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Research Paper

CLASSIFICATION OF GROUPS WHOSE VANISHING ELEMENTS ARE CONTAINED IN EXACTLY TWO CONJUGACY CLASSES

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Abstract. Let G be a finite group. We say that an element g in G is a vanishing element if there exists some irreducible character χ of G such that $\chi(g) = 0$. Moreover, the conjugacy class of a vanishing element is called a vanishing conjugacy class. In this paper, we classify groups with exactly two vanishing conjugacy classes and show that such groups are either Frobenius or quasi-Frobenius groups.

1. Introduction

Throughout this paper, G will be a finite group. Let $\mathrm{Van}(G)$ be the set of vanishing elements of G, where an element x in G is vanishing if $\chi(x) = 0$ for some irreducible character χ of G. And, a conjugacy class contained in $\mathrm{Van}(G)$ is called a vanishing conjugacy class. It is clear that $\mathrm{Van}(G)$ is the union of vanishing conjugacy classes. The well-known theorem of

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Burnside signifies that $\mathrm{Van}(G)=\emptyset$ if and only if G is abelian. Motivated by the results, we in [8] show that groups whose set of vanishing elements is the union of at most three conjugacy classes are solvable and in [10] assert that groups with at most six vanishing conjugacy classes are solvable or almost simple groups. Additionally, we classify groups whose set of vanishing elements is exactly a conjugacy class, see [9].

In this paper, we prove that every group whose set of vanishing elements is the union of exactly two conjugacy classes is either a Frobenius group or a quasi-Frobenius group.

We summarize our notations. Let $\mathrm{O}_{2'}(G)$ denote the unique maximal normal subgroup of G of odd order and $cl_G(a)$ denote the conjugacy class of a in G.

2. Preliminaries

In this section, we provide some results which are useful tools for the investigations later in the paper.

Theorem 2.1 ([11], Theorem 1)**.** *Let G be a finite group with a non-cyclic subgroup of order 4 which is its own centralizer in G. Then $G/\mathrm{O}_{2'}(G)$ is isomorphic to $PSL(3,3), M_{11}, GL(2,3), H(q), PGL(2,q), PSL(2,q)(q\ odd), A_7, D_{2^n}$, or SD_{2^n}.*

Theorem 2.2 ([11], Theorem 2)**.** *Let G be a finite group with a cyclic subgroup of order 4 which is its own centralizer in G. Then $G/\mathrm{O}_{2'}(G)$ is isomorphic to $SL(2,3), SL(2,5), PSL(2,7), PSL(2,9), PGL(2,3), PGL(2,5), H(9), J, A_7, D_8, Q_{2^n}, SD_{2^n}$ or Z_4.*

Before stating the main section, we mention some necessary results on vanishing and non-vanishing elements.

Theorem 2.3 ([5], Theorem B)**.** *Let G be a nilpotent group. Then each non-vanishing element of G is central.*

Lemma 2.4 ([2], Lemma 2.6)**.** *Let G ba a solvable group, and let $\mathrm{F}(G)$ be the Fitting subgroup of G. If $G/\mathrm{F}(G)$ is abelian, then $G\setminus \mathrm{F}(G)\subseteq \mathrm{Van}(G)$.*

Theorem 2.5 ([5], Theorem D)**.** *Let x be a non-vanishing element of the solvable group G. Then the image of x in $G/\mathrm{F}(G)$ has 2-power order, and in particular, if x has odd order, then $x\in \mathrm{F}(G)$. In any case, if G is not nilpotent, then x lies in the penultimate term of the ascending Fitting series.*

3. Main Theorem

It is clear that if G has at most n vanishing conjugacy classes and $N\lhd G$, then G/N has at most n vanishing conjugacy classes.

Lemma 3.1 ([7], Theorem 2.1)**.** *Let G be a nilpotent group and suppose that it has at most two non-central conjugacy classes of each size. Then G is an abelian group.*

Lemma 3.2. *Let G be a nilpotent group. The set of vanishing elements of G is the union of exactly two conjugacy classes if and only if G is an abelian group.*

Proof. Assume that G is a non-abelian nilpotent group. By Theorem 2.3, each non-central element is a vanishing element of G. Thus,

$$G = \mathrm{Van}(G) \cup \mathrm{Z}(G) = C_1 \cup C_2 \cup \mathrm{Z}(G),$$

for some conjugacy classes C_1 and C_2 of G. It is impossible by Lemma 3.1. □

Lemma 3.3. *Let G be a solvable group. Then there exists some proper normal subgroup N of G containing G' such that $G - N \subseteq \mathrm{Van}(G)$.*

