Algebraic Structures and Their Applications

Algebraic Structures and Their Applications Vol. 11 No. 4 (2024) pp 273-285.

Research Paper

ON LEFT WEAKLY JOINTLY PRIME (R, S)-MODULES

DIAN ARIESTA YUWANINGSIH* AND INDAH EMILIA WIJAYANTI

Abstract. Let R and S be commutative rings and M an (R, S)-module. A proper (R, S)-submodule P of M is called left weakly jointly prime if for each (R, S)-submodule N of M and elements a, b of R such that $abNS \subseteq P$ implies either $aNS \subseteq P$ or $bNS \subseteq P$. This paper defines left weakly jointly prime (R, S)-modules and presents some of their properties. On the other hand, a ring R is called fully prime if each proper ideal of R is prime. We extend this fact to (R, S)-modules. An (R, S)-module M is called fully left weakly jointly prime if each proper (R, S)-submodule of M is left weakly jointly prime. Moreover, we present some properties of fully left weakly jointly prime (R, S)-modules. At the end of this paper, we present our main results about the necessary and sufficient conditions for an arbitrary (R, S)-module to be fully left weakly jointly prime.

DOI: 10.22034/as.2024.19918.1630

MSC(2010): 13C13 Primary: 13C05, 16D20, 16D80.

Keywords: Fully prime, Prime module, (R, S)-submodule, Weakly prime.

Received: 01 April 2023, Accepted: 25 April 2024.

*Corresponding author

1. Introduction

The notion of prime submodules has been introduced by Dauns in [9]. A proper submodule P of an R-module M is said to be prime if for any element $r \in R$ and element $m \in M$ with $rm \in P$, we have $m \in P$ or $rM \subseteq P$. Moreover, a non-zero R-module M is considered prime if the zero submodules are prime. Some previous authors have studied the prime submodules and prime modules, for example, in papers [14, 4, 12, 10].

In module theory, an R-module has been generalized into an (R,S)-bimodule, where R and S are arbitrary rings. Khumprapussorn et al. in [13] have generalized the (R,S)-bimodule structure to an (R,S)-module structure. Let R and S be rings and M an abelian group under addition. Khumprapussorn et al. in [13] said M is an (R,S)-module if there exists a function $\_ \cdot \_ \cdot \_ : R \times M \times S \to M$ such that for all $r_1, r_2, r \in R$, $s_1, s_2, s \in S$, and $m, n \in M$ satisfied (1) $r \cdot (m+n) \cdot s = r \cdot m \cdot s + r \cdot m \cdot s$; (2) $(r_1 + r_2) \cdot m \cdot s = r_1 \cdot m \cdot s + r_2 \cdot m \cdot s$; (3) $r \cdot m \cdot (s_1 + s_2) = r \cdot m \cdot s_1 + r \cdot m \cdot s_2$; (4) $r_1 \cdot (r_2 \cdot m \cdot s_1) \cdot s_2 = (r_1 r_2) \cdot m \cdot (s_1 s_2)$. Moreover, the concepts around (R,S)-module have been studied in [17]. An (R,S)-module has an (R,S)-bimodule structure when both rings R and S have central idempotent elements.

According to [13], a proper (R,S)-submodule P of M is called a jointly prime (R,S)-submodule if for each left ideal I of R, right ideal J of S, and (R,S)-submodule K of M with $IKJ \subseteq P$ implies $IMJ \subseteq P$ or $K \subseteq P$. If R and S are commutative rings, then we have a proper (R,S)-submodule P of M is called a jointly prime (R,S)-submodule if for each ideal I of R, ideal J of S, and (R,S)-submodule K of M with $IKJ \subseteq P$ implies $IMJ \subseteq P$ or $K \subseteq P$. Furthermore, a non-zero (R,S)-module M is said to be jointly prime if its zero (R,S)-submodule is a jointly prime (R,S)-submodule of M.

