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Research Paper

ON COMMUTING AUTOMORPHISMS AND COMMUTATOR POLYGROUPS

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Abstract. We introduce the notions of commuting automorphism and commutator polygroups. The basic question that can be arose about the set of all commuting automorphisms is that for the assumed polygroup $(P, \cdot)$, under what conditions the set of all commuting automorphism $\boldsymbol{A}(P)$ is a subgroup of $Aut(P)$. In this paper basically the answer to this question is investigated for the class of polygroups.

1. Introduction

Hyperstructure theory was born in 1934 at the 8th congress of Scandinavian Mathematicians, where Marty [18] introduced the hypergroup notion as a generalization of groups. Surveys of the theory can be found in the books of Corsini [4], Davvaz and Leoreanu-Fotea [7] and Davvaz [6]. Polygroups or quasi-canonical hypergroups were introduced by P. Corsini and later, they were studied by P. Bonansinga and Ch.G. Massouros. They satisfy all the conditions

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of canonical hypergroups, except the commutativity. Later, S.D. Comer introduced this class of hypergroups independently, using the name of polygroups. He emphasized the importance of polygroups, by analyzing them in connections to graphs, relations, Boolean and cylindric algebras. Another connection between polygroups and artificial intelligence was considered and analyzed by G. Ligozat. The double cosets hypergroups are particular quasi-canonical hypergroups and they were analyzed by K. Drbohlav, D.K. Harrison and S.D. Comer.
Bettina Eick in [12] describes a new algorithm for computing automorphism groups and checking isomorphism of nilpotent finite dimensional associative algebras over a finite field. The algorithm can also be applied to modular group algebras and thus yields a new approach for checking the modular isomorphism problem. They report on its application to the groups of order dividing 2^8 and 3^6. In this paper we investigate a generalization of notion commuting automorphism for the class of polygroups which is introduced by M. Deaconescu et.al for the class of groups [8]. Using the notion of commuting automorphism we introduce the notion of commutator polygroups and a characterization of commutator polygroups extended by groups has been investigated.

We recall here some basic notions of hypergroup theory.

Let H be a non-empty set and $P^*(H)$ be the set of all non-empty subsets of H. Let $\cdot$ be a *hyperoperation* (or *join operation*) on H that $\cdot$ is a function from $H \times H$ into $P^*(H)$. If $(a, b) \in H \times H$, its image under $\cdot$ in $P^*(H)$ is denoted by $a \cdot b$. The join operation is extended to subsets of H in a natural way that is for non-empty subsets A, B of H, $A \cdot B = \bigcup\limits_{a\in A, b\in B} a \cdot b$. The notation $a \cdot A$ is used for $\{a\} \cdot A$ and $A \cdot a$ for $A \cdot \{a\}$. Generally, the singleton $\{a\}$ is identified with its member a. The structure $(H, \cdot)$ is called a *semihypergroup* if $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in H$, which means that

$$\bigcup_{u\in a\cdot b} u \cdot c = \bigcup_{v\in b\cdot c} a \cdot v,$$

A semihypergroup is a *hypergroup* if $a \cdot H = H \cdot a = H$ for all $a \in H$. A function $f : H \to H'$ is called a *homomorphism* if $f(a \cdot b) \subseteq f(a) \circ f(b)$ for all a and b in H. We say that f is a *good homomorphism* if for all a and b in H, $f(a \cdot b) = f(a) \circ f(b)$. If $(H, \cdot)$ is a hypergroup and $\rho \subseteq H \times H$ is an equivalent relation, then for all non-empty subsets A, B of H we set

$$A \overline{\overline{\rho}} B \Leftrightarrow a\rho b, \text{ for all } a \in A, \; b \in B.$$

The relation ρ is called *strongly regular on the left* (*on the right*) if $x\rho y \Rightarrow a \cdot x \overline{\overline{\rho}} a \cdot y$ ($x\rho y \Rightarrow x \cdot a \overline{\overline{\rho}} y \cdot a$, respectively), for all $(x, y, a) \in H^3$. Moreover, ρ is called *strongly regular* if it is strongly regular on the right and on the left.