Proof. Since G is solvable, so there exist some non-trivial linear character λ of G' and some irreducible character χ of G such that

$$[\chi_{G'}, \lambda] = [\chi, \lambda^G] \neq 0.$$

Moreover, we know that the restriction of each linear character of G to G' is the trivial character of G', therefore $\chi_{G'} \neq \lambda$ is non-linear and reducible. Now, since G/G' is abelian, it follows from Theorem 6.22 and Definition 6.21 of [4], there are some proper normal subgroup N of G containing G' and $\psi \in \mathrm{Irr}(N)$ such that $\chi = \psi^G$ and $\psi_{G'} \in \mathrm{Irr}(G')$. Hence, $\chi(g) = \psi^G(g) = 0$ for every $g \in G - N$. □

Lemma 3.4 ([1], Proposition 4.1)**.** *Let G be a Frobenius group and $p \leq 5$ be a prime number. If the order of every vanishing element of G is p, then the Frobenius kernel of G is abelian.*

Now, we ready to prove the main theorem.

Theorem 3.5. *Let G be a finite group. Then the set of vanishing elements of G is the union of exactly two conjugacy classes of G if and only if one of the following situation occurs:*

(1) *G is a Frobenius group with abelian kernel G' and complement of order* 3.
(2) *$G/\mathrm{Z}(G)$ is a Frobenius with abelian kernel $(G' \times \mathrm{Z}(G))/\mathrm{Z}(G)$ of odd order and complement of order* 2*. Moreover, $|\mathrm{Z}(G)| = 2$ and $G' \cap \mathrm{Z}(G) = 1$.*

Proof. We know that G is solvable, see Theorem 2.8 of [8]. It can derived from Lemma 3.3 that there exist some conjugacy classes C_1 and C_2 of G contained in $\mathrm{Van}(G)$ such that

$$G = \mathrm{Van}(G) \cup N = C_1 \cup C_2 \cup N,$$

in which N is a proper normal subgroup of G containing G'. As the set of left cosets of N in G is a partition of G and $|C_i| \leq |G'| \leq |N|$ for $i = 1, 2$, so we break the proof into three cases.

Case (1): $N = G'$ and $|G : N| = 3$.

In this case, $|C_1| = |C_2| = |G'|$ and so $|\mathrm{C}_G(x)| = 3$ for each $x \in C_1 \cup C_2$. Additionally, $\mathrm{C}_G(x) = \langle x \rangle$ and hence G is a Frobenius group with Frobenius kernel G' and complement $\langle x \rangle$ of order 3, see Problem 7.1 of [4]. By Theorem 13.8 of [3], we have

$$\mathrm{Van}(G) = cl(x) \cup cl(x^2) = xG' \cup x^2G'$$

which implies from Lemma 3.4 that G' is abelian.

Now, if $\chi \in \mathrm{Irr}(G)$ is non-linear, we have $\chi = \theta^G$ for some $\theta \in \mathrm{Irr}(G')$ and by Theorem 6.34 of [4], we can write that

$$\chi_{G'} = \theta_1 + \theta_2 + \theta_3$$

where $\theta_1 = \theta, \theta_2, \theta_3$ are the distinct conjugates of θ in G. As G' is abelian, so $\theta_i(y)$ for each $y \in G'$ is an nth root of unity for $n = |G'|$. We can check that $\chi_{G'}(y) \neq 0$ because $(n, 3) = 1$. Therefore, $G' \cap \mathrm{Van}(G) \neq \emptyset$ as desired.

Case (2): $N = G'$ and $|G : N| = 2$.

In this case, $|C_1| + |C_2| = |G'|$ and we can assume that $|C_1| \geq |G'|/2$ and so $|\mathrm{C}_G(x)| \leq 4$ for each $x \in C_1$. If $|\mathrm{C}_G(x)| = p$ and $p = 2$ or 3, we conclude that $\mathrm{C}_G(x) = \langle x \rangle$ and G is a Frobenius group with kernel G' and complement $\langle x \rangle$ of order p. Since $|G : G'| = 2$, so $p = 2$. Thus, by Lemma 13.3 of [3], G' is an abelian subgroup of odd order and so G has exactly one vanishing conjugacy class which is a contradiction. so $|\mathrm{C}_G(x)| = 4$ for each $x \in C_1 \cup C_2$ and $|C_1| = |C_2| = |G'|/2$.