Weakly prime submodules are generalizations of prime submodules. Weakly prime submodules have been introduced and studied over an associative ring with identity in [7, 6]. Assume that R is an associative ring with identity. According to [7], a proper submodule P of M is said to be weakly prime if for any $a, b \in R$ and submodule K of M with $aRbK \subseteq P$ implies either $aK \subseteq P$ or $bK \subseteq P$. If R is a commutative ring, then a proper submodule P of M is weakly prime if for each submodule K of M and elements a, b of R with $abK \subseteq P$, implies either $aK \subseteq P$ or $bK \subseteq P$. Moreover, weakly prime submodules over a commutative ring have been studied in [3, 5, 2, 1]. Next, we extend these facts to (R,S)-modules. Following to [16], a proper (R,S)-submodule P of M is said to be left weakly jointly prime if for each (R,S)-submodule N of M and element $a, b \in R$ such that $abNS \subseteq P$ implies either $aNS \subseteq P$ or $bNS \subseteq P$. According to [7], an R-module M is called a weakly prime module if its zero submodules is a weakly prime submodule of M. An R-module M is said to be weakly prime if the annihilator of any non-zero submodule of M is a prime ideal. Moreover, this work aims to define left weakly jointly prime (R,S)-modules and then investigate their properties.

In Section 2, we present the definition of left weakly jointly prime (R, S)-module and give some of their properties. First, we provide the necessary and sufficient condition for (R, S)-modules to be left weakly jointly prime. And then we present the set of $(0 :_R \langle m \rangle)$ for each $0 \neq m \in M$ is a chain of prime ideals if and only if M is a left weakly jointly prime (R, S)-module where $S^2 = S$ and $a \in RaS$ for all $a \in M$. At the end of this section, we present the sufficient condition for every non-zero summand of (R, S)-module M to be left weakly jointly prime.

According to [7], a ring R is called a fully prime ring if each proper ideal of R is prime. This ring type is fully investigated in [8, 15]. Based on [7], we have that an R-module M is called a fully weakly prime module if each proper submodule of M is weakly prime. When we extend these facts to (R, S)-module, we have an (R, S)-module of M is fully left weakly jointly prime if every proper (R, S)-submodule of M is a left weakly jointly prime (R, S)-submodule. Section 3 presents some properties of fully left weakly jointly prime (R, S)-modules. We develop some properties of fully weakly prime modules studied in [7]. Moreover, at the end of this section, we show our main results about the necessary and sufficient conditions for an arbitrary (R, S)-module to be fully left weakly jointly prime.

Throughout this paper, R and S are commutative rings unless stated otherwise, and M is an additive abelian group.

2. Left weakly jointly prime (R, S)-modules

In this section, we present the definition of left weakly jointly prime (R, S)-modules and give some of their properties. We begin by defining a left weakly jointly prime (R, S)-submodule as follows.

Definition 2.1. [16] Let M be an (R, S)-module. A proper (R, S)-submodule P of M is called a left weakly jointly prime (R, S)-submodule if for each (R, S)-submodule N of M and element $a, b \in R$ such that $abNS \subseteq P$ implies either $aNS \subseteq P$ or $bNS \subseteq P$.

When $a \in RaS$ for all $a \in M$, we have another definition of a left weakly jointly prime (R, S)-submodule as follows.

Definition 2.2. [16] Let M be an (R, S)-module satisfied $a \in RaS$ for all $a \in M$. A proper (R, S)-submodule P of M is called a left weakly jointly prime (R, S)-submodule if for each ideal I, J of R and (R, S)-submodule N of M with $IJNS \subseteq P$, implies either $INS \subseteq P$ or $JNS \subseteq P$.

Below, we give an example of left weakly jointly prime (R, S)-submodules.

Example 2.3. Let $\mathbb{Z}$ be a $(4\mathbb{Z}, 3\mathbb{Z})$-module. A proper $(4\mathbb{Z}, 3\mathbb{Z})$-submodule $12\mathbb{Z}$ of $\mathbb{Z}$ is a left weakly jointly prime $(4\mathbb{Z}, 3\mathbb{Z})$-submodule of $\mathbb{Z}$. Let $a, b \in 4\mathbb{Z}$ with $a = 4k$ and $b = 4l$ and N

be a $(4\mathbb{Z}, 3\mathbb{Z})$-submodule of $\mathbb{Z}$ with $N = x\mathbb{Z}$ for some $k, l, x \in \mathbb{Z}$. We have

$$abN(3\mathbb{Z}) = (4k)(4l)(x\mathbb{Z})(3\mathbb{Z}) = 48klx\mathbb{Z}^2 \subseteq 48klx\mathbb{Z} \subseteq 12\mathbb{Z}.$$

