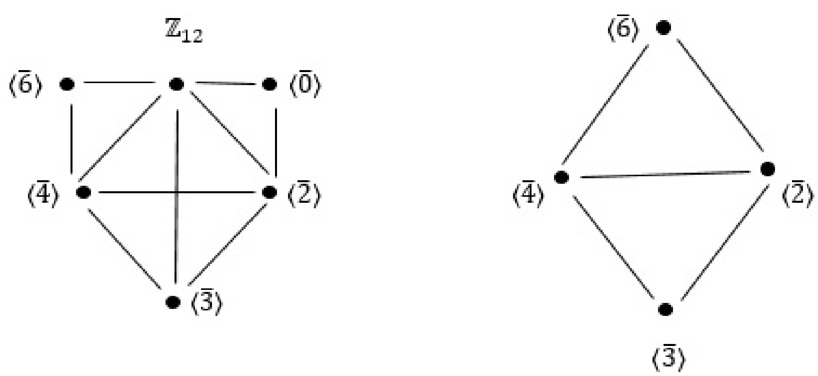
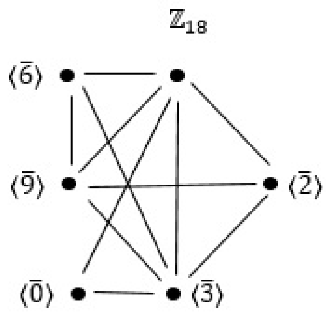


FIGURE 3. $\mathbb{AE}^0_{\mathbb{Z}}(\mathbb{Z}_{12})$ $\mathbb{AE}^1_{\mathbb{Z}}(\mathbb{Z}_{12})$.

(ii) Take $M = \mathbb{Z}_{18}$ as a $\mathbb{Z}$-module. Then $\langle\bar{2}\rangle = \mathrm{ann}_M(9\mathbb{Z})$, $\langle\bar{3}\rangle = \mathrm{ann}_M(6\mathbb{Z})$, $\langle\bar{6}\rangle = \mathrm{ann}_M(3\mathbb{Z})$ and $\langle\bar{9}\rangle = \mathrm{ann}_M(2\mathbb{Z})$. Clearly, $\langle\bar{3}\rangle$ is the only proper essential submodule of M. Also, one can check that the graphs $\mathbb{AE}^i_{\mathbb{Z}}(\mathbb{Z}_{18})$ $(i = 0, 1)$ are as Figure 4.

Proposition 2.6. *Let us see $M = \mathbb{Z}_{2^n}$ as a $\mathbb{Z}$-module. Then we have $\omega(\mathbb{AE}^0_{\mathbb{Z}}(M)) = n + 1$ and $\omega(\mathbb{AE}^1_{\mathbb{Z}}(M)) = n - 1$.*

Proof. Note that the uniserial $\mathbb{Z}$-module $M = \mathbb{Z}_{2^n}$ is an Artinian $\mathbb{Z}$-module such that $M \supset \langle\bar{2}\rangle \supset \langle\bar{4}\rangle \supset \cdots \supset \langle\overline{2^{n-1}}\rangle \supset 0$ is the only chain of all its submodules. One can check that $\mathbb{AE}^0_{\mathbb{Z}}(M) \cong K_{n+1}$ and $\mathbb{AE}^1_{\mathbb{Z}}(M) \cong K_{n-1}$, as needed. □



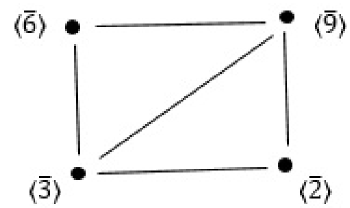


FIGURE 4. $\mathbb{AE}^0_{\mathbb{Z}}(\mathbb{Z}_{18})$ $\mathbb{AE}^1_{\mathbb{Z}}(\mathbb{Z}_{18})$.