First, we suppose that every $x \in C_1 \cup C_2$ is of order 2. Since $\mathrm{C}_G(x)$ is an elementary abelian 2-group, we can deduce that $\mathrm{C}_G(x) \leq \mathrm{C}_G(\mathrm{C}_G(x)) \leq \mathrm{C}_G(x)$ and $\mathrm{C}_G(\mathrm{C}_G(x)) = \mathrm{C}_G(x)$. It follows from Theorem 2.1 that $G/\mathrm{O}_{2'}(G)$ is a non-abelian 2-group. It is impossible by Lemma 3.2.

Now, consider $|x| = 4$ for some $x \in C_1 \cup C_2$. Thus, $\mathrm{C}_G(x) = \langle x \rangle$ is a cyclic subgroup of order 4. By our assumption and Theorem 2.2, $G/\mathrm{O}_{2'}(G)$ is isomorphic to Z_4 and $G \cong \mathrm{O}_{2'}(G) \rtimes \mathrm{Z}_4$. Furthermore, since $|cl_G(x)| = |\mathrm{O}_{2'}(G)| \leq |G'|$, we have $G' = \mathrm{O}_{2'}(G)$ which is a contradiction.

Case (3): $|N : G'| = 2$ and $|G : N| = 2$.

In this case, $|C_1| = |C_2| = |G'|$ and $N \cap \mathrm{Van(G)} = \emptyset$. Thus, $|\mathrm{C}_G(x)| = 4$ and $|x| = 2$ or 4. Using a proof similar to the proof of Case (2), we can write that

$$G = \mathrm{O}_{2'}(G) \rtimes \langle x \rangle,$$

where $\langle x\rangle \cong \mathrm{Z}_4$ and $G' = \mathrm{O}_{2'}(G)$. If G is a Frobenius group, we observe that G has three vanishing conjugacy classes $cl(x^i) = x^i G'$ for $i = 1, 2, 3$ which is impossible. Therefore, we can see that $N = G'\langle x^2\rangle$, $|\mathrm{C}_G(x)| > 4$, and x^2 is non-vanishing.

Theorem 2.5 tells us that the order of the image of each non-vanishing element of G in $G/\mathrm{F}(G)$ is a power of 2. Hence, we can conclude that $G/\mathrm{F}(G)$ is a 2-group with at most two vanishing conjugacy classes. As a consequent of Lemma 3.2 and Lemma 2.4, we have $G/\mathrm{F}(G)$ is abelian and

$$G - \mathrm{F}(G) \subseteq \mathrm{Van}(G) = C_1 \cup C_2,$$

which yields that $N \subseteq \mathrm{F}(G)$, $\mathrm{Z}(G) = \langle x^2\rangle$, and $N = G' \times \mathrm{Z}(G)$. Using Problem 7.1 and Lemma 7.21 of [4], we obtain that

$$G/\mathrm{Z}(G) = ((G' \times \mathrm{Z}(G))/\mathrm{Z}(G)) \rtimes \mathrm{Z}_2,$$

is a Frobenius group and G' is abelian. Let λ be the non-trivial linear character of G. By Corollary 6.17 of [4], we can get that

$$\mathrm{Irr}(G) = \{\chi, \lambda\chi | \chi \in \mathrm{Irr}(G/\mathrm{Z}(G)\}.$$

Moreover, Theorem 13.8 of [3] guarantees that $cl(x\mathrm{Z}(G))$ is the vanishing conjugacy class of $G/\mathrm{Z}(G)$ and so x and $x.x^2 = x^3$ are representatives of two vanishing conjugacy classes of G.

On the other hand, if $a\mathrm{Z}(G)$ is a non-vanishing element of $G/\mathrm{Z}(G)$, we conclude from Problem 3.12 of [4] that

$$0 \neq \chi(a)\chi(x^2) = \chi(1)\chi(ax^2).$$

Consequently, χ and $\lambda\chi$ do not vanish on a and ax^2 and so the set of vanishing elements of G is the union exactly two conjugacy classes of G. The proof is complete. □

Example 3.6. Using [6], observe that $D_{12} \cong \mathrm{S}_3 \times \mathrm{Z}_2$, an split central extension of S_3 by Z_2, and

$$T_{12} = \langle a, b | a^6 = 1, a^3 = b^2, b^{-1}ab = a^{-1}\rangle,$$

in which T_{12} is a non-split central extension of S_3 by Z_2 and $T_{12}/\mathrm{Z}(T_{12}) \cong S_3$. We can easily check that these groups satisfy Main Theorem.

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