



Research Paper

**CLASSIFICATION OF GROUPS WHOSE VANISHING
 ELEMENTS ARE CONTAINED IN EXACTLY TWO CONJUGACY
 CLASSES**

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ABSTRACT. Let G be a finite group. We say that an element g in G is a vanishing element if there exists some irreducible character χ of G such that $\chi(g) = 0$. Moreover, the conjugacy class of a vanishing element is called a vanishing conjugacy class. In this paper, we classify groups with exactly two vanishing conjugacy classes and show that such groups are either Frobenius or quasi-Frobenius groups.

1. INTRODUCTION

Throughout this paper, G will be a finite group. Let $\text{Van}(G)$ be the set of vanishing elements of G , where an element x in G is vanishing if $\chi(x) = 0$ for some irreducible character χ of G . And, a conjugacy class contained in $\text{Van}(G)$ is called a vanishing conjugacy class. It is clear that $\text{Van}(G)$ is the union of vanishing conjugacy classes. The well-known theorem of

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Burnside signifies that $\text{Van}(G) = \emptyset$ if and only if G is abelian. Motivated by the results, we in [8] show that groups whose set of vanishing elements is the union of at most three conjugacy classes are solvable and in [10] assert that groups with at most six vanishing conjugacy classes are solvable or almost simple groups. Additionally, we classify groups whose set of vanishing elements is exactly a conjugacy class, see [9].

In this paper, we prove that every group whose set of vanishing elements is the union of exactly two conjugacy classes is either a Frobenius group or a quasi-Frobenius group.

We summarize our notations. Let $O_{2'}(G)$ denote the unique maximal normal subgroup of G of odd order and $cl_G(a)$ denote the conjugacy class of a in G .

2. PRELIMINARIES

In this section, we provide some results which are useful tools for the investigations later in the paper.

Theorem 2.1 ([11], Theorem 1). *Let G be a finite group with a non-cyclic subgroup of order 4 which is its own centralizer in G . Then $G/O_{2'}(G)$ is isomorphic to $PSL(3, 3), M_{11}, GL(2, 3), H(q), PGL(2, q), PSL(2, q)(q \text{ odd}), A_7, D_{2^n}$, or SD_{2^n} .*

Theorem 2.2 ([11], Theorem 2). *Let G be a finite group with a cyclic subgroup of order 4 which is its own centralizer in G . Then $G/O_{2'}(G)$ is isomorphic to $SL(2, 3), SL(2, 5), PSL(2, 7), PSL(2, 9), PGL(2, 3), PGL(2, 5), H(9), J, A_7, D_8, Q_{2^n}, SD_{2^n}$ or Z_4 .*

Before stating the main section, we mention some necessary results on vanishing and non-vanishing elements.

Theorem 2.3 ([5], Theorem B). *Let G be a nilpotent group. Then each non-vanishing element of G is central.*

Lemma 2.4 ([2], Lemma 2.6). *Let G be a solvable group, and let $F(G)$ be the Fitting subgroup of G . If $G/F(G)$ is abelian, then $G \setminus F(G) \subseteq \text{Van}(G)$.*

Theorem 2.5 ([5], Theorem D). *Let x be a non-vanishing element of the solvable group G . Then the image of x in $G/F(G)$ has 2-power order, and in particular, if x has odd order, then $x \in F(G)$. In any case, if G is not nilpotent, then x lies in the penultimate term of the ascending Fitting series.*

3. MAIN THEOREM

It is clear that if G has at most n vanishing conjugacy classes and $N \triangleleft G$, then G/N has at most n vanishing conjugacy classes.