Theorem 2.7. *Suppose that M is a non-zero R-module. Then,*

(i) *An ideal I of R is a vertex of $\mathbb{AE}^1_R(M)$ whenever $I \subseteq \mathrm{Z}(M)$.*

(ii) *In the case of M is a MacCoy R-module, then for each finitely generated ideal I of R with $I \subseteq \mathrm{Z}(M)$, $\mathrm{ann}_M(I)$ is a vertex in $\mathbb{AE}^1_R(M)$. In particular, in a finitely generated R-module M such that R is a Noetherian ring, $I \subseteq \mathrm{Z}(M)$ implies that $\mathrm{ann}_M(I)$ is a vertex in $\mathbb{AE}^1_R(M)$.*

Proof. (i) If $I \nsubseteq \mathrm{Z}(M)$, then there exists a non-zero element $a \in I \cap (R - \mathrm{Z}(M))$. Consequently, if $\mathrm{ann}_M(a) = 0$, it follows that $\mathrm{ann}_M(I) = 0$. Therefore, $\mathrm{ann}_M(I)$ does not belong to the set of vertices of $\mathbb{AE}^1_R(M)$.
(ii) The first statement follows from definition. The second part is obtained by [13, Theorem 82], because when R is a Noetherian ring, then each finitely generated R-module M is a MacCoy module. □

Corollary 2.8. *If R is a ring with property (A), then for each finitely generated ideal $I \subseteq \mathrm{Z}(R)$, $\mathrm{ann}_R(I)$ is a vertex of $\mathbb{AE}^0(R)$.*

Corollary 2.9. *Let M be a super coprimal R-module. Then for every finitely generated ideal I of R with $I \subseteq \mathrm{Z}(M)$, $\mathrm{ann}_M(I)$ is a vertex in $\mathbb{AE}^0_R(M)$.*

Proof. The evidence is evident as each R-module that is super coprimal is also a McCoy R-module. □

Example 2.10. Consider $R = \mathbb{Z}_2 \times \mathbb{Z}_2$. Then R is a MacCoy ring and also $\mathrm{Z}(R) = \{(0,0),(1,0),(0,1)\}$. Note that all proper ideals of R are as follows: $I_1 = \{(0,0)\}$, $I_2 = \{(0,0),(1,0)\}$ and $I_3 = \{(0,0),(0,1)\}$. Also, $I_i \subseteq \mathrm{Z}(R)$ and $\mathrm{ann}_R(I_i) \neq 0$ for $i = 1,2,3$. Thus,

R is a McCoy ring. For $I_2 = \langle(1,0)\rangle$ and $I_3 = \langle(0,1)\rangle$, we have $\mathrm{ann}_R(I_2) = 0 \times \mathbb{Z}_2 = I_3 \neq 0_R$ and $\mathrm{ann}_R(I_3) = \mathbb{Z}_2 \times 0 = I_2 \neq 0_R$. Note that $I_2 + I_3 = \langle(1,0),(0,1)\rangle = R$ and $\mathrm{ann}(I_2 + I_3) = \mathrm{ann}(R) = 0$. Thus, $\mathrm{ann}(I_2 + I_3)$ is not a vertex of $\mathbb{AE}^1(R)$. Clearly, $\mathrm{ann}(I_2) + \mathrm{ann}(I_3) = I_3 + I_2 = R$ is essential in R. Since $I_2 \cap I_3 = \{(0,0)\} = 0_R$, hence neither I_2 nor I_3 is not essential ideal in R, see Figure 5.

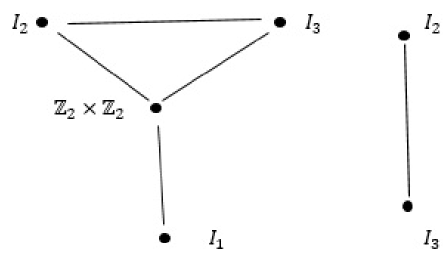


FIGURE 5. $\mathbb{AE}^0(\mathbb{Z}_2 \times \mathbb{Z}_2)$ $\mathbb{AE}^1(\mathbb{Z}_2 \times \mathbb{Z}_2)$.