In the other side, we obtain $aN(3\mathbb{Z}) = (4k)(x\mathbb{Z})(3\mathbb{Z}) = 12kx\mathbb{Z}^2 \subseteq 12kx\mathbb{Z} \subseteq 12\mathbb{Z}$ or $bN(3\mathbb{Z}) = (4l)(x\mathbb{Z})(3\mathbb{Z}) = 12lx\mathbb{Z}^2 \subseteq 12lx\mathbb{Z} \subseteq 12\mathbb{Z}$. Hence, $12\mathbb{Z}$ is a left weakly jointly prime $(4\mathbb{Z}, 3\mathbb{Z})$-submodule of $\mathbb{Z}$.

Now, we present the definition of left weakly jointly prime (R, S)-module.

Definition 2.4. An (R, S)-module M is called left weakly jointly prime if it's zero (R, S)-submodule is left weakly jointly prime.

Example 2.5. Let $\mathbb{Z}$ be an $(4\mathbb{Z}, 3\mathbb{Z})$-module. It is easy to show that the zero $(4\mathbb{Z}, 3\mathbb{Z})$-submodule of $\mathbb{Z}$ is a left weakly jointly prime $(4\mathbb{Z}, 3\mathbb{Z})$-submodule. Thus, $\mathbb{Z}$ is a left weakly jointly prime $(4\mathbb{Z}, 3\mathbb{Z})$-module.

Example 2.6. Let R and S are commutative rings with

$$R = \left\{ \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{0} & \bar{0} \end{pmatrix} \middle| \bar{a}, \bar{b} \in \mathbb{Z}_4 \right\} \text{ and } S = \left\{ \begin{pmatrix} \bar{a} & \bar{0} \\ \bar{b} & \bar{0} \end{pmatrix} \middle| \bar{a}, \bar{b} \in \mathbb{Z}_4 \right\}.$$

Let an (R, S)-module M with

$$M = \left\{ \begin{pmatrix} \bar{a} & \bar{0} \\ \bar{b} & \bar{c} \end{pmatrix} \middle| \bar{a}, \bar{b}, \bar{c} \in \mathbb{Z}_4 \right\}.$$

We can show that M is not a left weakly jointly prime (R, S)-module. Let (R, S)-submodule $N = \left\{ \begin{pmatrix} \bar{a} & \bar{0} \\ \bar{0} & \bar{b} \end{pmatrix} \middle| \bar{a}, \bar{b} \in \mathbb{Z}_4 \right\}$ and any element $a, b \in R$ with $a = \begin{pmatrix} \bar{0} & \bar{y} \\ \bar{0} & \bar{0} \end{pmatrix}$ and $b = \begin{pmatrix} \bar{0} & \bar{x} \\ \bar{0} & \bar{0} \end{pmatrix}$. Let any element $n \in N$ with $n = \begin{pmatrix} \bar{a} & \bar{0} \\ \bar{0} & \bar{b} \end{pmatrix}$. We obtain

$$abnS = \begin{pmatrix} \bar{0} & \bar{y} \\ \bar{0} & \bar{0} \end{pmatrix} \begin{pmatrix} \bar{0} & \bar{x} \\ \bar{0} & \bar{0} \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{0} \\ \bar{0} & \bar{b} \end{pmatrix} S = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} \right\}.$$

However, we have

$$anS = \begin{pmatrix} \bar{0} & \bar{y} \\ \bar{0} & \bar{0} \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{0} \\ \bar{0} & \bar{b} \end{pmatrix} S = \begin{pmatrix} \bar{0} & \overline{yb} \\ \bar{0} & \bar{0} \end{pmatrix} S \neq \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} \right\},$$

and

$$bnS = \begin{pmatrix} \bar{0} & \bar{x} \\ \bar{0} & \bar{0} \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{0} \\ \bar{0} & \bar{b} \end{pmatrix} S = \begin{pmatrix} \bar{0} & \overline{xb} \\ \bar{0} & \bar{0} \end{pmatrix} \neq \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} \right\}.$$

Thus, M is not a left weakly jointly prime (R, S)-module.