Theorem 1.1. *(Theorem 31, [4]). If $(H, \cdot)$ is a semihypergroup (hypergroup) and ρ is a strongly regular relation on H, then the quotient H/ρ is a semigroup (group) under the operation:*

$$\rho(x) \otimes \rho(y) = \rho(z), \text{ for all } z \in x \cdot y.$$

We denote $\rho(x)$ by $\bar{x}$ and instead of $\bar{x} \otimes \bar{y}$ we write $\bar{x}\bar{y}$. For all $n > 1$, we define the relation β_n on a semihypergroup H, as follows:

$$a\beta_n b \Leftrightarrow \exists (x_1, \ldots, x_n) \in H^n : \{a, b\} \subseteq \prod_{i=1}^{n} x_i,$$

and $\beta = \bigcup_{i=1}^{n} \beta_n$, where $\beta_1 = \{(x, x) \mid x \in H\}$ is the diagonal relation on H. This relation was introduced by Koskas [16] and studied mainly by Corsini [4]. Suppose that β^* is the transitive closure of β. The relation β^* is a strongly regular relation [4]. Also, we have:

Theorem 1.2. *(Freni, [14]). If H is hypergroup then $\beta = \beta^*$.*

Note that, in general, for a semihypergroup may be $\beta \neq \beta^*$. The relation β^* is the least equivalent relation on a hypergroup H, such that the quotient H/β^* is a group. The *heart* ω_H of a hypergroup H is the set of all elements x of H, for which the equivalence class $\beta^*(x)$ is the identity of the group H/β^*.

Remark 1.3. If G is a group then ω_G is the identity of G. Moreover $\beta^*(x) = \{x\}$, for all $x \in G$.

We say that A is a *complete part* of H if for any nonzero natural number n and for all $a_1, \ldots, a_n$ of H, the following implication holds:

$$A \cap \prod_{i=1}^{n} a_i \neq \emptyset \Rightarrow \prod_{i=1}^{n} a_i \subseteq A.$$

Let A be a nonempty part of H. The intersection of the parts of H which are complete and contain A is called the *complete closure* of A in H, it will be denoted by $C(A)$.

Proposition 1.4. *Let H be a hypergroup and $A = \prod_{i=1}^{n} a_i$ be a subset of H. Then we have:*

$$\omega_H \cdot A = A \cdot \omega_H = a \cdot \omega_H = \omega_H \cdot a,$$

for all $a \in A$.

Proof. Let $a \in A$, we know $C(a) = a$, thus $A \cap C(a) \neq \emptyset$ and $C(a)$ is complete, therfore $A \subseteq C(a)$, we have $A \cdot \omega_H \subseteq C(a) \cdot \omega_H = a \cdot \omega_H$. On the other hand, Let $a \in A$, thus $a \cdot \omega_H \subseteq A \cdot \omega_H$, therfore $A \cdot \omega_H = a \cdot \omega_H$. □

A hypergroup $(P, \cdot)$ is called *polygroup* if the following conditions hold:

(1) P has a scalar identity e (i.e., $e \cdot x = x \cdot e = x$, for every $x \in P$);

(2) every element x of P has a unique inverse x^{-1} in P;

(3) $x \in y \cdot z$ implies $y \in x \cdot z^{-1}$ and $z \in y^{-1} \cdot x$.

A non-empty subset K of a polygroup $(P, \cdot)$ is a *subpolygroup* of P if $x, y \in K$ implies $x \cdot y \in K$, and $x \in K$ implies $x^{-1} \in K$.

Example 1.5. The set $P = \{e, a, b, c\}$ under hyperoperation $*$ is a polygroup.

TABLE 1.