Proposition 2.11. *Suppose M is a non-zero semisimple R-module. Then, two different annihilating submodules N and K of M are connected in $\mathbb{AE}^i_R(M)$ $(i = 0,1)$ whenever $N + K = M$. Moreover, since M is comultiplication, then $\mathbb{AE}^i_R(M)$ $(i = 0,1)$ has no isolated vertex.*

Proof. (i) The initial portion is evident, as a semisimple module does not have any proper essential submodule. For second part, since M is comultiplication, so $\mathrm{V}(\mathbb{AE}^0_R(M)) = \mathbb{S}(M)$ and $\mathrm{V}(\mathbb{AE}^1_R(M)) = \mathbb{S}^*(M)$. Assume, $N = \mathrm{ann}_M(I)$, then there exists a submodule $K = \mathrm{ann}_M(J)$ of M with $N \oplus K = M$ where I and J are two separate ideals of R. By the first part, N is adjacent to K in $\mathbb{AE}^i_R(M)$ $(i = 0,1)$. It implies that for every $N \in \mathrm{V}(\mathbb{AE}^i_R(M))$, $\deg_i(N) \geq 1$ for $i = 0,1$. □

Recall that a ring where every two ideals are comparable is named a *chained ring*. For instance, localization of $\mathbb{Z}$ at each prime ideal or furthermore generally every valuation domain is a chained ring.

Proposition 2.12. *Let M be a comultiplication R-module, $N \in \mathrm{Max}(M)$. Then for every $m \in M \setminus N$ we have,*

(i) *If M is not a cyclic R-module, then $Rm - N$ is an edge of $\mathbb{AE}^1_R(M)$.*
(ii) *If R is a chained ring, then $Rm \notin \mathrm{V}(\mathbb{AE}^1_R(M))$.*

Proof. (i) Clearly, $Rm = \mathrm{ann}_M(\mathrm{ann}_R(Rm))$ and $N = \mathrm{ann}_M(\mathrm{ann}_R(N))$ are non-trivial submodules of M. Since $Rm + N = M$ is essential in M, so the proof is complete. Moreover, $\deg_1(N) \geq |\{m : m \in M \setminus N\}|$.

(ii) We emphasize that every comultiplication module M over a chained ring R is a compariable module. According to $Rm \nsubseteq N$, so $N \subseteq Rm$ and so $Rm = N + Rm = M$ is not a vertex of $\mathbb{AE}^1_R(M)$. $\square$

Theorem 2.13. *Let M be a non-zero R-module and $\text{ann}_M(I) \leq^e M$ for some proper ideal I of R. Then,*

(i) $\text{diam}(\mathbb{AE}^i_R(M)) \leq 2$ *for* $(i = 0, 1)$.

(ii) *Suppose M satisfies* DAC. *Whenever I and J are two separate comparable ideals in R, then* $\text{ann}_M(I)$ *and* $\text{ann}_M(J)$ *are adjacent in* $\mathbb{AE}^i_R(M)$ $(i = 0, 1)$. *Moreover, in the case that M is a uniserial module, then $\mathbb{AE}^i_R(M)$ forms a complete graph for $i = 0, 1$.*

(iii) *For every summand I in R such as J,* $\text{ann}_M(J)$ *is not a vertex of* $\mathbb{AE}^1_R(M)$.

Proof. (i) Note that $\text{ann}_M(I)$ is a universal vertex in $\mathbb{AE}^i_R(M)$ for $i = 0, 1$, because for every vertex $\text{ann}_M(J)$ of $\mathbb{AE}^i_R(M)$ $(i = 0, 1)$, $\text{ann}_M(I) + \text{ann}_M(J)$ is an essential submodule of M. In fact, we have

$$\deg_i(\text{ann}_M(I)) = |\text{V}(\mathbb{AE}^i_R(M))| - 1 \ \ (i = 0, 1).$$

Now if $N = \text{ann}_M(J)$ and $K = \text{ann}_M(T)$ are two distinct annihilating submodules of M, then $N - \text{ann}_M(I) - K$ is a path. Thus, $\mathbb{AE}^i_R(M)$ is a connected graph and $\text{diam}(\mathbb{AE}^i_R(M)) \leq 2$ for $(i = 0, 1)$.