According to [7], a proper submodule N of an R-module M is weakly prime if and only if the quotient R-module M/N is weakly prime. Proposition 2.7 extends this result to (R,S)-modules as follows.

Proposition 2.7. *Let M be an (R,S)-module. Then a proper (R,S)-submodule X of M is a left weakly jointly prime (R,S)-module if and only if M/X is a left weakly jointly prime (R,S)-module.*

Proof. Let X be a left weakly jointly prime (R,S)-submodule of M. We have M/X is a left weakly jointly prime (R,S)-module. Conversely, assume that M/X is a left weakly jointly prime (R,S)-module. Then X is a left weakly jointly prime (R,S)-submodule of M. □

According to [13], for each (R,S)-submodule N of M, let $(N:_R M) = \{r \in R \mid rMS \subseteq N\}$. Clearly that $(N:_R M)$ is only an additive subgroup of R. However, if we have the condition $S^2 = S$, clearly that $(N:_R M)$ is an ideal of R. We may also say that $(N:_R M)$ is the annihilator of the quotient (R,S)-module M/N over the ring R.

Before we present the next properties of left weakly jointly prime (R,S)-modules, we need the following properties.

Proposition 2.8. *Let M be an (R,S)-module with $S^2 = S$ and $a \in RaS$ for all $a \in M$ and N be a proper (R,S)-submodule of M. Then N is a left weakly jointly prime (R,S)-submodule of M if and only if $(N:_R K)$ is a prime ideal of R for each (R,S)-submodule K of M with $K \nsubseteq N$.*

Proof. Let K be an (R,S)-submodule of M with $K \nsubseteq N$. Since $S^2 = S$ and $a \in RaS$ for all $a \in M$, then $(N:_R K)$ is a proper ideal of R. Let any elements a and b of R such that $ab \in (N:_R K)$, so we have $abKS \subseteq N$. Since N is a left weakly jointly prime (R,S)-submodule, then $aKS \subseteq N$ or $bKS \subseteq N$. Thus, we obtain $a \in (N:_R K)$ or $b \in (N:_R K)$. Hence, $(N:_R K)$ is a prime ideal of R. Conversely, let a and b be elements of R and L be an (R,S)-submodule of M such that $abLS \subseteq N$ and $aLS \nsubseteq N$. Then $L \nsubseteq N$. So, we have $ab \in (N:_R L)$. Based on the hypothesis, $(N:_R L)$ is a prime ideal of R. Thus from $ab \in (N:_R L)$ and $aLS \nsubseteq N$ we obtain $a \notin (N:_R L)$ and $b \in (N:_R L)$. Thus, we have $bLS \subseteq N$. Thus, N is a left weakly jointly prime (R,S)-submodule of M. □

Proposition 2.9 presents the necessary and sufficient condition for an (R,S)-module to be left weakly jointly prime.

Proposition 2.9. *Let M be an (R,S)-module with $S^2 = S$ and $a \in RaS$ for all $a \in M$. Then M is a left weakly jointly prime if and only if $(0 :_R K)$ is a prime ideal of R, for each non-zero (R,S)-submodule K of M.*

Proof. Let K be a non-zero (R,S)-submodule of M. Clearly that $(0 :_R K) = \{r \in R \mid rKS = 0\}$ is a proper ideal of R. Since M is a left weakly jointly prime (R,S)-module, 0 is a left weakly jointly prime (R,S)-submodule of M. Thus, based on Proposition 2.8, we have $(0 :_R K)$ is a prime ideal of R. Conversely, it is known that for each non-zero (R,S)-submodule K of M satisfy $(0 :_R K)$ is a prime ideal of R. Using Proposition 2.8, 0 is a left weakly jointly prime (R,S)-submodule of M. Hence, M is a left weakly jointly prime (R,S)-module. □

Let M be an (R,S)-modules. Following to [13], for any non-empty subsets Y of M we define

$$\langle Y\rangle = \bigcap\{K \mid K \text{ is an } (R,S) - \text{submodule of } M \text{ containing } Y\}.$$

It is obvious that $\langle Y\rangle$ is an (R,S)-submodule of M containing Y. If $Y = \{a\}$, then we have

$$\langle\{a\}\rangle = \langle a\rangle = \bigcap\{K \mid K \text{ is an } (R,S) - \text{submodule of } M \text{ containing } a\}.$$

Clearly that $\langle a\rangle$ is an (R,S)-submodule of M for any element $a \in M$. Moreover, element a is contained in $\langle a\rangle$.

