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Research Paper

ϕ-(k,n)-ABSORBING (PRIMARY) HYPERIDEALS IN A KRASNER (m,n)-HYPERRING

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ABSTRACT. Various expansions of prime hyperideals have been studied in a Krasner (m,n)-hyperring R. For instance, a proper hyperideal Q of R is called weakly (k,n)-absorbing (primary) provided that for $r_1^{kn-k+1} \in R$, $g(r_1^{kn-k+1}) \in Q - \{0\}$ implies that there are $(k-1)n-k+2$ of the r_i's whose g-product is in Q $\Big(g(r_1^{(k-1)n-k+2}) \in Q$ or a g-product of $(k-1)n-k+2$ of r_i's ,except $g(r_1^{(k-1)n-k+2})$, is in $\boldsymbol{r}^{(m,n)}(Q) \Big)$. In this paper, we aim to extend the notions to the concepts of ϕ-(k,n)-absorbing and ϕ-(k,n)-absorbing primary hyperideals. Assume that ϕ is a function from $\mathcal{HI}(R)$ to $\mathcal{HI}(R) \cup \{\varphi\}$ such that $\mathcal{HI}(R)$ is the set of hyperideals of R and k is a positive integer. We call a proper hyperideal Q of R a ϕ-(k,n)-absorbing (primary) hyperideal if for $r_1^{kn-k+1} \in R$, $g(r_1^{kn-k+1}) \in Q - \phi(Q)$ implies that there are $(k-1)n-k+2$ of the r_i's whose g-product is in Q $\Big(g(r_1^{(k-1)n-k+2}) \in Q$ or a g-product of $(k-1)n-k+2$ of r_i's ,except $g(r_1^{(k-1)n-k+2})$, is in $\boldsymbol{r}^{(m,n)}(Q) \Big)$. Several properties and characterizations of them are presented.

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1. Introduction

Extensions of prime and primary ideals to the context of ϕ-prime and ϕ-primary ideals were studied in [7, 12]. Afterwards, Khaksari in [20] and Badawi et al. in [9] introduced ϕ-2-prime and ϕ-2-primary ideals, respectively. Let R be a commutative ring. Suppose that ϕ is a function from $\mathcal{I}(R)$ to $\mathcal{I}(R) \cup \{\varphi\}$ where $\mathcal{I}(R)$ is the set of ideals of R. A proper ideal I of R is said to be a ϕ-2-absorbing ideal if whenever $x, y, z \in R$, with $xyz \in I - \phi(I)$ implies that $xy \in I$ or $xz \in I$ or $yz \in I$. Also, A proper ideal I of R is called a ϕ-2-absorbing primary ideal if for every $x, y, z \in R$, $xyz \in I - \phi(I)$ implies that $xy \in I$ or $xz \in \boldsymbol{r}(I)$ or $yz \in \boldsymbol{r}(I)$.

Hyperstructures are algebraic structures equipped with at least one multi-valued operation, called a hyperoperation. A hyperoperation on a non-empty set is a mapping from to the nonempty power set. Hundreds of papers and several books have been written on this topic (for more details see [2, 10, 11, 13, 17, 21, 26, 30, 32, 33, 34]). An n-ary extension of algebraic structures is the most natural method for deeper understanding of their fundamental properties. Mirvakili and Davvaz in [28] introduced (m, n)-hyperrings and gave several results in this respect. They defined and described a generalization of the notion of a hypergroup and a generalization of an n-ary group, which is called n-ary hypergroup [14]. Some review of the n-ary structures can be found in in [22, 23, 24, 25, 31]. One important class of hyperrings, where the addition is a hyperoperation, while the multiplication is an ordinary binary operation, is Krasner hyperring. An extension of the Krasner hyperrings, which is a subclass of (m, n)-hyperrings, was presented by Mirvakili and Davvaz [27], which is called Krasner (m, n)-hyperring. Some important hyperideals namely Jacobson radical, nilradical, n-ary prime and primary hyperideals and n-ary multiplicative subsets of Krasner (m, n)-hyperrings were defined by Ameri and Norouzi in [1]. Afterward, the concept of (k, n)-absorbing (primary) hyperideals was studied by Hila et al. [18]. Norouzi et al. gave a new definition for normal hyperideals in Krasner (m, n)-hyperrings, with respect to that one given in [27] and they showed that these hyperideals correspond to strongly regular relations [29]. Direct limit of a direct system was defined and analysed by Asadi and Ameri in the category of Krasner (m, n)-hyperrigs [8]. The notion of δ-primary hyperideals in Krasner (m, n)-hyperrings, which unifies the prime and primary hyperideals under one frame, was presented in [4]. Recently, Davvaz et al. introduced new expansion classes, namely weakly (k, n)-absorbing (primary) hyperideals in a Krasner (m, n)-hyperring [16].

In this paper, we introduce and study the notions of ϕ-(k, n)-absorbing and ϕ-(k, n)-absorbing primary hyperideals in a commutative Krasner (m, n)-hyperring. A number of main results are given to explain the general framework of these structures. Among many results in this paper, it is shown (Theorem 3.6) that if Q is a ϕ-(k, n)-absorbing hyperideal of R, then Q is a ϕ-(s, n)-absorbing hyperideal for all $s \geq k$. Although every ϕ-(k, n)-absorbing of a Krasner

(m,n)-hyperring is ϕ-(k,n)-absorbing primary, Example 4.3 shows that the converse may not be always true. It is shown (Theorem 4.13) that Q is a ϕ-(k,n)-absorbing primary hyperideal of R if and only if $Q/\phi(Q)$ is a weakly (k,n)-absorbing primary hyperideal of $R/\phi(Q)$. In Theorem 4.16, we show that if Q is a ϕ-(k,n)-absorbing primary hyperideal of R but is not a (k,n)-absorbing primary, then $g(Q^{k(n-1)+1}) \subseteq \phi(Q)$. As a result of the theorem we conclude that if Q is a ϕ-(k,n)-absorbing primary hyperideal of R that is not a (k,n)-absorbing primary hyperideal of R, then $\boldsymbol{r}^{(m,n)}(Q) = \boldsymbol{r}^{(m,n)}(\phi(Q))$.

2. Krasner (m,n)-hyperrings

In this section, we summarize the preliminary definitions which are related to Krasner (m,n)-hyperrings.

Let A be a non-empty set and $P^*(A)$ the set of all the non-empty subsets of A. An n-ary hyperoperation on A is a map $f : A^n \longrightarrow P^*(A)$ and the couple (A,f) is called an n-ary hypergroupoid. The notation a_i^j will denote the sequence $a_i, a_{i+1}, ..., a_j$ for $j \geq i$ and it is the empty symbol for $j < i$. If $G_1, ..., G_n$ are non-empty subsets of A, then we define $f(G_1^n) = f(G_1, ..., G_n) = \bigcup\{f(a_1^n) \mid a_i \in G_i, 1 \leq i \leq n\}$. If $b_{i+1} = ... = b_j = b$, we write $f(a_1^i, b_{i+1}^j, c_{j+1}^n) = f(a_1^i, b^{(j-i)}, c_{j+1}^n)$. If f is an n-ary hyperoperation, then t-ary hyperoperation $f_{(l)}$ is given by

$$f_{(l)}(a_1^{l(n-1)+1}) = f\Bigg(f(..., f(f(a_1^n), a_{n+1}^{2n-1}), ...), a_{(l-1)(n-1)+1}^{l(n-1)+1}\Bigg),$$

where $t = l(n-1)+1$.

Definition 2.1. [27] (R,f,g), or simply R, is defined as a Krasner (m,n)-hyperring if the following statements hold:

(1) (R,f) is a canonical m-ary hypergroup;

(2) (R,g) is a n-ary semigroup;

(3) The n-ary operation g is distributive with respect to the m-ary hyperoperation f , i.e., for every $a_1^{i-1}, a_{i+1}^n, x_1^m \in R$, and $1 \leq i \leq n$,

$$g\Bigg(a_1^{i-1}, f(x_1^m), a_{i+1}^n\Bigg) = f\Bigg(g(a_1^{i-1}, x_1, a_{i+1}^n), ..., g(a_1^{i-1}, x_m, a_{i+1}^n)\Bigg);$$

(4) 0 is a zero element of the n-ary operation g, i.e., for each $a_1^n \in R$, $g(a_1^{i-1}, 0, a_{i+1}^n) = 0$.