(ii) Let $J \subsetneq I$, then $\text{ann}_M(I) \subsetneq \text{ann}_M(J)$. By assumption, $\text{ann}_M(J) \leq^e M$. Clearly, $\text{ann}_M(J) \neq 0$, so $\text{ann}_M(J) \in \text{V}(\mathbb{AE}^1_R(M))$. Due to this $\text{ann}_M(I) + \text{ann}_M(J) = \text{ann}_M(J)$, it is essential in M, so $\text{ann}_M(I) - \text{ann}_M(J)$ is an edge in $\mathbb{AE}^0_R(M)$. Now if $I \subsetneq J$ (especially, $J = R$) for some ideal J of R, then $\text{ann}_M(I) + \text{ann}_M(J) = \text{ann}_M(I)$ is again an essential submodule of M, as needed. The second part is clear, see Example 2.4 (iii).

(iii) Assume, $R = I + J$ for some ideal J of R. Be careful that, $\text{ann}_M(I) \cap \text{ann}_M(J) = \text{ann}_M(I + J) = 0$. By assumption, $\text{ann}_M(J) = 0$ since $\text{ann}_M(I) \leq^e M$. So $\text{ann}_M(J)$ is not a vertex of $\mathbb{AE}^1_R(M)$. $\square$

Corollary 2.14. *Let M be a strong comultiplication R-module under condition* $\text{J}(R) \neq 0$. *Then three parts of Theorem 2.13 are true.*

Proof. Since $\text{J}(R) \neq 0$, so R has a non-zero superfluous submodule J. Set $N = \text{ann}_M(J)$. Claim that N is a non-zero essential submodule of M only when J is a superfluous ideal of R. Clearly, since M is strong comultiplication and $J \neq R$, so N is a non-zero submodule in M. Propose that $N \cap L = 0$ for some submodule L of M. Based on the assumption, there is an ideal X in R such that $L = \text{ann}_M(X)$ and so $N \cap L = \text{ann}_M(J) \cap \text{ann}_M(X) = \text{ann}_M(J + X) = 0$. By hypothesis, $J + X = \text{ann}_R(\text{ann}_M(J + X)) = R$. Since J is superfluous, hence $X = R$ and so $L = 0$, as needed. The converse is similar. Hence, N is essential in M and the conditions of Theorem 2.13 satisfy. $\square$

Corollary 2.15. *If M is a non-zero uniform R-module, then $\mathbb{AE}^i_R(M)$ $(i = 0, 1)$ are complete graphs.*

Note that Corollary 2.15 is not established for comultiplication modules, refer to Example 2.4 (ii).

Corollary 2.16. *Let M be a non-zero R-module. If one of the following situations holds, then $\mathbb{AE}^1_R(M)$ is a null graph and $\mathbb{AE}^0_R(M)$ is the complete graph $0 - M$.*

(i) *R is a field.*
(ii) *M is simple.*
(iii) *M is a strong comultiplication R-module with* $\mathrm{J}(R) = 0$

Theorem 2.17. *Consider M as a non-zero R-module with* DAC. *In addition, let there exists an ideal I of R with* $\mathrm{card}(I) \geq 2$ *such that* $\mathrm{ann}_M(I) \leq^e M$ *for some proper ideal I of R. Then* $\mathrm{gr}(\mathbb{AE}^i_R(M)) = 3$.

Proof. Assume that $\{a, b\}$ is a subset of I. By assumption, $\mathrm{ann}_M(a)$ and $\mathrm{ann}_M(b)$ are distinct annihilating essential submodules in M. Let $T \cap (\mathrm{ann}_M(a) + \mathrm{ann}_M(b)) = 0$ for some submodule T of M. Then $T \cap \mathrm{ann}_M(I) = 0$. By assumption, $T = 0$. It conclude that $\mathrm{ann}_M(a) - \mathrm{ann}_M(b)$ is an edge. Hence, $\mathrm{ann}_M(a) - \mathrm{ann}_M(b) - \mathrm{ann}_M(I) - \mathrm{ann}_M(a)$, is a triangle, as needed. $\square$

Corollary 2.18. *Let M be a strong comultiplication R-module. In addition, if there are non-comparable ideals I and J in R such that $I + J$ is superfluous in R, then* $\mathrm{gr}(\mathbb{AE}^i_R(M)) = 3$.