Next, we give a proposition explaining the elements' form in (R,S)-submodule $\langle Y\rangle$.

Theorem 2.10. *Let M be an (R,S)-module and the set $Y \subseteq M$. If $Y = \emptyset$, then $\langle Y\rangle = \{0\}$. If $Y \neq \emptyset$, then we get*

$$\langle Y\rangle = \Big\{\sum_{i=1}^{t} r_i y_i s_i + \sum_{j=1}^{k} n_j y'_j \Big| r_i \in R, y_i, y'_j \in Y, s_i \in S, n_j \in \mathbb{Z}, \forall i = 1,2,...,t, \forall j = 1,2,...,k\Big\}.$$

Proof. We assume $Y \neq \emptyset$ and

$$A = \Big\{\sum_{i=1}^{t} r_i y_i s_i + \sum_{j=1}^{k} n_j y'_j \Big| r_i \in R, y_i, y'_j \in Y, s_i \in S, n_j \in \mathbb{Z}, \forall i = 1,2,...,t, \forall j = 1,2,...,k\Big\}.$$

We will prove that $\langle Y\rangle = A$. Since $\langle Y\rangle$ is the intersection of all (R,S)-submodules of M that contain Y, it is clear that $Y \subseteq \langle Y\rangle$. Since $\langle Y\rangle$ is closed to the scalar addition and multiplication operations, then $A \subseteq Y \subseteq \langle Y\rangle$. Next, we will prove $\langle Y\rangle \subseteq A$. It is equivalent to show that A is an (R,S)-submodule of M containing Y. Let any $y \in Y$, we have $y = 0y0 + 1y \in A$, so $Y \subseteq A$. Let any $\Big(\sum_{i=1}^{t} r_i y_i s_i + \sum_{j=1}^{k} n_j y'_j\Big), \Big(\sum_{i=1}^{q} r'_i y'_i s'_i + \sum_{j=1}^{l} n'_j y"_j\Big) \in A$, we have

$$\Big(\sum_{i=1}^{t} r_i y_i s_i + \sum_{j=1}^{k} n_j y'_j\Big) - \Big(\sum_{i=1}^{q} r'_i y'_i s'_i + \sum_{j=1}^{l} n'_j y"_j\Big) = \Big(\sum_{i=1}^{t} r_i y_i s_i - \sum_{i=1}^{q} r'_i y'_i s'_i\Big) + \Big(\sum_{j=1}^{k} n_j y'_j - \sum_{j=1}^{l} n'_j y"_j\Big) \in A.$$

Next, let any element $r \in R$ and $s \in S$, we obtain

$$r\Big(\sum_{i=1}^{t} r_i y_i s_i + \sum_{j=1}^{k} n_j y'_j\Big)s = \sum_{i=1}^{t}(rr_i)y_i(s_i s) + \sum_{j=1}^{k}(rn_j)y'_j s \in A.$$

Thus, A is an (R, S)-submodule of M containing Y. So, we obtain $\langle Y\rangle \subseteq A$. Hence, it has proved that $\langle Y\rangle = A$. □

Following to Theorem 2.10, If $Y = \{a\}$, then we have

$$\langle a\rangle = \Big\{\sum_{i=1}^{t} r_i a s_i + \sum_{j=1}^{k} n_j x \Big| r_i \in R, s_i \in S, n_j \in \mathbb{Z}, \forall i = 1, 2, ..., t, \forall j = 1, 2, ..., k\Big\}.$$

If $a \in RaS$ for all $a \in M$, then we have

$$\langle a\rangle = \Big\{\sum_{i=1}^{t} r_i a s_i \Big| r_i \in R, s_i \in S, \forall i = 1, 2, ..., t\Big\}.$$

Next, we define cyclic (R, S)-submodules as follows.