$*$	e	a	b	c
e	e	a	b	c
a	a	e, b	a, c	b
b	b	a, c	e, b	a
c	c	b	a	e

Example 1.6. (Double coset algebra) Suppose that H is a subgroup of a group G. Define a system

$$G//H = \langle \{HxH | x \in G\}, *, H, -I \rangle,$$

where $(HxH)^{-I} = Hx^{-1}H$ and $(HxH) * (HyH) = \{HxhyH | h \in H\}$. The algebra of double cosets $G//H$ is a polygroup introduced by Dresher and Ore [11].

Example 1.7. ([9]) Suppose $(G, \cdot)$ is a group and $\overline{G}$ is a set of all G conjugate classes. Both members c_i, c_j from $\overline{G}$ hyperoperation $*$ as follows consider. $c_i * c_j = \{c_k : c_k \subseteq c_i \cdot c_j\}$ In this case, $(\overline{G}, *)$ is a polygroup, which was first proposed by Campaigne in [2]. In the special case, suppose G binary group D_4 as $D_4 = \{r^0 = 1, r, r^2 = s, r^3 = t, h, hr = d, hr^2 = v, hr^3 = f\}$. In this case, because this group has five conjugation classes, $c_1 = \{1\}, c_2 = \{s\}, c_3 = \{r, t\}, c_4 = \{d, f\}, c_5 = \{h, v\}$ the table of the group. $(\overline{D_4}, *)$ is as follows.

TABLE 2.

$*$	c_1	c_2	c_3	c_4	c_5
c_1	c_1	c_2	c_3	c_4	c_5
c_2	c_2	c_1, c_2	c_3	c_4	c_5
c_3	c_3	c_3	c_1, c_2	c_5	c_4
c_4	c_4	c_4	c_5	c_1, c_2	c_3
c_5	c_5	c_5	c_4	c_3	c_1, c_2

Proposition 1.8. *([15]) In each category of the following groups we have:*

a) non-abelian p-groups G that $\frac{G}{Z(G)} \leq p^3$,

b) p-groups that have a cyclic maximal subgroup in which p is the prime odd number,

c) the category of groups that has a non-abelian minimal subgroup.

In the following, we introduce the class of $(P_{G,n}, \odot)$, the polygroups which can be considered the extension of each classes groups mentioned.

Proposition 1.9. *([21]) Suppose G is a group, and n is a natural number, in which case* $G \cap \{a_1, a_2, \cdots, a_n\} = \emptyset$, *then* $P_{G,n} = G \cup \{a_1, a_2, \cdots, a_n\}$. *The hyperoperation* $\odot$ *is as follows:*

a) $a_i \odot a_i = \{e, a_{i+1}, \cdots, a_n\}$ *and* $a_n \odot a_n = \{e\}$, *for every* $1 \leq i \leq n-1$. *Also* $a_i \odot a_j = a_j \odot a_i = a_i$ *for every* $1 \leq i < j \leq n$,

b) $e \odot x = x \odot e = x$, *for every* $x \in P_{G,n}$,

c) $a_i \odot x = x \odot a_i = x$, *for every* $1 \leq i \leq n$ *and* $x \in P_{G,n} - \{e, a_1, a_2, \cdots, a_n\}$,

d) $x \odot y = xy$, *for every* $x, y \in G$ *that* $x^{-1} \neq y$,

e) $x \odot x^{-1} = \{e, a_1, a_2, \cdots, a_n\}$, *for every* $x \in P_{G,n} - \{e, a_1, a_2, \cdots a_n\}$.

Then the hyperstructure $(P_{G,n}, \odot)$ *is a polygroup which is named a polygroup extended of group G.*

Example 1.10. Suppose G is a binary group D_4. Considering the assumptions of Example 2.3 the table of the polygroup extended $P_{D_4,1}$ as follows:

TABLE 3.