Throughout this paper, R denotes a commutative Krasner (m,n)-hyperring with the scalar identity 1.

A non-empty subset T of R is called a subhyperring of R if (T,f,g) is a Krasner (m,n)-hyperring. The non-empty subset I of R is a hyperideal if (I,f) is an m-ary subhypergroup of (R,f) and $g(x_1^{i-1}, I, x_{i+1}^n) \subseteq I$, for each $x_1^n \in R$ and $1 \leq i \leq n$.

Definition 2.2. [1] Let I be a proper hyperideal of R. I refers to a prime hyperideal if for hyperideals I_1^n of R, $g(I_1^n) \subseteq P$ implies $I_i \subseteq I$ for some $1 \leq i \leq n$.

Lemma 4.5 in [1] shows that the proper hyperideal I of R is prime if for all $a_1^n \in R$, $g(a_1^n) \in I$ implies $a_i \in I$ for some $1 \leq i \leq n$.

Definition 2.3. [1] The radical of a proper hyperideal I of R, denoted by $\boldsymbol{r}^{(m,n)}(I)$ is the intersection of all prime hyperideals of R containing I. If the set of all prime hyperideals which contain I is empty, then $\boldsymbol{r}^{(m,n)}(I) = R$.

It was shown (Theorem 4.23 in [1]) that if $a \in \boldsymbol{r}^{(m,n)}(I)$ then there exists $s \in \mathbb{N}$ with $g(a^{(s)}, 1_R^{(n-s)}) \in I$ for $s \leq n$, or $g_{(l)}(a^{(s)}) \in I$ for $s = l(n-1)+1$.

Definition 2.4. [1] A proper hyperideal I of R is primary if $g(a_1^n) \in I$ for $a_1^n \in R$ implies $a_i \in I$ or $g(a_1^{i-1}, 1_R, a_{i+1}^n) \in \boldsymbol{r}^{(m,n)}(I)$ for some $1 \leq i \leq n$.

Theorem 4.28 in [1] shows that the radical of a primary hyperideal of R is prime.

Definition 2.5. [18] Let I be a proper hyperideal of R. I refers to an

(1) (k,n)-absorbing hyperideal if for $r_1^{kn-k+1} \in R$, $g(r_1^{kn-k+1}) \in I$ implies that there exist $(k-1)n-k+2$ of the $r_i^,$s whose g-product is in I. In this case, if $k=1$, then I is an n-ary prime hyperideal of R. If $n=2$ and $k=1$, then I is a classic prime hyperideal of R.

(2) (k,n)-absorbing primary hyperideal if for $r_1^{kn-k+1} \in R$, $g(r_1^{kn-k+1}) \in I$ implies that $g(r_1^{(k-1)n-k+2}) \in I$ or a g-product of $(k-1)n-k+2$ of the $r_i^,$s, except $g(r_1^{(k-1)n-k+2})$, is in $\boldsymbol{r}^{(m,n)}(I)$.

3. ϕ-(k,n)-ABSORBING HYPERIDEALS

In his paper [16], Davvaz et al. introduced a generalization of the n-ary prime hyperideals in a Krasner (k,n)-hyperring, which they defined as weakly (k,n)-absorbing hyperideals. In this section, we generalize this notion to the context of ϕ-(k,n)-absorbing hyperideals.

Definition 3.1. Assume that $\mathcal{HI}(R)$ is the set of hyperideals of R and $\phi : \mathcal{HI}(R) \longrightarrow \mathcal{HI}(R) \cup \{\varphi\}$ is a function. Let k be a positive integer. A proper hyperideal Q of R is said to be ϕ-(k,n)-absorbing provided that for $r_1^{kn-k+1} \in R$, $g(r_1^{kn-k+1}) \in Q - \phi(Q)$ implies that there are $(k-1)n-k+2$ of the $r_i^,$s whose g-product is in Q.

Example 3.2. Consider the Krasner $(2,2)$-hyperring $R = \{0,1,x\}$ with the hyperaddition and multiplication defined by

$+$	0	1	x
0	0	1	x
1	1	R	1
x	x	1	$\{0,x\}$

$\cdot$	0	1	x
0	0	0	0
1	0	1	x
x	0	x	0

Assume that ϕ is a function from $\mathcal{HI}(R)$ to $\mathcal{HI}(R) \cup \{\varphi\}$ defined $\phi(I) = g(I^{(2)})$ for $I \in \mathcal{HI}(R)$. Then the hyperideal $Q = \{0, x\}$ is a ϕ-$(2,2)$-absorbing hyperideal of R.

Example 3.3. Let $t > 4$. Consider Krasner $(4,3)$-hyperring $(\mathbb{Z}_{5^{5t}}, +, \cdot)$ where $+$ and $\cdot$ are usual addition and multiplication. Defined $\phi(I) = I^5$ for $I \in \mathcal{HI}(\mathbb{Z}_{5^{5t}})$. Then $I = \langle 5^t \rangle$ is not a $(2,3)$-absorbing hyperideal of $\mathbb{Z}_{5^{5t}}$ since $5.5.5.5.5^{t-4} \in I - \phi(I)$ but $5.5.5, 5.5.5^{t-4} \notin I$.

Let $\phi : \mathcal{HI}(R) \longrightarrow \mathcal{HI}(R) \cup \{\varphi\}$ be a function. Clearly, every (k,n)-absorbing hyperideal in a Krasner (m,n)-hyperring is a ϕ-(k,n)-absorbing hyperideal. But, the following example shows that the converse does not necessarily hold.

Example 3.4. Assume that R is the Krasner $(2,4)$-hyperring given in Example 4.7 in [1]. In [16], it was shown that $\langle 0 \rangle$ is not a $(1,4)$-absorbing hyperideal of R. Now, defined $\phi(I) = g(I^{(4)})$ for $I \in \mathcal{HI}(R)$. In this hyperring, $\langle 0 \rangle$ is a ϕ-$(1,4)$-absorbing hyperideal of R.

Theorem 3.5. *Let $\phi_1, \phi_2 : \mathcal{HI}(R) \longrightarrow \mathcal{HI}(R) \cup \{\varphi\}$ be two functions such that for all $I \in \mathcal{HI}(R)$, $\phi_1(I) \subseteq \phi_2(I)$. If Q is a ϕ_1-(k,n)-absorbing hyperideal of R, then Q is a ϕ_2-(k,n)-absorbing hyperideal.*

Proof. Suppose that $g(r_1^{kn-k+1}) \in Q - \phi_2(Q)$ for $r_1^{kn-k+1} \in R$. From $\phi_1(Q) \subseteq \phi_2(Q)$, it follows that $g(r_1^{kn-k+1}) \in Q - \phi_1(Q)$. Since Q is a ϕ_1-(k,n)-absorbing hyperideal of R, we conclude that there are $(k-1)n - k + 2$ of the $r_i^,$s whose g-product is in Q, as needed. □

Theorem 3.6. *Let $\phi : \mathcal{HI}(R) \longrightarrow \mathcal{HI}(R) \cup \{\varphi\}$ be a function. If Q is a ϕ-(k,n)-absorbing hyperideal of R, then Q is a ϕ-(s,n)-absorbing hyperideal for all $s \geq k$.*

Proof. Let us use the induction on k that if Q is ϕ-(k,n)-absorbing hyperideal of R, then Q is ϕ-$(k+1,n)$-absorbing. Assume that Q is ϕ-$(2,n)$-absorbing and $g(r_1^{2n-2}, g(r_{2n-1}^{3n-2})) \in Q - \phi(Q)$ for some $r_1^{3n-2} \in R$. Since Q is ϕ-$(2,n)$-absorbing, then there are n of the $r_i^,$s except $g(r_{2n-1}^{3n-2})$ whose g-product is in Q and so there are $2n-1$ of the $r_i^,$s whose g-product is in Q. This shows that Q is ϕ-$(3,n)$-absorbing. Assume that Q is ϕ-(k,n)-absorbing and $g(g(r_1^{2n-2}), r_{2n-1}^{(k+1)n-(k+1)+1}) \in Q - \phi(Q)$ for some $r_1^{(k+1)n-(k+1)+1} \in R$. Since Q is ϕ-(k,n)-absorbing, we conclude that $g(g(r_1^{2(n-1)}), r_{2n-1}, \cdots, \widehat{r_i}, \cdots, r_{(k+1)n-(k+1)+1}) \in Q$ for some $2(n-1) \leq i \leq (k+1)n-(k+1)+1$ or $g(r_{2n-1}^{(k+1)n-(k+1)+1}) \in Q$. The former case shows that Q is ϕ-$(k+1,n)$-absorbing. In the latter case, we obtain $g(r_1^{n-1}, r_{2n-1}^{(k+1)n-(k+1)+1}) \in Q$ since $g(r_1^{2(n-1)}) \in Q$. Thus Q is ϕ-$(k+1,n)$-absorbing. □