Definition 2.11. Let M be an (R, S)-module. An (R, S)-submodule N of M is called a cyclic (R, S)-submodule of M if N is generated by an element $x \in M$, i.e. $N = \langle x\rangle$.

Clearly that (R, S)-module M is cyclic if M is generated by an element $a \in M$, i.e. $M = \langle a\rangle$. We have the following properties according to Proposition 2.9.

Corollary 2.12. *Let M be an (R, S)-module with $S^2 = S$ and $a \in RaS$ for all $a \in M$. Then M is a left weakly jointly prime (R, S)-module if and only if for each $0 \neq m \in M$ satisfy $(0 :_R \langle m\rangle)$ is a prime ideal of R.*

Proof. Let $m \in M\backslash\{0\}$ and we form the set $(0 :_R \langle m\rangle)$. Let $x, y \in R$ such that $xy \in (0 :_R \langle m\rangle)$, so we have $xy\langle m\rangle S = 0$. Since M is a left weakly jointly prime (R, S)-modules, we have $x\langle m\rangle S = 0$ or $y\langle m\rangle S = 0$. Thus, we obtain $x \in (0 :_R \langle m\rangle)$ or $y \in (0 :_R \langle m\rangle)$. Hence, $(0 :_R \langle m\rangle)$ is a prime ideal of R. Conversely, let any (R, S)-submodule N of M and elements $a, b \in R$ such that $abNS = 0$ and $aNS \neq 0$. Let element $n \in N$. If element $n = 0$, clearly that from $abnS \subseteq abNS = 0$ we have $bnS = 0$. If element $n \neq 0$, then we get a cyclic (R, S)-submodule $\langle n\rangle$. Clearly that $\langle n\rangle \subseteq N$. So, we have $ab\langle n\rangle S \subseteq abNS = 0$. Based on the hypothesis, we obtain $b\langle n\rangle S = 0$. Since $n \in \langle n\rangle$, then we have $bnS = 0$. Thus, we get $bnS = 0$ for all element $n \in N$, so $bNS = 0$. Hence, M is left weakly jointly prime (R, S)-modules. □

Based on Corollary 2.12, we have the following properties.

Proposition 2.13. *Let M be an (R, S)-module with $S^2 = S$ and $a \in RaS$ for all $a \in M$. Then M is a left weakly jointly prime (R, S)-module if and only if the set $\mathfrak{J} = \{(0 :_R \langle m\rangle) \mid 0 \neq m \in M\}$ is a chain of prime ideals of R.*

Proof. Let M be a left weakly jointly prime (R, S)-module. Then for each $0 \neq m \in M$ satisfy $(0 :_R \langle m\rangle)$ is a prime ideal of R. We have to show that $\mathfrak{J}$ is a chain of prime ideals of R. Let $m, n \in M \setminus \{0\}$. Clearly, $(0 :_R \langle m\rangle) \cap (0 :_R \langle n\rangle) \subseteq (0 :_R \langle m\rangle + \langle n\rangle)$. Since M is a left weakly jointly prime (R, S)-module, then $(0 :_R \langle m\rangle + \langle n\rangle)$ is a prime ideal of R. Since $(0 :_R \langle m\rangle)(0 :_R \langle n\rangle) \subseteq (0 :_R \langle m\rangle) \cap (0 :_R \langle n\rangle) \subseteq (0 :_R \langle m\rangle + \langle n\rangle)$ then $(0 :_R \langle m\rangle) \subseteq (0 :_R \langle m\rangle + \langle n\rangle)$ or $(0 :_R \langle n\rangle) \subseteq (0 :_R \langle m\rangle + \langle n\rangle)$. So, we have $(0 :_R \langle m\rangle) = (0 :_R \langle m\rangle) \cap (0 :_R \langle m\rangle + \langle n\rangle) \subseteq (0 :_R \langle n\rangle)$ or $(0 :_R \langle n\rangle) = (0 :_R \langle n\rangle) \cap (0 :_R \langle m\rangle + \langle n\rangle) \subseteq (0 :_R \langle m\rangle)$. Thus, $\mathfrak{J}$ is a chain of prime ideals of R. Conversely, assume that $\mathfrak{J}$ is a chain prime ideal of R. It means that for each $0 \neq m \in M$, $(0 :_R \langle m\rangle)$ is a prime ideal of R. Thus, using Corollary 2.12, we have M a left weakly jointly prime (R, S)-module. □