Proof. First let $i = 0$, then clearly $0 - \mathrm{ann}_M(I) - M - 0$ is a triangle in $\mathbb{AE}^0_R(M)$, since $\mathrm{ann}_M(I)$ is essential in M. In the case, $i = 1$, suppose that $x \in I \setminus J$ and $y \in J \setminus I$. In virtue of Corollary 2.14, $\mathrm{ann}_M(x)$, $\mathrm{ann}_M(y)$ and $\mathrm{ann}_M(I + J)$ are essential submodules of M. Then the proof results from Theorem 2.17 and Corollary 2.14, because $\{x, y\} \subseteq I + J$. $\square$

Lemma 2.19. *If $I \in \mathbb{I}(R)$, then* $\mathrm{ann}_R(M/\mathrm{ann}_M(I)) = \mathrm{ann}_R(IM)$.

Proof. Clearly, if I is a subset of $\mathrm{ann}_R(M)$, then $IM = 0$ and the proof is clear. Now assume that $I \nsubseteq \mathrm{ann}_R(M)$ and $r \in \mathrm{ann}_R(M/\mathrm{ann}_M(I))$. Then, $rM \subseteq \mathrm{ann}_M(I)$ and so $rIM = 0$. Hence, $r \in \mathrm{ann}_R(IM)$. The converse is similar. $\square$

Theorem 2.20. *If* $\mathrm{ann}_M(I)$ *is a prime submodule of* M *such that* $I^2 \nsubseteq \mathrm{ann}_R(M)$*, then* $\mathrm{ann}_M(I)$ *is the collection of all elements* m *in* M *where* $rm \in \mathrm{ann}_R(IM)M$ *for some* $r \in R \setminus \mathrm{ann}_R(IM)$*. Furthermore,* $\mathrm{ann}_M(I)$ *is a minimal prime submodule of* M*.*

Proof. By assumption, $\mathrm{ann}_M(I)$ is a prime submodule of M, so $\mathfrak{p} = (\mathrm{ann}_M(I) :_R M) = \mathrm{ann}_R(M/\mathrm{ann}_M(I))$ is a prime ideal of R. By Lemma 2.19, $\mathrm{ann}_R(IM) = \mathfrak{p} \in \mathrm{Spec}(R)$. Let $H := \{m \in M : rm \in \mathfrak{p}M \textit{ for some } r \notin \mathfrak{p}\}$ and $m \in H$. Then, there exists $s \in R \setminus \mathfrak{p}$ such that $sm \in \mathfrak{p}M = \mathrm{ann}_R(IM)M$. This implies that $sm = \sum_{i=1}^{k} s_i m_i$, where $s_i \in \mathfrak{p}$ and $m_i \in M$ for $1 \leq i \leq k$. Thus, $sIm = \sum_{i=1}^{k} s_i I m_i = 0$ and so $sm \in \mathrm{ann}_M(I)$. Since $s \notin \mathfrak{p}$, it follows that $m \in \mathrm{ann}_M(I)$. Therefore, $H \subseteq \mathrm{ann}_M(I)$. Conversely, let $m \in \mathrm{ann}_M(I)$. Then, $Im = 0 \subseteq \mathfrak{p}M$. If $I \nsubseteq \mathfrak{p}$, then there exists an element r in $I \setminus \mathfrak{p}$ so that $0 = rm \in \mathfrak{p}M$ and so $m \in H$. Now, if $I \subseteq \mathfrak{p} = \mathrm{ann}_R(IM)$, then $I^2M = 0$ and so $I^2 \subseteq \mathrm{ann}_R(M)$, a contradiction. Assume that $P \in \mathrm{Spec}(M)$ and $P \subseteq \mathrm{ann}_M(I)$. Let $m \in \mathrm{ann}_M(I)$. Then, $Im = 0 \subseteq P$ which implies that $I \subseteq \mathrm{ann}_R(M/P)$ or $m \in P$. If $IM \subseteq P \subseteq \mathrm{ann}_M(I)$, then $I^2 \subseteq \mathrm{ann}_R(M)$, a contradiction. It implies that, $m \in P$ and so $P = \mathrm{ann}_M(I)$. □

Theorem 2.21. *Let* M *be an* R*-module. Then,*

(i) *Assume that for some proper non-nilpotent ideal* I *of* R*,* $\mathrm{ann}_M(I)$ *is essential in* M*. When* R *is an Artinian ring or* M *is a Noetherian module, then* $\mathbb{AE}^i_R(M)$ $(i = 0, 1)$ *contains a complete subgraph.*