Recall from [15] that if (R_1, f_1, g_1) and (R_2, f_2, g_2) are two Krasner (m,n)-hyperrings such that 1_{R_1} and 1_{R_2} are scalar identities of R_1 and R_2, respectively, then $(R_1 \times R_2, f_1 \times f_2, g_1 \times g_2)$ is a Krasner (m,n)-hyperring where

$$\begin{aligned} f &= f_1 \times f_2((a_1,b_1),\cdots,(a_m,b_m)) = \{(a,b) \mid \; a \in f_1(a_1^m), b \in f_2(b_1^m)\}, \\ g &= g_1 \times g_2((x_1,y_1),\cdots,(x_n,y_n)) = (g_1(x_1^n), g_2(y_1^n)), \end{aligned}$$

for all $a_1^m, x_1^n \in R_1$ and $b_1^m, y_1^n \in R_2$.

Theorem 3.7. *Let (R_i, f_i, g_i) be a commutative Krasner (m,n)-hyperring for each $1 \leq i \leq kn-k+1$ and $\phi_i : \mathcal{HI}(R_i) \longrightarrow \mathcal{HI}(R_i) \cup \{\varphi\}$ be a function. Let Q_i be a hyperideal of R_i for each $1 \leq i \leq kn-k+1$ and $\phi = \phi_1 \times \cdots \times \phi_{kn-k+1}$. If $Q = Q_1 \times \cdots \times Q_{kn-k+1}$ is a ϕ-$(k+1,n)$-absorbing hyperideal of $R = R_1 \times \cdots \times R_{kn-k+1}$, then Q_i is a ϕ_i-(k,n)-absorbing hyperideal of R_i and $Q_i \neq R_i$ for all $1 \leq i \leq kn-k+1$.*

Proof. Let $r_1^{kn-k+1} \in R_i$ such that $g(r_1^{kn-k+1}) \in Q_i - \phi_i(Q_i)$. Suppose by contradiction that Q_i is not a ϕ_i-(k,n)-absorbing hyperideal of R_i. Define

$$\begin{aligned} a_1 &= (1_{R_1},\cdots,1_{R_{i-1}},r_1,1_{R_{i+1}},\cdots,1_{R_{kn-k+1}}), \\ a_2 &= (1_{R_1},\cdots,1_{R_{i-1}},r_2,1_{R_{i+1}},\cdots,1_{R_{kn-k+1}}), \\ &\vdots \\ a_{kn-k+1} &= (1_{R_1},\cdots,1_{R_{i-1}},r_{kn-k+1},1_{R_{i+1}},\cdots,1_{R_{kn-k+1}}), \\ a_{kn-k} &= (1_{R_1},\cdots,1_{R_{i-1}},1_{R_i},1_{R_{i+1}},\cdots,1_{R_{kn-k+1}}), \\ a_{(k+1)n-(k+1)+1} &= (0,\cdots,0,1_{R_i},0,\cdots,0). \end{aligned}$$

Hence $g(a_1^{(k+1)n-(k+1)+1)}) \in Q - \phi(Q)$ but $g(a_1^{kn-k+1}) \notin Q$. Since Q is a ϕ-$(k+1,n)$-absorbing hyperideal of R, we conclude that one of g-productions of $kn-k+1$ of $a_i^,$s except $g(a_1^{(k+1)n-(k+1)+1})$ is in Q. This implies that there exist $(k-1)n-k+2$ of $r_i^,$s whose g-product is in Q_i which is a contradiction. Consequently, Q_i is a ϕ_i-(k,n)-absorbing hyperideal of R_i. □

Assume that (R_1, f_1, g_1) and (R_2, f_2, g_2) are two Krasner (m,n)-hyperrings. Recall from [27] that a mapping $h : R_1 \longrightarrow R_2$ is called a homomorphism if for all $a_1^m \in R_1$ and $b_1^n \in R_1$ we have $(1) h(f_1(a_1,...,a_m)) = f_2(h(a_1),...,h(a_m))$, $(2) h(g_1(b_1,...,b_n)) = g_2(h(b_1),...,h(b_n))$. Moreover, recall from [19] that a function $\phi : \mathcal{HI}(R) \longrightarrow \mathcal{HI}(R) \cup \{\varphi\}$ is called a reduction function of $\mathcal{HI}(R)$ if $\phi(P) \subseteq P$ and $P \subseteq Q$ implies that $\phi(P) \subseteq \phi(Q)$ for all $P, Q \in \mathcal{HI}(R)$.

Now, assume that R_1 and R_2 are two Krasner (m,n)-hyperring such that $h : R_1 \longrightarrow R_2$ is a homomorphism. Suppose that ϕ_1 and ϕ_2 are two reduction functions of $\mathcal{HI}(R_1)$ and $\mathcal{HI}(R_2)$, respectively. If $\phi_1(h^{-1}(I_2)) = h^{-1}(\phi_2(I_2))$ for all $I_2 \in \mathcal{HI}(R_2)$, then we say h is a ϕ_1-ϕ_2-homomorphism. Let h be a ϕ_1-ϕ_2-epimorphism from R_1 to R_2 and let I_1 be a hyperideal of R_1 with $Ker(h) \subseteq I_1$. It is easy to see that $\phi_2(h(I_1)) = h(\phi_1(I_1))$.

Example 3.8. Let R_1 and R_2 be two Krasner (m,n)-hyperrings and ϕ_1 and ϕ_2 be two empty reduction functions of $\mathcal{HI}(R_1)$ and $\mathcal{HI}(R_2)$, respectively. Then every homorphism h from R_1 to R_2 is a ϕ_1-ϕ_2-homomorphism.

Theorem 3.9. *Let $h : R_1 \longrightarrow R_2$ be a ϕ_1-ϕ_2-homomorphism, where ϕ_1 and ϕ_2 are two reduction functions of $\mathcal{HI}(R_1)$ and $\mathcal{HI}(R_2)$, respectively. Then*

(1) *If Q_2 is a ϕ_2-(k,n)-absorbing hyperideal of R_2, then $h^{-1}(Q_2)$ is a ϕ_1-(k,n)-absorbing of R_1.*

(2) *If h is surjective and Q_1 is a ϕ_1-(k,n)-absorbing hyperideal of R_1 with $Ker(h) \subseteq Q_1$, then $h(Q_1)$ is a ϕ_2-(k,n)-absorbing hyperideal of R_2.*

Proof. (1) Let Q_2 be a ϕ_2-(k,n)-absorbing hyperideal of R_2 and $g(r_1^{kn-k+1}) \in h^{-1}(Q_2) - \phi_1(h^{-1}(Q_2))$ for some $r_1^{kn-k+1} \in R_1$. Then we get $h(g(r_1^{kn-k+1})) = g(h(r_1), \cdots, h(r_{kn-k+1})) \in Q_2 - \phi_2(Q_2)$. Since Q_2 is a ϕ_2-(k,n)-absorbing hyperideal of R_2, we conclude that the image of h of $(k-1)n-k+2$ of $r_i^{,}$s whose g-product is in Q_2. Then there exist $(k-1)n-k+2$ of $r_i^{,}$s whose g-product is in $h^{-1}(Q_2)$. Thus $h^{-1}(Q_2)$ is a ϕ_1-(k,n)-absorbing of R_1.