Now, we recall from [7] that each summand of a weakly prime R-module is a weakly prime R-module. Next, we present the generalization of these properties to (R, S)-modules.

Proposition 2.14. *An (R, S)-module M is left weakly jointly prime if and only if every direct summand of M, including the zero summands, is a left weakly jointly prime (R, S)-submodule.*

Proof. Assume that $M = N \oplus K$. Let $a, b \in R$ and $x \in M \setminus N$ such that $ab\langle x\rangle S = 0$. Since M is a left weakly jointly prime (R, S)-module, then we have $a\langle x\rangle S = 0$ or $b\langle x\rangle S = 0$. Since $0 \subseteq N$, then for each $a, b \in R$ that satisfy $ab\langle x\rangle S \subseteq N$ implies either $a\langle x\rangle S \subseteq N$ or $b\langle x\rangle S \subseteq N$. Thus, N is a left weakly jointly prime (R, S)-submodule of M. Hence, every direct summand of M is a left weakly jointly prime (R, S)-submodule. Conversely, let $M = M \oplus \{0\}$. By our hypothesis, $\{0\}$ is a left weakly jointly prime (R, S)-submodule, i.e., M is a left weakly jointly prime (R, S)-module. □

From Proposition 2.14, we have that each direct summand of left weakly jointly prime (R, S)-modules is a left weakly jointly prime (R, S)-module. Therefore it is natural to consider (R, S)-modules which are not indecomposable and not left weakly jointly prime, but their non-zero summands are left weakly jointly prime (R, S)-modules.

Corollary 2.15. *Let M be an (R,S)-module with $S^2 = S$ and $a \in RaS$ for all $a \in M$, M is not a left weakly jointly prime (R,S)-module and not indecomposable. If every decomposition of M is of the form $M = N \oplus K$, where N and K are non-zero indecomposable left weakly jointly prime (R,S)-module, then every non-zero summand of M is left weakly jointly prime (R,S)-modules.*

Proof. Assume that $M = N \oplus K$ where N, K are non-zero indecomposable left weakly jointly prime (R,S)-submodule. Following Proposition 2.14, the non-zero summand of M is a left weakly jointly prime (R,S)-module. □

3. Fully left weakly jointly prime (R,S)-modules

In this section, we present the definition of fully left weakly jointly prime (R,S)-modules and give some of their properties. We recall the definition of the fully prime ring and fully weakly prime modules as follows.

Definition 3.1. [8] Let R be an associative ring with identity. A ring R is called a fully prime ring if each proper ideal of R is prime.

Definition 3.2. [7] Let R be an associative ring with identity. An R-module M is a fully weakly prime module if every proper submodule of M is a weakly prime submodule.

We extend the definition of fully weakly prime modules to (R,S)-modules as follows.

Definition 3.3. An (R,S)-module M is called fully left weakly jointly prime if every proper (R,S)-submodule of M is left weakly jointly prime.

Example 3.4. Let $\mathbb{Z}_6$ be a $(2\mathbb{Z},\mathbb{Z})$-module. The proper $(2\mathbb{Z},\mathbb{Z})$-submodules of $\mathbb{Z}_6$ are $\{\bar{0}\}$, $\{\bar{0},\bar{2},\bar{4}\}$, and $\{\bar{0},\bar{3}\}$. Those proper submodules are left weakly jointly prime $(2\mathbb{Z},\mathbb{Z})$-submodules. Thus, $\mathbb{Z}_6$ is a fully left weakly jointly prime $(2\mathbb{Z},\mathbb{Z})$-module.