(ii) *If* $IJ = 0$ *for some ideal* J *in* $\mathbb{I}^*(R)$*, then* $\mathrm{ann}_M(I) - \mathrm{ann}_M(J)$ *is an edge of* $\mathbb{AE}^i_R(M)$*.*

(iii) *If* I *is a finitely generated ideal of* R *and* I *is a subset of* $\mathrm{rad}(\mathrm{ann}_R(M))$*, then* $\mathbb{AE}^i_R(M)$ $(i = 0, 1)$ *has a universal vertex.*

(iv) *Let* $a, b \in R$*. If* $ab \notin \mathrm{rad}(\mathrm{ann}_R(M))$ *and* $\mathrm{ann}_M(ab)$ *is a prime submodule of* M*, then* $\mathrm{ann}_M(a)$ *is not connected to* $\mathrm{ann}_M(b)$ *in* $\mathbb{AE}^i_R(M)$*.*

Proof. (i) Consider the descending chain $I \supseteq I^2 \supseteq I^3 \supseteq \cdots$ from the ideals of R such that $I \in \mathbb{I}^*(R)$. According to assumption, there exists the smallest natural number $t \in \mathbb{N}$ such that $I^t = I^{t+k}$ for $k \geq 1$. Then $0 \subsetneq \mathrm{ann}_M(I) \subseteq \mathrm{ann}_M(I^2) \subseteq \cdots \subseteq \mathrm{ann}_M(I^t)$ is an ascending chain of submodules of M. By assumption, for every $1 \leq s \leq t$, $\mathrm{ann}_M(I^s)$ is an essential submodule of M. Thus for every $1 \leq i \neq j \leq t$, $\mathrm{ann}_M(I^i) - \mathrm{ann}_M(I^j)$ is an edge of $\mathbb{AE}^i_R(M)$ and so $\mathbb{AE}^i_R(M)$ contains the complete subgraph K_t. For a Noetherian R-module M the proof is similar.

(ii) For each $I \in \mathbb{I}^*(R)$, $IM + \mathrm{ann}_M(I)$ is essential in M. Let N be a submodule of M and $I \in \mathbb{I}^*(R)$. Then, $IN \subseteq IM \cap N \subseteq (IM + \mathrm{ann}_M(I)) \cap N$. If $(IM + \mathrm{ann}_M(I)) \cap N = 0$, then $IN = 0$ which implies that $N \subseteq \mathrm{ann}_M(I)$. Hence, $N \subseteq (IM + \mathrm{ann}_M(I)) \cap N$ and so $N = 0$. Therefore, $IM + \mathrm{ann}_M(I)$ is essential in M. Let $IJ = 0$ for some ideal J of R so $IJM = 0$,

thus $IM \subseteq \mathrm{ann}_M(J)$ and so $IM+\mathrm{ann}_M(I) \subseteq \mathrm{ann}_M(J)+\mathrm{ann}_M(I)$. This implies that $\mathrm{ann}_M(I)$ is adjacent to $\mathrm{ann}_M(J)$ in $\mathbb{AE}^i_R(M)$ $(i = 0, 1)$, as needed.
(iii) By assumption, there exists the smallest number $t \in \mathbb{N}$, such that $I^tM = 0$, since I is finitely generated. Thus, $IM \subseteq \mathrm{ann}_M(I^{t-1})$ and so $IM + \mathrm{ann}_M(I) \subseteq \mathrm{ann}_M(I^{t-1})$. So, $\mathrm{ann}_M(I^{t-1})$ is an essential submodule of M by (ii). This implies that $\mathrm{ann}_M(I^{t-1}) + \mathrm{ann}_M(J)$ is essential in M for every $\mathrm{ann}_M(J) \in \mathrm{V}(\mathbb{AE}^i_R(M))$ $(i = 0, 1)$, i.e., $\mathrm{ann}_M(I^{t-1})$ is a universal vertex of $\mathbb{AE}^i_R(M)$, and the proof is complete.
(iv) Note that

$$\mathrm{ann}_M(a) + \mathrm{ann}_M(b) \subseteq \mathrm{ann}_M(Ra \cap Rb) \subseteq \mathrm{ann}_M(RaRb) = \mathrm{ann}_M(ab).$$