(2) Suppose that Q_1 is a ϕ_1-(k,n)-absorbing hyperideal of R_1 with $Ker(h) \subseteq Q_1$ and h is surjective. Let $g(s_1^{kn-k+1}) \in h(Q_1) - \phi_2(h(Q_1))$ for some $s_1^{kn-k+1} \in R_2$. Then there exists $r_i \in R_1$ for every $1 \leq i \leq kn-k+1$ such that $h(r_i) = s_i$. Hence we get $h(g(r_1^{kn-k+1}) = g(h(r_1), \cdots, h(r_{kn-k+1})) = g(s_1^{kn-k+1}) \in h(Q_1)$. Since $Ker(h) \subseteq Q_1$ and h is a ϕ_1-ϕ_2-epimorphism, we have $g(r_1^{kn-k+1}) \in Q_1 - \phi_1(Q_1)$. Since Q_1 is a ϕ_1-(k,n)-absorbing hyperideal of R_1, there are $(k-1)n-k+2$ of $r_i^{,}$s whose g-product is in Q_1. Now, since h is a homomorphism, we are done. □

Let P be a hyperideal of R. Then the set $R/P = \{f(a_1^{i-1}, P, a_{i+1}^m) \mid a_1^{i-1}, a_{i+1}^m \in R\}$ with m-ary hyperoperation f and n-operation g is the quotient Krasner (m,n)-hyperring of R by P. Theorem 3.2 in [1] shows that the projection map π from R to R/P, defined by $\pi(r) = f(r, P, 0^{(m-2)})$, is homomorphism. Let P be a hyperideal of R and ϕ be a reduction function of $\mathcal{HI}(R)$. Then the function ϕ_q from $\mathcal{HI}(R/P)$ to $\mathcal{HI}(R/P) \cup \{\varphi\}$ defined by $\phi_q(I/P) = \phi(I)/P$ is a reduction function. Now, we have the following theorem as a result of Theorem 3.9 that is easily verified, and hence we omit the proof.

Theorem 3.10. *Let Q and P be two hyperideals of R and ϕ be a reduction function of $\mathcal{HI}(R)$ such that $P \subseteq \phi(Q) \subseteq Q$. If Q is a ϕ-(k,n)-absorbing hyperideal of R, then Q/P is a ϕ_q-(k,n)-absorbing hyperideal of R/P.*

4. ϕ-(k,n)-absorbing primary hyperideals

Definition 4.1. Suppose that $\mathcal{HI}(R)$ is the set of hyperideals of R and $\phi : \mathcal{HI}(R) \longrightarrow \mathcal{HI}(R) \cup \{\varphi\}$ is a function. Let k be a positive integer. A proper hyperideal Q of R is called ϕ-(k,n)-absorbing primary if $g(r_1^{kn-k+1}) \in Q - \phi(Q)$ for $r_1^{kn-k+1} \in R$ implies that $g(r_1^{(k-1)n-k+2}) \in Q$ or a g-product of $(k-1)n-k+2$ of $r_i^,$s ,except $g(r_1^{(k-1)n-k+2})$, is in $\boldsymbol{r}^{(m,n)}(Q)$.

Example 4.2. Every ϕ-(k,n)-absorbing of a Krasner (m,n)-hyperring is ϕ-(k,n)-absorbing primary.

The converse may not be always true as it is shown in the following example.

Example 4.3. Consider the Krasner $(2,2)$-hyperring $R=[0,1]$ with the 2-ary hyperoperation defined by

$$a \oplus b = \begin{cases} \{\max\{a,b\}\}, & \text{if } a \neq b, \\ [0,a], & \text{if } a = b, \end{cases}$$

and multiplication is the usual multiplication on real numbers. Suppose that ϕ is a function from $\mathcal{HI}(R)$ to $\mathcal{HI}(R) \cup \{\varphi\}$ defined $\phi(I) = \cap_{i=1}^{\infty} g(I^{(i)})$ for $I \in \mathcal{HI}(R)$.Then the hyperideal $Q=[0,0.5]$ is a ϕ-$(2,2)$-absorbing primary hyperideal of R but it is not ϕ-$(2,2)$-absorbing.

The next theorem provides us how to determine ϕ-(k,n)-absorbing primary hyperideal to be (k,n)-absorbing primary.

Theorem 4.4. *Assume that Q is a hyperideal of R and $\phi : \mathcal{HI}(R) \longrightarrow \mathcal{HI}(R) \cup \{\varphi\}$ is a reduction function such that $\phi(Q)$ is a (k,n)-absorbing primary huperideal of R. If Q is a ϕ-(k,n)-absorbing primary hyperideal of R, then Q is a (k,n)-absorbing primary hyperideal of R.*

Proof. Let $r_1^{kn-k+1} \in R$ such that $g(r_1^{kn-k+1}) \in Q$ and $g(r_1^{(k-1)n-k+2}) \notin Q$. Assume that $g(r_1^{kn-k+1}) \in \phi(Q)$. Since $\phi(Q)$ is a (k,n)-absorbing primary hyperideal of R and $g(r_1^{(k-1)n-k+2}) \notin \phi(Q)$, we conclude that a g-product of $(k-1)n-k+2$ of the $r_i^,$s, except $g(r_1^{(k-1)n-k+2})$ is in $\boldsymbol{r}^{(m,n)}(\phi(Q)) \subseteq \boldsymbol{r}^{(m,n)}(Q)$, as needed. Suppose that $g(r_1^{kn-k+1}) \notin \phi(Q)$. Since Q is a ϕ-(k,n)-absorbing primary hyperideal of R, we are done. □

In the following, the relationship between a ϕ-(k,n)-absorbing primary hyperideal of R and its radical is considered.

Theorem 4.5. *Let Q be a hyperideal of R and $\phi : \mathcal{HI}(R) \longrightarrow \mathcal{HI}(R) \cup \{\varphi\}$ be a function such that $\boldsymbol{r}^{(m,n)}(\phi(Q)) = \phi(\boldsymbol{r}^{(m,n)}(Q))$. If Q is a ϕ-(k,n)-absorbing primary hyperideal of R, then $\boldsymbol{r}^{(m,n)}(Q)$ is a ϕ-(k,n)-absorbing hyperideal of R.*

Proof. Let $r_1^{kn-k+1} \in R$ such that $g(r_1^{kn-k+1}) \in \boldsymbol{r}^{(m,n)}(Q) - \phi(\boldsymbol{r}^{(m,n)}(Q))$. Assume that all products of $(k-1)n-k+2$ of the r_i's except $g(r_1^{(k-1)n-k+2})$ are not in $\boldsymbol{r}^{(m,n)}(Q)$. Since $g(r_1^{kn-k+1}) \in \boldsymbol{r}^{(m,n)}(Q)$, then there exists $s \in \mathbb{Z}^+$ with $g(g(r_1^{kn-k+1})^{(s)}, 1^{(n-s)}) \in Q$, for $s \leq n$ or $g_{(l)}(g(r_1^{kn-k+1})^{(s)}) \in Q$, for $s > n$, $s = l(n-1)+1$. In the former case, we get $g(g(r_1)^{(s)}, g(r_2)^{(s)}, \cdots, g(r_{kn-k+1})^{(s)}, 1^{(n-s)}) \in Q$. If $g(g(r_1)^{(s)}, g(r_2)^{(s)}, \cdots, g(r_{kn-k+1})^{(s)}, 1^{(n-s)}) \in \phi(Q)$, we obtain $g(r_1^{kn-k+1}) \in \boldsymbol{r}^{(m,n)}(\phi(Q)) = \phi(\boldsymbol{r}^{(m,n)}(Q))$, a contradiction. Since Q is a ϕ-(k,n)-absorbing primary hyperideal of R, then $g(g(r_1)^{(s)}, g(r_2)^{(s)}, \cdots, g(r_{(k-1)n-k+2})^{(s)}), 1^{(n-s)}) = g(g(r_1^{(k-1)n-k+2)})^{(s)}, 1^{(n-s)}) \in Q$ which means $g(r_1^{(k-1)n-k+2}) \in \boldsymbol{r}^{(m,n)}(Q)$. For the other case, we have a similar argument. Consequently, $\boldsymbol{r}^{(m,n)}(Q)$ is a ϕ-(k,n)-absorbing hyperideal of R. □

Example 4.6. Assume that $H = \mathbb{Z}_3[X,Y,Z]$ and $Q = \langle X^3Y^3Z^3 \rangle$. Then $R = H/Q$ is a Krasner (m,n)-hyperring with ordinary addition and ordinary multiplication. Defined $\phi(I/Q) = 0_R$ for $I/Q \in \mathcal{HI}(R)$. In the hyperring, Q/Q is a ϕ-$(1,3)$-absorbing primary hyperideal of R and $\boldsymbol{r}^{(m,n)}(\phi(Q/Q)) \neq \phi(\boldsymbol{r}^{(m,n)}(Q/Q))$. Note that $\boldsymbol{r}^{(m,n)}(Q/Q)$ is not a ϕ-$(1,3)$-absorbing hyperideal of R because $2XYZ + Q = (2X+Q)(Y+Q)(Z+Q) \in \boldsymbol{r}^{(m,n)}(Q/Q) - \phi(\boldsymbol{r}^{(m,n)}(Q/Q))$ but none of the elements $(2X+Q), (Y+Q)$ and $(Z+Q)$ are not in $\boldsymbol{r}^{(m,n)}(Q/Q))$.