$\odot$	1	r	s	t	h	d	v	f	a_1
1	1	r	s	t	h	d	v	f	a_1
r	r	s	t	$1, a_1$	f	h	d	v	r
s	s	t	$1, a_1$	r	v	f	h	d	s
t	t	$1, a_1$	r	s	d	v	f	h	t
h	h	d	v	f	$1, a_1$	r	s	t	h
d	d	v	f	h	t	$1, a_1$	r	s	d
v	v	f	h	d	s	t	$1, a_1$	r	v
f	f	h	d	v	r	s	t	$1, a_1$	f
a_1	a_1	r	s	t	h	d	v	f	1

2. Commuting automorphism and commutator polygroups

In this section, we investigate a generalization of notion commuting automorphism for the class of polygroups and we introduce the notion of commutator polygroups and also a characterization of commutator polygroups extended by groups has been investigated.

Definition 2.1. Suppose $(P, \cdot)$ is a polygroup and α corresponds one to one from P to P. α is called an automorphism whenever in the condition $\alpha(xy) = \alpha(x)\alpha(y)$ holds true. We display all automorphisms of $(P, \cdot)$ with the symbol $Aut(P)$ and the automorphism α is called a commuting automorphism whenever the following condition holds:

$$\forall x \in P, \quad x\alpha(x) \cap \alpha(x)x \neq \emptyset.$$

We will display the set of all commuting automorphisms with the symbol $\boldsymbol{A}(P)$. It is clear that the set $\boldsymbol{A}(P)$ is closed relative to inverse. In other words, $\alpha \in \boldsymbol{A}(P)$ if and only if $\alpha^{-1} \in \boldsymbol{A}(P)$.

Example 2.2. Suppose $(P, *)$ is a four-member polygroup of Example 1.5. In this case $Aut(P) = \{i_P\}$. Because if α is an automorphism then we have: $\alpha(e) = x$ results in $\alpha(e * e) = \alpha(e) * \alpha(e)$ then $x = e$ *or* c. Also $\alpha(c) = y$ results that $x = \alpha(c * c) = \alpha(c) * \alpha(c)$. Therefore $y = c$, $x = e$. Similarly, $\alpha(a) = a$ and $\alpha(b) = b$. In this example we have $Aut(P) = \boldsymbol{A}(P) = \{i_P\}$.

The basic question that can be arose about the set of all commuting automorphisms is that for the assumed polygroup $(P, \cdot)$, under what conditions $\boldsymbol{A}(P)$ is a subgroup of $Aut(P)$. The answer to this question has been considered by researchers for the class of groups, which can be referred to the references [8, 13, 10, 17, 19]. In the following we investigate this question for the class of polygroups.

Definition 2.3. ([1]) Suppose $(P, \cdot)$ is a polygroup. For the pair $(x, y) \in P^2$, the left and right commutator are defined as follows:

a) $[x, y]_l = \{h : xy \cap hyx \neq \emptyset\}$,

b) $[x, y]_r = \{h : xy \cap yxh \neq \emptyset\}$,

c) $[x, y] = [x, y]_r \cup [x, y]_l$.

Proposition 2.4. *Suppose $(P, \cdot)$ is a polygroup. In this case the automorphism α is a commuting automorphism if and only if for every $x \in P$ we have:*

$$e \in \ [x, \alpha(x)].$$

Proof. The proof is straightforward. □

Proposition 2.5. *If $(P,\cdot)$ is a polygroup then for every $x,y \in P$ and for every commuting automorphism α we have*

a) $[x,\alpha(x)]_l \subseteq \omega_P$,

b) $[x,\alpha(x)]_r \subseteq \omega_P$,

c) $[x,\alpha(y)]_l \cdot \omega_P = [\alpha(x),y]_l \cdot \omega_P$,

d) $[x,\alpha(y)]_r \cdot \omega_P = [\alpha(x),y]_r \cdot \omega_P$.