By virtue of [6, Theorem 5 (iii)], $\mathrm{ann}_M(ab)$ is not an essential submodule of M and so $\mathrm{ann}_M(a)+\mathrm{ann}_M(b)$ is not an essential submodule of M, as we stated. □

Corollary 2.22. *Let M be a non-zero module on a ring R with DAC and $N, K \in \mathrm{V}(\mathbb{AE}^0_R(M))$. Then,*

(i) *If $\mathrm{rad}(\mathrm{ann}_R(M))$ is not zero, then $\mathbb{AE}^0_R(M)$ contains a complete subgraph.*
(ii) *If M is comultiplication with condition $|\mathrm{Min}(M)| \geq 3$ and $\mathrm{Min}(M) \cap ess(M) \neq \emptyset$, then $\mathrm{gr}(\mathbb{AE}^1(M)) = 3$.*

Proof. (i) Assume that $a \in \mathrm{rad}(\mathrm{ann}_R(M))$, thus there exists a smallest natural number t, such that $a^tM = 0$, so $0 \neq a^iM \subseteq \mathrm{ann}_M(a^{t-i})$ for $1 \leq i \leq t-1$ and so $\mathrm{ann}_M(a^{t-i}) \in \mathrm{V}(\mathbb{AE}^0(M))$ for $1 \leq i \leq t-1$. By Theorem 2.21 (ii), $\mathrm{ann}_M(a)$ is an essential submodule of M. Now since $\mathrm{ann}_M(a) \subseteq \mathrm{ann}_M(a^i)$ for $2 \leq i \leq t-1$ so the annihilating submodules $\mathrm{ann}_M(a^i)$ $(2 \leq i \leq t-1)$ are essential submodules of M. Thus, for every $1 \leq i \neq j \leq t-1$, $\mathrm{ann}_M(a^i) + \mathrm{ann}_M(a^j)$ is an essential submodule of M and so $\mathrm{ann}_M(a^i) - \mathrm{ann}_M(a^j)$ is an edge of $\mathbb{AE}^0(M)$. Therefore $\mathbb{AE}^0(M)$ contains the complete subgraph K_{t-1}.
(ii) Let $\{K_1, K_2, K_3\} \subseteq \mathrm{Min}(M)$. By [1, Theorem 3.2], $K_i \in \mathrm{Min}(M)$ if and only if there exists $\mathfrak{m}_i \in \mathrm{Max}(R)$ such that $K_i = (0 :_M \mathfrak{m}_i) \neq 0$ for all $1 \leq i \leq 3$. Therefore $\mathrm{Min}(M) \subseteq \mathrm{V}(\mathbb{AE}^1(M))$ and since $\mathfrak{m}_i \cap \mathfrak{m}_j = 0$ for $1 \leq i \neq j \leq 3$ hence $K_i + K_j = \mathrm{ann}_M(\mathfrak{m}_i) + \mathrm{ann}_M(\mathfrak{m}_j) = \mathrm{ann}_M(\mathfrak{m}_i \cap \mathfrak{m}_j) = M$ is essential in M. Therefore $K_1 - K_2 - K_3 - K_1$ is a 3-cyclic in $\mathbb{AE}^1(M)$, as needed. Note that, if I is a non-zero ideal of R, then there is a maximal ideal $\mathfrak{m}$ such that I is contained in $\mathfrak{m}$. Thus, $0 \neq \mathrm{ann}_M(\mathfrak{m}) \subseteq \mathrm{ann}_M(I)$ and so $\mathrm{ann}_M(I)$ is a vertex of graph $\mathbb{AE}^1(M)$. □

3. Conclusions

In this paper, the basic properties of sum-annihilating essential submodule graph are examined, and related results presented. Additionally, the interaction between the graph-theoretic properties and the corresponding algebraic structures are investigated. In Definition 2.2 we represented the sum-annihilating essential submodule graph of $\mathbb{AE}^0_{\mathbb{R}}(M)$ (resp., its subgraph $\mathbb{AE}^1_{\mathbb{R}}(M)$) with the vertex set of all (resp., non-zero proper) annihilating submodules of M and two separate vertices N and K are adjacent in $\mathbb{AE}^i_R(M)$ $(i = 0, 1)$, whenever $N + K$ is an essential submodule of M.