Theorem 4.7. *Assume that $\phi : \mathcal{HI}(R) \longrightarrow \mathcal{HI}(R) \cup \{\varphi\}$ is a function. If Q is a ϕ-(k,n)-absorbing primary hyperideal of R, then Q is a ϕ-(s,n)-absorbing primary hyperideal for all $s \geq k$.*

Proof. Let Q be a ϕ-(k,n)-absorbing primary hyperideal of R. Suppose that $g(g(r_1^{n+2}), r_{n+3}^{(k+1)n-(k+1)+1}) \in Q - \phi(Q)$ for some $r_1^{(k+1)n-(k+1)+1} \in R$. Put $g(r_1^{n+2}) = a_1$. Then we conclude that $g(a_1, \cdots, r_{(k+1)n-(k+1)+1}) \in Q$ or a g-product of $kn-k+1$ of the r_i's, except $g(a_1, \cdots, r_{(k+1)n-(k+1)+1})$ is in $\boldsymbol{r}^{(m,n)}(Q)$ as Q is a ϕ-(k,n)-absorbing primary hyperideal of R. Since $\boldsymbol{r}^{(m,n)}(Q)$ is a hyperideal of R and $r_1^{n+2} \in R$, we conclude that $g(r_1, r_{n+3}, \cdots, r_{(k+1)n-(k+1)+1}) \in \boldsymbol{r}^{(m,n)}(Q)$ or $\cdots$ or $g(r_{n+2}, r_{n+3}, \cdots, r_{(k+1)n-(k+1)+1}) \in \boldsymbol{r}^{(m,n)}(Q)$ and so Q is $(k+1,n)$-absorbing primary. □

Theorem 4.8. *Let $\phi_1, \phi_2 : \mathcal{HI}(R) \longrightarrow \mathcal{HI}(R) \cup \{\varphi\}$ be two functions such that for all $I \in \mathcal{HI}(R)$, $\phi_1(I) \subseteq \phi_2(I)$. If Q is a ϕ_1-(k,n)-absorbing primary hyperideal of R, then Q is a ϕ_2-(k,n)-absorbing primary hyperideal.*

Proof. It is proved in a similar way to Theorem 3.5. □

Theorem 4.9. *Let $\phi : \mathcal{HI}(R) \longrightarrow \mathcal{HI}(R) \cup \{\varphi\}$ be a function. If Q is a ϕ-$(1,n)$-absorbing primary hyperideal of R, then Q is a ϕ-$(2,n)$-absorbing primary hyperideal.*

Proof. Let Q be a ϕ-$(1,n)$-absorbing primary hyperideal and $g(g(r_1^n), \cdots, r_{2n-1}) \in Q - \phi(Q)$ for some $r_1^{2n-1} \in R$. Then we get $g(r_1^n) \in Q$ or $g(r_{n+1}^{2n-1}) \in \boldsymbol{r}^{(m,n)}(Q)$. By definition of hyperideal, we conclude that $g(r_1, r_{n+1}, \cdots, r_{2n-1}) \in \boldsymbol{r}^{(m,n)}(Q)$ or $\cdots$ or $g(r_1, r_{n+1}, \cdots, r_{2n-1}) \in \boldsymbol{r}^{(m,n)}(Q)$ since $r_1^n \in R$. Consequently, Q is a ϕ-$(2,n)$-absorbing primary hyperideal of R. □

Let Q be a proper hyperideal of R and $\phi : \mathcal{HI}(R) \longrightarrow \mathcal{HI}(R) \cup \{\varphi\}$ be a function. Q refers to a strongly ϕ-(k,n)-absorbing primary hyperideal of R if $g(Q_1^{kn-k+1}) \subseteq Q - \phi(Q)$ for some hyperideals Q_1^{kn-k+1} of R implies that $g(Q_1^{(k-1)n-k+2}) \subseteq Q$ or a g-product of $(k-1)n-k+2$ of Q_i's, except $g(Q_1^{(k-1)n-k+2})$, is a subset of $\boldsymbol{r}^{(m,n)}(Q)$. In the sequel, we assume that all ϕ-(k,n)-absorbing primary hyperideals of R are strongly ϕ-(k,n)-absorbing primary hyperideal. Recall from [16] that a proper hyperideal Q of R is called weakly (k,n)-absorbing primary if $0 \neq g(r_1^{kn-k+1}) \in Q$ for $r_1^{kn-k+1} \in R$ implies that $g(r_1^{(k-1)n-k+2}) \in Q$ or a g-product of $(k-1)n-k+2$ of r_i's ,except $g(r_1^{(k-1)n-k+2})$, is in $\boldsymbol{r}^{(m,n)}(Q)$.

Theorem 4.10. *Suppose that Q is a proper hyperideal of a commutative Krasner $(m,2)$-hyperring R and $\phi : \mathcal{HI}(R) \longrightarrow \mathcal{HI}(R) \cup \{\varphi\}$ is a function. Then the followings are equivalent:*

(1) *Q is a ϕ-$(2,2)$-absorbing primary hyperideal of R.*

(2) *$Q/\phi(Q)$ is a weakly $(2,2)$-absorbing primary hyperideal of $R/\phi(Q)$.*

Proof. (1) $\Longrightarrow$ (2) Let Q be ϕ-$(2,2)$-absorbing primary and for $a_{11}^{1m}, a_{21}^{2m}, a_{31}^{3m} \in R$,

$$\begin{aligned}
\phi(Q) &\neq g(f(a_{11}^{1(i-1)}, \phi(Q), a_{1(i+1)}^{1m}), f(a_{21}^{2(i-1)}, \phi(Q), a_{2(i+1)}^{2m}), f(a_{31}^{3(i-1)}, \phi(Q), a_{3(i+1)}^{3m})) \\
&= f(g(a_{11}^{31}), \cdots, g(a_{1(i-1)}^{3(i-1)}), \phi(Q), g(a_{1(i+1)}^{3(i+1)}), \cdots, g(a_{1m}^{3m})) \\
&\in Q/\phi(Q).
\end{aligned}$$

Then

$$\begin{aligned}
&f(g(a_{11}^{31}), \cdots, g(a_{1(i-1)}^{3(i-1)}), 0, g(a_{1(i+1)}^{3(i+1)}), \cdots, g(a_{1m}^{3m})) \\
&\qquad = g(f(a_{11}^{1(i-1)}, 0, a_{1(i+1)}^{1m}), f(a_{21}^{2(i-1)}, 0, a_{2(i+1)}^{2m}), f(a_{31}^{3(i-1)}, 0, a_{3(i+1)}^{3m})) \\
&\qquad \in Q - \phi(Q).
\end{aligned}$$

Since Q is a ϕ-$(2,2)$-absorbing primary hyperideal of R, we get

$$\begin{aligned} & g(f(a_{11}^{1(i-1)},0,a_{1(i+1)}^{1m}),f(a_{21}^{2(i-1)},0,a_{2(i+1)}^{2m})) \\ &\qquad = f(g(a_{11}^{21}),\cdots,g(a_{1(i-1)}^{2(i-1)}),0,g(a_{1(i+1)}^{2(i+1)}),\cdots,g(a_{1m}^{2m})) \subseteq Q, \\ & g(f(a_{21}^{2(i-1)},0,a_{2(i+1)}^{2m}),f(a_{31}^{3(i-1)},0,a_{3(i+1)}^{3m})) \\ &\qquad = f(g(a_{21}^{31}),\cdots,g(a_{2(i-1)}^{3(i-1)}),0,g(a_{2(i+1)}^{3(i+1)}),\cdots,g(a_{2m}^{3m})) \subseteq \boldsymbol{r}^{(m,n)}(Q), \end{aligned}$$

or

$$\begin{aligned} & g(f(a_{11}^{1(i-1)},0,a_{1(i+1)}^{1m}),f(a_{31}^{3(i-1)},0,a_{3(i+1)}^{3m})) \\ &\qquad = f(g(a_{11}^{31}),\cdots,g(a_{1(i-1)}^{3(i-1)}),0,g(a_{1(i+1)}^{3(i+1)}),\cdots,g(a_{1m}^{3m})) \subseteq \boldsymbol{r}^{(m,n)}(Q). \end{aligned}$$