Lemma 3.1 ([7], Theorem 2.1). *Let G be a nilpotent group and suppose that it has at most two non-central conjugacy classes of each size. Then G is an abelian group.*

Lemma 3.2. *Let G be a nilpotent group. The set of vanishing elements of G is the union of exactly two conjugacy classes if and only if G is an abelian group.*

Proof. Assume that G is a non-abelian nilpotent group. By Theorem 2.3, each non-central element is a vanishing element of G . Thus,

$$G = \text{Van}(G) \cup Z(G) = C_1 \cup C_2 \cup Z(G),$$

for some conjugacy classes C_1 and C_2 of G . It is impossible by Lemma 3.1. \square

Lemma 3.3. *Let G be a solvable group. Then there exists some proper normal subgroup N of G containing G' such that $G - N \subseteq \text{Van}(G)$.*

Proof. Since G is solvable, so there exist some non-trivial linear character λ of G' and some irreducible character χ of G such that

$$[\chi_{G'}, \lambda] = [\chi, \lambda^G] \neq 0.$$

Moreover, we know that the restriction of each linear character of G to G' is the trivial character of G' , therefore $\chi_{G'} \neq \lambda$ is non-linear and reducible. Now, since G/G' is abelian, it follows from Theorem 6.22 and Definition 6.21 of [4], there are some proper normal subgroup N of G containing G' and $\psi \in \text{Irr}(N)$ such that $\chi = \psi^G$ and $\psi_{G'} \in \text{Irr}(G')$. Hence, $\chi(g) = \psi^G(g) = 0$ for every $g \in G - N$. \square

Lemma 3.4 ([1], Proposition 4.1). *Let G be a Frobenius group and $p \leq 5$ be a prime number. If the order of every vanishing element of G is p , then the Frobenius kernel of G is abelian.*

Now, we ready to prove the main theorem.

Theorem 3.5. *Let G be a finite group. Then the set of vanishing elements of G is the union of exactly two conjugacy classes of G if and only if one of the following situation occurs:*

- (1) G is a Frobenius group with abelian kernel G' and complement of order 3.
- (2) $G/Z(G)$ is a Frobenius with abelian kernel $(G' \times Z(G))/Z(G)$ of odd order and complement of order 2. Moreover, $|Z(G)| = 2$ and $G' \cap Z(G) = 1$.

Proof. We know that G is solvable, see Theorem 2.8 of [8]. It can derived from Lemma 3.3 that there exist some conjugacy classes C_1 and C_2 of G contained in $\text{Van}(G)$ such that

$$G = \text{Van}(G) \cup N = C_1 \cup C_2 \cup N,$$

in which N is a proper normal subgroup of G containing G' . As the set of left cosets of N in G is a partition of G and $|C_i| \leq |G'| \leq |N|$ for $i = 1, 2$, so we break the proof into three cases.

Case (1): $N = G'$ and $|G : N| = 3$.

In this case, $|C_1| = |C_2| = |G'|$ and so $|C_G(x)| = 3$ for each $x \in C_1 \cup C_2$. Additionally, $C_G(x) = \langle x \rangle$ and hence G is a Frobenius group with Frobenius kernel G' and complement $\langle x \rangle$ of order 3, see Problem 7.1 of [4]. By Theorem 13.8 of [3], we have

$$\text{Van}(G) = cl(x) \cup cl(x^2) = xG' \cup x^2G'$$

which implies from Lemma 3.4 that G' is abelian.

Now, if $\chi \in \text{Irr}(G)$ is non-linear, we have $\chi = \theta^G$ for some $\theta \in \text{Irr}(G')$ and by Theorem 6.34 of [4], we can write that

$$\chi_{G'} = \theta_1 + \theta_2 + \theta_3$$

where $\theta_1 = \theta, \theta_2, \theta_3$ are the distinct conjugates of θ in G . As G' is abelian, so $\theta_i(y)$ for each $y \in G'$ is an n th root of unity for $n = |G'|$. We can check that $\chi_{G'}(y) \neq 0$ because $(n, 3) = 1$. Therefore, $G' \cap \text{Van}(G) \neq \emptyset$ as desired.

Case (2): $N = G'$ and $|G : N| = 2$.

In this case, $|C_1| + |C_2| = |G'|$ and we can assume that $|C_1| \geq |G'|/2$ and so $|C_G(x)| \leq 4$ for each $x \in C_1$. If $|C_G(x)| = p$ and $p = 2$ or 3 , we conclude that $C_G(x) = \langle x \rangle$ and G is a Frobenius group with kernel G' and complement $\langle x \rangle$ of order p . Since $|G : G'| = 2$, so $p = 2$. Thus, by Lemma 13.3 of [3], G' is an abelian subgroup of odd order and so G has exactly one vanishing conjugacy class which is a contradiction. so $|C_G(x)| = 4$ for each $x \in C_1 \cup C_2$ and $|C_1| = |C_2| = |G'|/2$.