In Examples 2.4, 2.5, 2.10, we presented some examples of such graphs. In Proposition 2.6, we expressed that for the uniserial module $M = \mathbb{Z}_{2^n}$ as a $\mathbb{Z}$-module the clique numbers of $\mathbb{AE}^0_{\mathbb{Z}}(M)$ and $\mathbb{AE}^1_{\mathbb{Z}}(M)$ are $n+1$ and $n-1$, respectively. In Theorem 2.7, we have provided conditions under which for an ideal I of R, $\mathrm{ann}_M(I)$ is a vertex of graph $\mathbb{AE}^1_R(M)$. In Corollaries 2.8, 2.9, we concluded that whenever either R is a ring with property (A) or M is a super coprimal R-module, then for each finitely generated ideal I of R such that I is a subset of $\mathrm{Z}(R)$, $\mathrm{ann}_M(I)$ is a vertex in $\mathbb{AE}^0_R(M)$. In Proposition 2.11, we demonstrated that if M is a non-zero semisimple module over R, then two distinct annihilating submodules N and K of M are connected in $\mathbb{AE}^i_R(M)$ $(i = 0, 1)$ whenever $N + K = M$. Moreover, if M is comultiplication, then $\mathbb{AE}^i_R(M)$ $(i = 0, 1)$ has no isolated vertex. In Proposition 2.12, we expressed that if M is a comultiplication module over R and N as its maximal submodule, then for every $m \in M \setminus N$ whenever M is not cyclic, $Rm - N$ is an edge of $\mathbb{AE}^1_R(M)$. Also, if R is a chained ring, then $Rm \notin \mathrm{V}(\mathbb{AE}^1_R(M))$. Among various results, in theorem 2.13, it was demonstrated that when M is a non-zero R-module with $\mathrm{ann}_M(I)$ essential in M for a certain proper ideal I in R, then $\mathrm{diam}(\mathbb{AE}^i_R(M)) \leq 2$ $(i = 0, 1)$. Three results of theorem 2.13 are Corollaries 2.14, 2.15 and 2.16 so that the last result states the conditions under which $\mathbb{AE}^1_R(M)$ is a null graph and $\mathbb{AE}^0_R(M)$ is the complete graph $0 - M$. In Theorem 2.17 and Corollary 2.18 we gave some conditions on the ring R, R-module M and ideals of R such that $\mathrm{gr}(\mathbb{AE}^i_R(M)) = 3$ $(i = 0, 1)$. In Theorem 2.20, we concluded that when $\mathrm{ann}_M(I)$ is a prime submodule of M in such a way that I^2 is not included in $\mathrm{ann}_R(M)$, then $\mathrm{ann}_M(I)$ is the set of all elements m in M where $rm \in \mathrm{ann}_R(IM)M$ for some $r \in R \setminus \mathrm{ann}_R(IM)$. Additionally, in this scenario, $\mathrm{ann}_M(I)$ is a minimal prime submodule of M.

Finally, in Theorem 2.21 among various results, we proved that if for some proper non-nilpotent ideal I of R, $\mathrm{ann}_M(I)$ is an essential submodule of M, whenever either R is an Artinian ring or M is a Noetherian module, then $\mathbb{AE}^i_R(M)$ $(i = 0, 1)$ contains a complete subgraph. As a result of this theorem in Corollary 2.22 we concluded that if M is a non-zero module on a ring R with DAC and N, K are vertices of $\mathbb{AE}^0_R(M)$ with $\mathrm{rad}(\mathrm{ann}_R(M)) \neq 0$, then $\mathbb{AE}^0_R(M)$ contains a complete subgraph. Also, in this case, if M is comultiplication with $|\mathrm{Min}(M)| \geq 3$ and $\mathrm{Min}(M) \cap ess(M) \neq \emptyset$, then $\mathrm{gr}(\mathbb{AE}^1(M)) = 3$.

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