It implies that

$$\begin{aligned} & f(g(a_{11}^{21}),\cdots,g(a_{1(i-1)}^{2(i-1)}),\phi(Q),g(a_{1(i+1)}^{2(i+1)}),\cdots,g(a_{1m}^{2m})) \\ &\qquad = g(f(a_{11}^{1(i-1)},\phi(Q),a_{1(i+1)}^{1m}),f(a_{21}^{2(i-1)},\phi(Q),a_{2(i+1)}^{2m})) \in Q/\phi(Q), \end{aligned}$$

or

$$\begin{aligned} & f(g(a_{21}^{31}),\cdots,g(a_{2(i-1)}^{3(i-1)}),\phi(Q),g(a_{2(i+1)}^{3(i+1)}),\cdots,g(a_{2m}^{3m})) \\ &\qquad = g(f(a_{21}^{2(i-1)},\phi(Q),a_{2(i+1)}^{2m}),f(a_{31}^{3(i-1)},\phi(Q),a_{3(i+1)}^{3m})) \\ &\qquad \in \boldsymbol{r}^{(m,n)}(Q)/\phi(Q) = \boldsymbol{r}^{(m,n)}(Q/\phi(Q)), \end{aligned}$$

or

$$\begin{aligned} & f(g(a_{11}^{31}),\cdots,g(a_{1(i-1)}^{3(i-1)}),\phi(Q),g(a_{1(i+1)}^{3(i+1)}),\cdots,g(a_{1m}^{3m})) \\ &\qquad = g(f(a_{11}^{1(i-1)},\phi(Q),a_{1(i+1)}^{1m}),f(a_{31}^{3(i-1)},\phi(Q),a_{3(i+1)}^{3m})) \\ &\qquad \in \boldsymbol{r}^{(m,n)}(Q)/\phi(Q) = \boldsymbol{r}^{(m,n)}(Q/\phi(Q)). \end{aligned}$$

(2) $\Longrightarrow$ (1) Let $g(r_1^3) \in Q-\phi(Q)$ for some $r_1^3 \in R$. Therefore we obtain $f(g(r_1^3),\phi(Q),0^{(m-2)}) \neq \phi(Q)$. It follows that

$$\phi(Q) \neq g(f(r_1,\phi(Q),0^{(m-2)}),f(r_2,\phi(Q),0^{(m-2)}),f(r_3,\phi(Q),0^{(m-2)})) \in Q/\phi(Q).$$

By the hypothesis, we get

$$g(f(r_1,\phi(Q),0^{(m-2)}),f(r_2,\phi(Q),0^{(m-2)})) = f(g(r_1^2),\phi(Q),0^{(m-2)}) \in Q/\phi(Q).$$

or

$$g(f(r_2,\phi(Q),0^{(m-2)}),f(r_3,\phi(Q),0^{(m-2)})) = f(g(r_2^3),\phi(Q),0^{(m-2)}) \in \boldsymbol{r}^{(m,n)}(Q)/\phi(Q).$$

or

$$g(f(r_1,\phi(Q),0^{(m-2)}),f(r_3,\phi(Q),0^{(m-2)}))=f(g(r_1^3),\phi(Q),0^{(m-2)})\in \boldsymbol{r}^{(m,n)}(Q)/\phi(Q).$$

This shows that $g(r_1^2)\in Q$ or $g(r_2^3)\in \boldsymbol{r}^{(m,n)}(Q)$ or $g(r_1^3)\in \boldsymbol{r}^{(m,n)}(Q)$. Consequently, Q is a ϕ-$(2,2)$-absorbing primary hyperideal of R. □

Suppose that I is a weakly $(2,2)$-absorbing primary hyperideal of a commutative Krasner $(m,2)$-hyperring R. Recall from [16] that (x,y,z) is said to be $(2,2)$-zero primary of I for $x,y,z\in R$, if $g(x,y,z)=0$, $g(x,y)\notin I$, $g(y,z)\notin \boldsymbol{r}^{(m,n)}(I)$ and $g(x,z)\notin \boldsymbol{r}^{(m,n)}(I)$. Now, assume that Q is a ϕ-$(2,2)$-absorbing primary hyperideal of a commutative Krasner $(m,2)$-hyperring R. Then we say (x,y,z) is a ϕ-$(2,2)$ primary of Q for some $x,y,z\in R$ if $g(x,y,z)\in \phi(Q)$, $g(x,y)\notin Q$, $g(y,z)\notin \boldsymbol{r}^{(m,n)}(Q)$ and $g(x,z)\notin \boldsymbol{r}^{(m,n)}(Q)$. It is easy to see that a proper hyperideal Q of R is ϕ-$(2,2)$-absorbing primary that is not $(2,2)$-absorbing primary if and only if Q has a ϕ-$(2,2)$ primary (x,y,z) for some $x,y,z\in R$.

Theorem 4.11. *Let R be a commutative Krasner $(m,2)$-hyperring and let $\phi:\mathcal{HI}(R)\longrightarrow \mathcal{HI}(R)\cup\{\varphi\}$ be a function. Let Q be a ϕ-$(2,2)$-absorbing primary hyperideal of R and $x,y,z\in R$. Then the followings are equivalent:*

(1) *(x,y,z) is a ϕ-$(2,2)$ primary of Q.*

(2) *$(f(x,\phi(Q),0^{(m-2)}),f(y,\phi(Q),0^{(m-2)}),f(z,\phi(Q),0^{(m-2)})$ is a $(2,2)$-zero primary of $Q/\phi(Q)$.*

Proof. (1) $\Longrightarrow$ (2) Let (x,y,z) be a ϕ-$(2,2)$ primary of Q. This means that $g(x,y,z)\in\phi(Q)$, $g(x,y)\notin Q$, $g(y,z)\notin \boldsymbol{r}^{(m,n)}(Q)$ and $g(x,z)\notin \boldsymbol{r}^{(m,n)}(Q)$. This implies that $f(g(x,y),Q,0^{(m-2)})\notin Q/\phi(Q)$, $f(g(y,z),\phi(Q),0^{(m-2)})\notin \boldsymbol{r}^{(m,n)}(Q)/\phi(Q)$ and $f(g(x,z),\phi(Q),0^{(m-2)})\notin \boldsymbol{r}^{(m,n)}(Q)/\phi(Q)$. By Theorem 4.10, we conclude that $(f(x,\phi(Q),0^{(m-2)}),f(y,\phi(Q),0^{(m-2)}),f(z,\phi(Q),0^{(m-2)})$ is a $(2,2)$-zero primary of $Q/\phi(Q)$.

(2) $\Longrightarrow$ (1) Assume that $(f(x,\phi(Q),0^{(m-2)}),f(y,\phi(Q),0^{(m-2)}),f(z,\phi(Q),0^{(m-2)})$ is a $(2,2)$-zero primary of $Q/\phi(Q)$. Thus $g(x,y,z)\in\phi(Q)$ but $f(g(x,y),Q,0^{(m-2)})\notin Q/\phi(Q)$, $f(g(y,z),\phi(Q),0^{(m-2)})\notin \boldsymbol{r}^{(m,n)}(Q)/\phi(Q)$ and $f(g(x,z),\phi(Q),0^{(m-2)})\notin \boldsymbol{r}^{(m,n)}(Q)/\phi(Q)$. Hence $g(x,y,z)\in\phi(Q)$, $g(x,y)\notin Q$, $g(y,z)\notin \boldsymbol{r}^{(m,n)}(Q)$ and $g(x,z)\notin \boldsymbol{r}^{(m,n)}(Q)$. It implies that (x,y,z) is a ϕ-$(2,2)$ primary of Q. □

Theorem 4.12. *Let R be a commutative Krasner $(m,2)$-hyperring and let $\phi:\mathcal{HI}(R)\longrightarrow \mathcal{HI}(R)\cup\{\varphi\}$ be a function. Let Q be a ϕ-$(2,2)$-absorbing primary hyperideal of R. If (x,y,z) is a ϕ-$(2,2)$ primary of Q for some $x,y,z\in R$, then*

(1) *$g(x,y,Q),g(y,z,Q),g(x,z,Q)\subseteq\phi(Q)$.*

(2) $g(x, Q^{(2)}), g(y, Q^{(2)}), g(z, Q^{(2)}) \subseteq \phi(Q)$.