Proof. Suppose $(P,\cdot)$ is a polygroup and $x \in P$ and α is a commuting automorphism. Since ω_P is a subpolygroup, we conclude from Proposition 1.4 and Proposition 2.4 $[x,\alpha(x)]_r \cdot \omega_P = e\cdot\omega_P = \omega_P$ and as a result $[x,\alpha(x)]_r \subseteq \omega_P$. In a similar way $[x,\alpha(x)]_l \subseteq \omega_P$. Now suppose that $x,y \in P$ and α be a commuting automorphism. In this case we have $xy\alpha(xy)\cap\alpha(xy)xy \neq \emptyset$. According to Proposition 1.4, we conclude that $xy\alpha(xy)\cdot\omega_P = \alpha(xy)xy\cdot\omega_P$. The following results are also obtained:

$$xy\alpha(x)\alpha(y)\cdot\omega_P = \alpha(x)\alpha(y)xy\cdot\omega_P.$$

Because $\frac{H}{\beta*}$ is a group we have:

$$\overline{x}\;\overline{y}\;\overline{\alpha(x)}\;\overline{\alpha(y)} = \overline{\alpha(x)}\;\overline{\alpha(y)}\;\overline{x}\;\overline{y}.$$

As a result, the following equation is obtained.

$$\overline{y}\;\overline{\alpha(x)}\;\overline{\alpha(y)}\;\overline{y}^{-1} = \overline{x}^{-1}\overline{\alpha(x)}\;\overline{\alpha(y)}\;\overline{x}.$$

But because $\overline{\alpha(y)}\;\overline{y}^{-1} = \overline{y}^{-1}\;\overline{\alpha(y)}$ and $\overline{\alpha(x)}\;\overline{x}^{-1} = \overline{x}^{-1}\;\overline{\alpha(x)}$ we have

$$\overline{y}\;\overline{\alpha(x)}\;\;\overline{y}^{-1}\;\overline{\alpha(y)} = \overline{\alpha(x)}\;\overline{x}^{-1}\;\overline{\alpha(y)}\;\overline{x},$$

$$\overline{\alpha(x)}^{-1}\overline{y}\;\overline{\alpha(x)}\;\;\overline{y}^{-1} = \overline{x}^{-1}\;\overline{\alpha(y)}\;\overline{x}\;\overline{\alpha(y)}^{-1}.$$

Therefore

$$[\overline{x},\;\overline{\alpha(y)}]_l = [\overline{\alpha(x)},\overline{y}]_l \quad , \quad [\overline{x},\;\overline{\alpha(y)}]_r = [\overline{\alpha(x)},\overline{y}]_r.$$

And we have $[x,\alpha(y)]_l\cdot\omega_P = [\alpha(x),y]_l\cdot\omega_P$, $[x,\alpha(y)]_r\cdot\omega_P = [\alpha(x),y]_r\cdot\omega_P$. □

Proposition 2.6. *If $(P,\cdot)$ is a polygroup and $\alpha,\partial \in Aut(P)$, then we have $\alpha\partial^{-1} \in \boldsymbol{A}(P)$ if and only if $e \in [\,\partial(x),\;\alpha(x)] \subseteq \omega_P$, for every x of P.*

Proof. Suppose $\alpha,\partial \in Aut(P)$ such that $\alpha\partial^{-1} \in \boldsymbol{A}(P)$ and $x \in P$. In this case, for every y of P we have $e \in [y,\alpha\partial^{-1}(y)]_r \cap [y,\alpha\partial^{-1}(y)]_l$. Now assume $y = \partial(x)$. We get $e \in [\partial(x),\alpha(x)]$. So $e \in [\partial(x),\;\alpha(x)] \subseteq \omega_P$. Conversely, $e \in [\partial(x),\;\alpha(x)] \subseteq \omega_P$. for every x of P. By putting $\partial^{-1}(x)$ instead of x, we conclude that

$$e \in [x,\alpha\partial^{-1}(x)].$$

So we have $\alpha\partial^{-1} \in \boldsymbol{A}(P)$. □

Symbolization. Suppose $(P, \cdot)$ is a polygroup. In this case, for every $\alpha \in Aut(P)$ we define a map $\overline{\alpha} : \frac{P}{\beta^*} \longrightarrow \frac{P}{\beta^*}$ with rule $\overline{\alpha}(\overline{x}) = \overline{\alpha(x)}$. Prove that $\overline{\alpha}$ is an automorphism for the group $\frac{P}{\beta^*}$. In addition, we set for every B subset of $Aut(P)$:

$$\overline{\boldsymbol{B}} = \{\overline{\alpha}{:}\alpha \in \boldsymbol{A}(P)\}.$$