First, we suppose that every $x \in C_1 \cup C_2$ is of order 2. Since $C_G(x)$ is an elementary abelian 2-group, we can deduce that $C_G(x) \leq C_G(C_G(x)) \leq C_G(x)$ and $C_G(C_G(x)) = C_G(x)$. It follows from Theorem 2.1 that $G/O_{2'}(G)$ is a non-abelian 2-group. It is impossible by Lemma 3.2.

Now, consider $|x| = 4$ for some $x \in C_1 \cup C_2$. Thus, $C_G(x) = \langle x \rangle$ is a cyclic subgroup of order 4. By our assumption and Theorem 2.2, $G/O_{2'}(G)$ is isomorphic to Z_4 and $G \cong O_{2'}(G) \rtimes Z_4$. Furthermore, since $|cl_G(x)| = |O_{2'}(G)| \leq |G'|$, we have $G' = O_{2'}(G)$ which is a contradiction.

Case (3): $|N : G'| = 2$ and $|G : N| = 2$.

In this case, $|C_1| = |C_2| = |G'|$ and $N \cap \text{Van}(G) = \emptyset$. Thus, $|C_G(x)| = 4$ and $|x| = 2$ or 4 . Using a proof similar to the proof of Case (2), we can write that

$$G = O_{2'}(G) \rtimes \langle x \rangle,$$

where $\langle x \rangle \cong Z_4$ and $G' = O_{2'}(G)$. If G is a Frobenius group, we observe that G has three vanishing conjugacy classes $cl(x^i) = x^i G'$ for $i = 1, 2, 3$ which is impossible. Therefore, we can see that $N = G' \langle x^2 \rangle$, $|C_G(x)| > 4$, and x^2 is non-vanishing.

Theorem 2.5 tells us that the order of the image of each non-vanishing element of G in $G/F(G)$ is a power of 2. Hence, we can conclude that $G/F(G)$ is a 2-group with at most two vanishing conjugacy classes. As a consequent of Lemma 3.2 and Lemma 2.4, we have $G/F(G)$ is abelian and

$$G - F(G) \subseteq \text{Van}(G) = C_1 \cup C_2,$$

which yields that $N \subseteq F(G)$, $Z(G) = \langle x^2 \rangle$, and $N = G' \times Z(G)$. Using Problem 7.1 and Lemma 7.21 of [4], we obtain that

$$G/Z(G) = ((G' \times Z(G))/Z(G)) \rtimes Z_2,$$

is a Frobenius group and G' is abelian. Let λ be the non-trivial linear character of G . By Corollary 6.17 of [4], we can get that

$$\text{Irr}(G) = \{\chi, \lambda\chi \mid \chi \in \text{Irr}(G/Z(G))\}.$$

Moreover, Theorem 13.8 of [3] guarantees that $cl(xZ(G))$ is the vanishing conjugacy class of $G/Z(G)$ and so x and $x.x^2 = x^3$ are representatives of two vanishing conjugacy classes of G .

On the other hand, if $aZ(G)$ is a non-vanishing element of $G/Z(G)$, we conclude from Problem 3.12 of [4] that

$$0 \neq \chi(a)\chi(x^2) = \chi(1)\chi(ax^2).$$

Consequently, χ and $\lambda\chi$ do not vanish on a and ax^2 and so the set of vanishing elements of G is the union exactly two conjugacy classes of G . The proof is complete. \square

Example 3.6. Using [6], observe that $D_{12} \cong S_3 \times Z_2$, an split central extension of S_3 by Z_2 , and

$$T_{12} = \langle a, b \mid a^6 = 1, a^3 = b^2, b^{-1}ab = a^{-1} \rangle,$$

in which T_{12} is a non-split central extension of S_3 by Z_2 and $T_{12}/Z(T_{12}) \cong S_3$. We can easily check that these groups satisfy Main Theorem.

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