(3) $g(Q^{(3)}) \subseteq \phi(Q)$.

Proof. (1) Let (x, y, z) be a ϕ-$(2,2)$ primary of a ϕ-$(2,2)$-absorbing primary hyperideal Q. By Theorem 4.11, $(f(x, \phi(Q), 0^{(m-2)}), f(y, \phi(Q), 0^{(m-2)}), f(z, \phi(Q), 0^{(m-2)})$ is a $(2,2)$-zero primary of $Q/\phi(Q)$ since (x, y, z) is a ϕ-$(2,2)$ primary of Q. Thus

$$f(g(x, y, Q), \phi(Q), 0^{(m-2)}) = f(g(y, z, Q), \phi(Q), 0^{(m-2)}) = f(g(x, z, Q), \phi(Q), 0^{(m-2)}) = \phi(Q),$$

by Theorem 4.9 in [16], which implies $g(x, y, Q)$, $g(y, z, Q)$ and $g(x, z, Q)$ are subsets of $\phi(Q)$.

(2) Theorem 4.11 shows that $(f(x, \phi(Q), 0^{(m-2)}), f(y, \phi(Q), 0^{(m-2)}), f(z, \phi(Q), 0^{(m-2)})$ is a $(2,2)$-zero primary of $Q/\phi(Q)$. Moreover, Theorem 4.10 shows that $Q/\phi(Q)$ is a weakly $(2,2)$-absorbing primary of $R/\phi(Q)$. Then $f(g(x, Q^{(2)}), \phi(Q), 0^{(m-2)}) = f(g(y, Q^{(2)}), \phi(Q), 0^{(m-2)}) = f(g(z, Q^{(2)}), \phi(Q), 0^{(m-2)}) = \phi(Q)$, by Theorem 4.9 of [16]. Consequently, $g(x, Q^{(2)}), g(y, Q^{(2)}), g(z, Q^{(2)})$ are subsets of $\phi(Q)$.

(3) Again, $(f(x, \phi(Q), 0^{(m-2)}), f(y, \phi(Q), 0^{(m-2)}), f(z, \phi(Q), 0^{(m-2)})$ is a $(2,2)$-zero primary of $Q/\phi(Q)$ and $Q/\phi(Q)$ is a weakly $(2,2)$-absorbing primary of $R/\phi(Q)$ by Theorem 4.11 and Theorem 4.10, respectively, then $f(g(Q^{(3)}), \phi(Q), 0^{(m-2)}) = \phi(Q)$ by Theorem 4.10 in [16]. Thus $g(Q^{(3)})$ is a subset of $\phi(Q)$. □

Theorem 4.13. *Suppose that Q is a proper hyperideal of a commutative Krasner (m, n)-hyperring R and $\phi : \mathcal{HI}(R) \longrightarrow \mathcal{HI}(R) \cup \{\varphi\}$ is a function. Then the followings are equivalent:*

(1) *Q is a ϕ-(k, n)-absorbing primary hyperideal of R.*

(2) *$Q/\phi(Q)$ is a weakly (k, n)-absorbing primary hyperideal of $R/\phi(Q)$.*

Proof. It can be easily proved in a similar manner to the proof of Theorem 4.10. □

Suppose that Q is a ϕ-(k, n)-absorbing primary hyperideal of R. Then we say $(r_1^{k(n-1)+1})$ is a ϕ-(k, n) primary of Q for some $r_1^{k(n-1)+1} \in R$ if $g(r_1^{k(n-1)+1}) \in \phi(Q)$, $g(r_1^{(k-1)n-k+2}) \notin Q$ and a g-product of $(k-1)n-k+2$ of r_i's, except $g(r_1^{(k-1)n-k+2})$, is not in $\boldsymbol{r}^{(m,n)}(Q)$.

Theorem 4.14. *Let R be a commutative Krasner $(m, 2)$-hyperring and let $\phi : \mathcal{HI}(R) \longrightarrow \mathcal{HI}(R) \cup \{\varphi\}$ be a function. Let Q be a ϕ-(k, n)-absorbing primary hyperideal of R and $r_1^{k(n-1)+1} \in R$. Then the followings are equivalent:*

(1) *$(r_1^{k(n-1)+1})$ is a ϕ-(k, n) primary of Q.*

(2) *$(f(r_1, \phi(Q), 0^{(m-2)}), \cdots, f(r_{k(n-1)+1}, \phi(Q), 0^{(m-2)})$ is a (k, n)-zero primary of $Q/\phi(Q)$.*

Proof. It is seen to be true in a similar manner to Theorem 4.11. □

Theorem 4.15. *Let R be a commutative Krasner (m,n)-hyperring and let $\phi : \mathcal{HI}(R) \longrightarrow \mathcal{HI}(R) \cup \{\varphi\}$ be a function. Let Q be a ϕ-(k,n)-absorbing primary hyperideal of R. If $(r_1^{k(n-1)+1})$ is a ϕ-(k,n) primary of Q for some $r_1^{k(n-1)+1} \in R$, then $g(r_1, \cdots, \widehat{r_{i_1}}, \cdots, \widehat{r_{i_2}}, \cdots, \widehat{r_{i_s}}, \cdots, r_{k(n-1)+1}, Q^{(s)}) \subseteq \phi(Q)$ for every $i_1, \cdots, i_s \in \{1, \cdots, k(n-1)+1\}$ and $1 \leq s \leq (k-1)n-k+2$.*

Proof. $(f(r_1, \phi(Q), 0^{(m-2)}), \cdots, f(r_{k(n-1)+1}, \phi(Q), 0^{(m-2)})$ is a (k,n)-zero primary of $Q/\phi(Q)$ by Theorem 4.14 and $Q/\phi(Q)$ is a weakly (k,n)-absorbing primary of $R/\phi(Q)$ by Theorem 4.13. Then we conclude that

$$f(g(f(r_1, \phi(Q), 0^{(m-2)}), \cdots, f(\widehat{r_{i_1}}, \phi(Q), 0^{(m-2)}), \cdots, f(\widehat{r_{i_2}}, \phi(Q), 0^{(m-2)}), \cdots,$$
$$f(\widehat{r_{i_s}}, \phi(Q), 0^{(m-2)}), \cdots, f(r_{k(n-1)+1}, \phi(Q), 0^{(m-2)}), Q^{(s)}), \phi(Q), 0^{(m-2)}) = \phi(Q).$$

for every $i_1, \cdots, i_s \in \{1, \cdots, k(n-1)+1\}$ and $1 \leq s \leq (k-1)n-k+2$, by Theorem 4.9 of [16]. Thus, $g(r_1, \cdots, \widehat{r_{i_1}}, \cdots, \widehat{r_{i_2}}, \cdots, \widehat{r_{i_s}}, \cdots, r_{k(n-1)+1}, Q^{(s)}) \subseteq \phi(Q)$. □

Theorem 4.16. *Let R be a commutative Krasner (m,n)-hyperring and let $\phi : \mathcal{HI}(R) \longrightarrow \mathcal{HI}(R) \cup \{\varphi\}$ be a function. Let Q be a ϕ-(k,n)-absorbing primary hyperideal of R but is not a (k,n)-absorbing primary. Then $g(Q^{k(n-1)+1}) \subseteq \phi(Q)$.*

Proof. This can be proved, by using Theorem 4.15, in a very similar manner to the way in which 4.12 was proved. □

Now, let give some related corollaries.