Theorem 2.7. *α is a commuting automorphism if and only if $\overline{\alpha}$ is also a commuting automorphism.*

Proof. $(\Longrightarrow)$ Suppose $(P, \cdot)$ is a polygroup, $\alpha \in \boldsymbol{A}(P)$ and $\overline{x}$, $\overline{y} \in \frac{P}{\beta^*}$. If $\overline{x} = \overline{y}$ then $(h_1, h_2, \cdots, h_n)$ in P^n exists so that $\{x, y\} \subseteq h_1 h_2 \cdots h_n$ therefore $\{\alpha(x), \alpha(y)\} \subseteq \alpha(h_1)\alpha(h_2)\cdots\alpha(h_n)$ and so on $\overline{\alpha(x)} = \overline{\alpha(y)}$ and thus $\overline{\alpha}(\overline{x}) = \overline{\alpha}(\overline{y})$. In addition for $z \in xy$ we have:

$$\overline{\alpha}(\overline{x}\,\overline{y}) = \overline{\alpha}(\overline{z}) = \overline{\alpha(z)} = \overline{\alpha(xy)} = \overline{\alpha(x)}\;\overline{\alpha(y)} = \overline{\alpha}(\overline{x})\overline{\alpha}(\overline{y}).$$

So $\overline{\alpha}$ is an automorphism for the group $\frac{P}{\beta^*}$. Moreover, we have:

$$\forall x \in P, \quad x\alpha(x) \cap \alpha(x)x \neq \emptyset.$$

Then for every $\overline{x} \in \frac{P}{\beta^*}$, $\overline{x}\;\overline{\alpha}(\overline{x}) = \overline{\alpha}(\overline{x})\overline{x}$. Thus $\overline{\alpha} \in \boldsymbol{A}(\frac{P}{\beta^*})$.
$(\Longleftarrow)$ Let $\overline{\alpha} \in \boldsymbol{A}(\frac{P}{\beta^*})$ and for every $\bar{x} \in \frac{P}{\beta^*}$, $\overline{x}\;\overline{\alpha}(\overline{x}) = \overline{\alpha}(\overline{x})\overline{x}$. Then

$$\forall x \in P, \quad x\alpha(x) \cap \alpha(x)x \neq \emptyset.$$

□

Definition 2.8. Let's say the polygroup $(P, \cdot)$ applies to the cover condition whenever for every φ of $\boldsymbol{A}(\frac{P}{\beta^*})$ and any $\overline{x}$ of $\frac{P}{\beta^*}$ element $\overline{\alpha}$ exists in $\overline{\boldsymbol{A}(P)}$ such that $\varphi(\overline{x}) = \overline{\alpha}(\overline{x})$.

Theorem 2.9. *Suppose the polygroup $(P, \cdot)$ applies in the cover condition. In this case, if $\boldsymbol{A}(P)$ is a subgroup of $Aut(P)$ then $\boldsymbol{A}(\frac{P}{\beta^*})$ is a subgroup of $Aut(\frac{P}{\beta^*})$.*

Proof. Suppose $\varphi, \psi \in \boldsymbol{A}(\frac{P}{\beta^*})$. Because the inverse of each member of $\boldsymbol{A}(\frac{P}{\beta^*})$ is in itself. It is enough to prove that $\varphi\psi \in \boldsymbol{A}(\frac{P}{\beta^*})$. If $\overline{x} \in \frac{P}{\beta^*}$ then elements $\overline{\alpha}$ and $\overline{\mu}$ are in $\overline{\boldsymbol{A}(P)}$ so that $\overline{\alpha}(\psi(\overline{x})) = \varphi(\psi(\overline{x}))$, $\overline{\mu}(\overline{x}) = \psi(\overline{x})$. Since $\boldsymbol{A}(P)$ is a subgroup of $Aut(P)$, we conclude that $\overline{\boldsymbol{A}(P)}$ also is subgroup of $Aut(\frac{P}{\beta^*})$, so we have:

$$\overline{x}\varphi\psi(\overline{x}) = \overline{x}\;\overline{\alpha}(\psi(\overline{x})) = \overline{x}\;\overline{\alpha}(\overline{\mu}(\overline{x})) = \overline{\alpha}(\overline{\mu}(\overline{x}))\overline{x} = \overline{\alpha}(\psi(\overline{x}))\overline{x} = \varphi(\psi(\overline{x}))\overline{x} = \varphi\psi(\overline{x})\overline{x}.$$

□

Example 2.10. Suppose $(P, *)$ is the polygroup of Example 1.5. In this case $Aut(P) = A(P) = \{i_p\}$. We have (P/β^*) isomorphism is by $\mathbb{Z}_2$ therefore $Aut(P/\beta^*) = A(P/\beta^*) = \{i_{p/\beta^*}\}$.

Definition 2.11. Suppose $(P, \cdot)$ is a polygroup and $x \in P$. Then

a) the element x is called central whenever for y from P, $xy = yx$,

b) display the centralizer x with $C(\ x)$, which is $C(\ x) = \{y \in P : xy = yx\}$,

c) we represent the weak centralizer x with $C_w(x)$, which is $C_w(x) = \{y \in P : xy \cap yx \neq \emptyset\}$,

d) the non-central element x is called the commutator when $C(x)$ is a commutative subpolygroup. We also call the non-central element x is called the weak commutator when $C_w(x)$ is commutative weak subpolygroup. That is, for every a, b from the subpolygroup $C_w(x)$, the condition $ab \cap ba \neq \emptyset$ holds.

Example 2.12. Suppose G is a binary group $S_3 = \{a^0 = e, a, a^2 = b, c, ac = d, a^2c = fg\}$. The table of the polygroup extended $P_{s_3,1}$ as follows:

TABLE 4.

$*$	e	a	b	c	d	f	h
e	e	a	b	c	d	f	h
a	a	b	e,h	d	f	c	a
b	b	e,h	a	f	c	d	b
c	c	f	d	e,h	b	a	c
d	d	c	f	a	e,h	b	d
f	f	d	c	b	a	e,h	f
h	h	a	b	c	d	f	e

(i) the element e is called central,

(ii) display the centralizer a with $C(a)$, which is $C(a) = \{e, a, b, h\}$, $C_w(b) = \{e, a, b, h\}$,

(iii) we have $C(a) = \{e, a, b, h\}$, $C(b) = \{e, a, b, h\}$, $C(c) = \{e, c, h\}$, $C(d) = \{e, d, h\}$, $C(f) = \{e, f, h\}$, Therefore a, b, c, d, f are elements commutator in $P_{s_3,1}$.

Definition 2.13. A polygroup is called a commutator polygroup (resp. weak commutator polygroup) when any of its non-central elements are commutator (resp. weak commutator polygroup).

Example 2.14. Suppose $P_{s_3,1}$ is polygroup of Example 2.12 then is a commutator polygroup and also weak commutator polygroup.

Theorem 2.15. *If G is a commutator group then $\boldsymbol{A}(G)$ is a subgroup of $Aut(G)$.*

Proof. See lemma 2.3 of [13]. □

Theorem 2.16. *If $(P, \cdot)$ is a weak commutator polygroup then $\boldsymbol{A}(P)$ is a subgroup of $Aut(P)$.*