Corollary 4.17. *Let $\phi : \mathcal{HI}(R) \longrightarrow \mathcal{HI}(R) \cup \{\varphi\}$ be a function. If Q is a ϕ-(k,n)-absorbing primary hyperideal of R such that $g(Q^{k(n-1)+1}) \nsubseteq \phi(Q)$, then Q is a (k,n)-absorbing primary hyperideal of R.*

Corollary 4.18. *Let $\phi : \mathcal{HI}(R) \longrightarrow \mathcal{HI}(R) \cup \{\varphi\}$ be a function and let Q be a ϕ-(k,n)-absorbing primary hyperideal of R that is not a (k,n)-absorbing primary hyperideal of R. Then $\boldsymbol{r}^{(m,n)}(Q) = \boldsymbol{r}^{(m,n)}(\phi(Q))$.*

Proof. By Theorem 4.16, we have $g(Q^{k(n-1)+1}) \subseteq \phi(Q)$ as Q is not a (k,n)-absorbing primary. This means $\boldsymbol{r}^{(m,n)}(Q) \subseteq \boldsymbol{r}^{(m,n)}(\phi(Q))$. On the other hand, from $\phi(Q) \subseteq Q$, it follows that $\boldsymbol{r}^{(m,n)}(\phi(Q)) \subseteq \boldsymbol{r}^{(m,n)}(Q)$. Hence $\boldsymbol{r}^{(m,n)}(Q) = \boldsymbol{r}^{(m,n)}(\phi(Q))$. □

Corollary 4.19. *Let $\phi : \mathcal{HI}(R) \longrightarrow \mathcal{HI}(R) \cup \{\varphi\}$ be a function and let Q be a proper hyperideal of R such that $\boldsymbol{r}^{(m,n)}(\phi(Q))$ is a (k,n)-absorbing hyperideal of R. Then Q is a ϕ-$(k+1,n)$-absorbing primary hyperideal of R if and only if Q is a $(k+1,n)$-absorbing primary hyperideal of R.*

Proof. ($\Longrightarrow$) Let Q be a ϕ-$(k+1,n)$-absorbing primary hyperideal of R. If Q is not a $(k+1,n)$-absorbing primary hyperideal of R. Hence $\boldsymbol{r}^{(m,n)}(Q) = \boldsymbol{r}^{(m,n)}(\phi(Q))$ by Corollary 4.18. Then $\boldsymbol{r}^{(m,n)}(Q)$ is a (k,n)-absorbing hyperideal of R which implies that Q is is a $(k+1,n)$-absorbing primary hyperideal of R by Theorem 4.9 in [18].

($\Longleftarrow$) It is clear. □

Theorem 4.20. *Let $h : R_1 \longrightarrow R_2$ be a ϕ_1-ϕ_2-homomorphism, where ϕ_1 and ϕ_2 are two reduction functions of $\mathcal{HI}(R_1)$ and $\mathcal{HI}(R_2)$, respectively. Then*

(1) *If Q_2 is a ϕ_2-(k,n)-absorbing primary hyperideal of R_2, then $h^{-1}(Q_2)$ is a ϕ_1-(k,n)-absorbing primary hyperideal of R_1.*

(2) *If h is surjective and Q_1 is a ϕ_1-(k,n)-absorbing primary hyperideal of R_1 with $Ker(h) \subseteq Q_1$, then $h(Q_1)$ is a ϕ_2-(k,n)-absorbing primary hyperideal of R_2.*

Proof. (1) Let Q_2 be a ϕ_2-(k,n)-absorbing primary hyperideal of R_2. Assume that $r_1^{kn-k+1} \in R_1$ such that $g(r_1^{kn-k+1}) \in h^{-1}(Q_2) - \phi_1(h^{-1}(Q_2))$. Then we get $h(g(r_1^{kn-k+1})) = g(h(r_1), \cdots, h(r_{kn-k+1})) \in Q_2 - \phi_2(Q_2)$. Since Q_2 is a ϕ_2-(k,n)-absorbing primary hyperideal of R_2, we obtain either $g(h(r_1), \cdots, h(r_{(k-1)n-k+2})) = h(g(r_1^{(k-1)n-k+2})) \in Q_2$ which means $g(r_1^{(k-1)n-k+2}) \in h^{-1}(Q_2)$, or $g(h(r_1), \cdots, \widehat{h(r_i)}, \cdots, h(r_{kn-k+1})) = h(g(r_1, \cdots, \widehat{r_i}, \cdots, r_{kn-k+1})) \in \boldsymbol{r}^{(m,n)}(Q_2)$ which means $g(r_1, \cdots, \widehat{r_i}, \cdots, r_{kn-k+1})) \in h^{-1}(\boldsymbol{r}^{(m,n)}(Q_2)) = \boldsymbol{r}^{(m,n)}(h^{-1}(Q_2))$ for some $1 \leq i \leq n$. Hence $h^{-1}(Q_2)$ is a ϕ_1-(k,n)-absorbing primary hyperideal of R_1.

(2) Assume that h is surjective and Q_1 is a ϕ_1-(k,n)-absorbing primary hyperideal of R_1 with $Ker(h) \subseteq Q_1$. Let $s_1^{kn-k+1} \in R_2$ such that $g(s_1^{kn-k+1}) \in h(Q_1) - \phi_2(h(Q_1))$. Therefore there exist $r_1^{kn-k+1} \in R_1$ with $h(r_1) = s_1, \cdots, h(r_{kn-k+1}) = s_{kn-k+1}$. Hence we get $h(g(r_1^{kn-k+1}) = g(h(r_1), \cdots, h(r_{kn-k+1})) = g(s_1^{kn-k+1}) \in h(Q_1)$. Since h is a ϕ_1-ϕ_2-epimorphism and $Ker(h) \subseteq Q_1$, we have $g(r_1^{kn-k+1}) \in Q_1 - \phi_1(Q_1)$. Since Q_1 is a ϕ_1-(k,n)-absorbing primary hyperideal of R_1, we conclude that $g(r_1^{(k-1)n-k+2)}) \in Q_1$ which implies

$$h(g(r_1^{(k-1)n-k+2)}) = g(h(r_1), \cdots, h(r_{(k-1)n-k+2)}) = g(s_1^{(k-1)n-k+2)}) \in h(Q_1),$$

or $g(r_1, \cdots, \widehat{r_i}, \cdots, r_{kn-k+1}) \in \boldsymbol{r}^{(m,n)}(Q_1)$ implies $h(g(r_1, \cdots, \widehat{r_i}, \cdots, r_{kn-k+1}) = g(h(r_1), \cdots, \widehat{h(r_i)}, \cdots, h(r_{kn-k+1})) = g(s_1, \cdots, \widehat{s_i}, \cdots, s_{kn-k+1}) \in h(\boldsymbol{r}^{(m,n)}(Q_1)) \subseteq \boldsymbol{r}^{(m,n)}(h(Q_1))$ for some $1 \leq i \leq (k-1)n - k + 2$. Consequently, $h(Q_1)$ is a ϕ_2-(k,n)-absorbing primary hyperideal of R_2. □

As an instant consequence of the previous theorem, we get the following explicit result.

Theorem 4.21. *Let Q and P be two hyperideals of R and ϕ be a reduction function of $\mathcal{HI}(R)$ such that $P \subseteq \phi(Q) \subseteq Q$. If Q is a ϕ-(k,n)-absorbing primary hyperideal of R, then Q/P is a ϕ_q-(k,n)-absorbing primary hyperideal of R/P.*

Theorem 4.22. *Let (R_i, f_i, g_i) be a commutative Krasner (m,n)-hyperring for each $1 \leq i \leq kn-k+1$ and $\phi_i : \mathcal{HI}(R_i) \longrightarrow \mathcal{HI}(R_i) \cup \{\varphi\}$ be a function. Let Q_i be a hyperideal of R_i for each $1 \leq i \leq kn-k+1$ and $\phi = \phi_1 \times \cdots \times \phi_{kn-k+1}$. If $Q = Q_1 \times \cdots \times Q_{kn-k+1}$ is a ϕ-$(k+1,n)$-absorbing primary hyperideal of $R = R_1 \times \cdots \times R_{kn-k+1}$, then Q_i is a ϕ_i-(k,n)-absorbing primary hyperideal of R_i and $Q_i \neq R_i$ for all $1 \leq i \leq kn-k+1$.*

Proof. By using an argument similar to that in the proof of Theorem 3.7, one can easily complete the proof. □

5. Conclusion

In this paper, motivated by the research works on ϕ-2-absorbing (primary) ideals of commutative rings, we propsed and investigated the notions of ϕ-(k,n)-absorbing and ϕ-(k,n)-absorbing primary hyperideals in a Krasner (m,n)-hyperring. Some of their essential characteristics were analysed. Moreover, the stabilty of the notions were examined in some hyperring-theoretic constructions. As a new research subject, we suggest the concept of ϕ-(k,n)-absorbing δ-primary hyperideals, where δ is an expansion function of $\mathcal{HI}(R)$.

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