Proof. Let $x \in P$ and $\varphi, \psi \in \boldsymbol{A}(P)$. It is enough to prove $x\varphi\psi(x) \cap \varphi\psi(x)x \neq \emptyset$ or equivalent

$$\varphi^{-1}(x)\psi(x) \cap \psi(x)\varphi^{-1}(x) \neq \emptyset.$$

We consider two cases. If x is a central element, then $\varphi^{-1}(x)$ is also central, so in this case the claim clearly holds. But if x is not a central element, since the weak centralizer x is commutative subpolygroup and $\{\varphi^{-1}(x), \psi(x)\} \subseteq C_w(x)$ we have:

$$\varphi^{-1}(x)\psi(x) \cap \psi(x)\varphi^{-1}(x) \neq \emptyset.$$

Therefore, in this case the sentence valids. □

Proposition 2.17. *Suppose G is a group and $(P_{G,n}, \odot)$ be polygroup extended of group G, in this case:*

a) $Z(P_{G,n}) = Z(G) \cup \{a_1, a_2, \cdots, a_n\}$, where $Z(P_{G,n})$is the set of central elements of $P_{G,n}$.

b) The set of non-central elements $P_{G,n}$ and the set of non-central elements of G are equal.

Proof. a) According to the hyperoperation $\odot$ every element of the set $\{a_1, a_2, \cdots, a_n\}$ is a central element. Now according to (d) we can conclude that $Z(P_{G,n}) = Z(G) \cup \{a_1, a_2, \cdots, a_n\}$. Part (b) with respect to (a) is clearly established. □

Theorem 2.18. *If G is a commutator group then $\boldsymbol{A}(P_{G,n})$ is a subgroup of $Aut(P_{G,n})$.*

Proof. It is enough to show that $(P_{G,n.}, \odot)$ is a weak commutator polygroup. For this suppose $x \in P_{G,n}$ and x is not central. Prove that $C_w(x)$ is a weak commutative subpolygroup of $P_{G,n}$. Using the Proposition 2.17 $Z(P_{G,n}) = Z(G) \cup \{a_1, a_2, \cdots, a_n\}$. Also non-central elements $P_{G,n}$ and the non-central elements of G are equal. Now because x is not central, then $x \in G$ and

$$C_w(x) = C_G(x) \cup \{a_1, a_2, \cdots, a_n\},$$

where $C_G(x)$ is the centralizer of x in G. Therefore $C_w(x)$ is a commutative subpolygroup. Thus $(P_{G,n}, \odot)$ is a weak commutator polygroup. □

Proposition 2.19. *Let G be a group and polygroup $(P_{G,n}, \odot)$ be the extended group of G. Then for every φ of $Aut(P_{G,n})$ and every $x \in \{a_1, a_2, \cdots, a_n\}$, $\varphi(x) = x$ valid.*

Proof. Suppose $a_i \in \{a_1, a_2, \cdots, a_n\}$. If $\varphi(a_i) = x$ then $e = \varphi(e) = \varphi(a_i a_i) = \varphi(a_i)\varphi(a_i) = xx$ consequently $x \in \{a_1, a_2, \cdots, a_n\}$. Now suppose k is the largest number that $\varphi(a_k) \neq a_k$. So $\varphi(a_k) = a_i$, where $i<k$. On the other hand $\varphi(a_j) = a_k$, where $j<k$. Therefore we have:

$$a_k = \varphi(a_j) = \varphi(a_k a_j) = \varphi(a_k)\varphi(a_j) = a_i a_k = a_i,$$

which is a contradiction. □

Corollary 2.20. *If $\varphi \in Aut(P_{G,n})$ then $\alpha \in Aut(G)$ exists such that for every $x \in P_{G,n}$:*

$$\varphi(x) = \begin{cases} \alpha(x), & x \in G, \\ x, & x \in \{a_1, a_2, \cdots, a_n\}. \end{cases}$$

Corollary 2.21. *$\boldsymbol{A}(P_{G,n})$ is a subgroup of $Aut(P_{G,n})$ if and only if $\boldsymbol{A}(G)$ is a subgroup of $Aut(G)$.*

3. Conclusion

We investigate a generalization of notion commuting automorphism for the class of polygroups which is introduced by M. Deaconescu et.al for the class of groups [8]. Using the notion of commuting automorphism we introduce the notion of commutator polygroups and a characterization of commutator polygroups extended by groups has been investigated. Researchers who would like to work on this field can apply my paper similar to the work done by Bettina Eick in [12].

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