Example 3.5. Let $4\mathbb{Z}$ be a $(2\mathbb{Z},3\mathbb{Z})$-module. The proper $(2\mathbb{Z},3\mathbb{Z})$-submodules of $4\mathbb{Z}$ are $\{\bar{0}\}$ and $(4n)\mathbb{Z}$ for all $n > 1$. Both of them are left weakly jointly prime $(2\mathbb{Z},3\mathbb{Z})$-submodules. Thus, $4\mathbb{Z}$ is a fully left weakly jointly prime $(2\mathbb{Z},3\mathbb{Z})$-module.

Example 3.6. Following to Example 2.6, M is not a fully left weakly jointly prime (R,S)-module since $\left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix} \right\}$ is not a left weakly jointly prime (R,S)-submodule of M.

Next, this proposition below shows that fully prime rings give us a big set of fully left weakly jointly prime (R,S)-modules.

Proposition 3.7. *Let R be a fully prime ring. Then each (R,S)-module M with $S^2 = S$ and $a \in RaS$ for all $a \in M$ is fully left weakly jointly prime (R,S)-module.*

Proof. Let R be a fully prime ring. We will show that M is a fully left weakly jointly prime (R,S)-module. Let N be a proper (R,S)-submodule of M. We will prove that N is a left weakly jointly prime (R,S)-submodule of M. Let K be an (R,S)-submodule of M that is not contained in N, so we have $(N :_R K)$ is an ideal of R. Since $a \in RaS$ for all $a \in M$, then $(N :_R K)$ is a proper ideal of R. Since R is a fully prime ring, $(N :_R K)$ is a prime ideal of R. Using Proposition 2.8, N is a left weakly jointly prime (R,S)-submodule of M. □

Proposition 3.8. *Let M be an (R,S)-module with $S^2 = S$ and $a \in RaS$ for all $a \in M$. Then the following statements are equivalent. (i) Each (R,S)-module M is fully left weakly jointly prime.*

(ii) If $R^2 = R$ and $a \in RaR$ for all $a \in R$, then the (R,R)-module R is fully left weakly jointly prime.

(iii) R is a fully prime ring.

Proof. $(i) \Rightarrow (ii)$. The proof is trivial.
$(ii) \Rightarrow (iii)$. Let N be an ideal of R. Then N is an (R,R)-submodule of R. Let $a,b \in R$ with $ab \in N$. Thus, we get $abR \subseteq NR \subseteq N$ so $abRR \subseteq NR \subseteq N$. Since N is an (R,R)-submodule of R and R is fully left weakly jointly prime, we have N is a left weakly jointly prime (R,R)-submodule. Consequently, we obtain $aRR \subseteq N$ or $bRR \subseteq N$. Since R is commutative and $a,b \in R$ then we have $RaR \subseteq N$ or $RbR \subseteq N$. Since $a \in RaR$ for all $a \in R$, then we obtain $a \in N$ or $b \in N$. Thus, N is a prime ideal of R.
$(iii) \Rightarrow (i)$. The proof is equal to the proof of Proposition 3.7. □

The following result extends a fact of fully prime rings to fully left weakly jointly prime (R,S)-modules. This result is based on papers [8, 15].

Proposition 3.9. *Let M be an (R,S)-module with $a \in RaS$ for all $a \in M$. Then M is a fully left weakly jointly prime (R,S)-module if and only if for each (R,S)-submodule K of M and each ideal I of R, $IKS = I^2KS$ and also for any two ideals A and B of R satisfy AKS and BKS are comparable.*

Proof. Let M be a fully left weakly jointly prime (R,S)-module. Let K be an (R,S)-submodule of M and I be an ideal of R. If $I^2KS = M$, then clearly $M = I^2KS \subseteq IKS \subseteq M$ so that $M = I^2KS = IKS$. Thus, we may assume that $I^2KS \neq M$, then I^2KS is a left weakly jointly prime (R,S)-submodule. Consequently, from $IIKS \subseteq I^2KS$ implies that $IKS = I^2KS$.

Now, if A and B are two ideals of R, we may assume that $AKS \neq M \neq BKS$. However, $AKS \cap BKS$ is a left weakly jointly prime (R, S)-submodule of M. Consequently, from $ABKS \subseteq AKS \cap BKS$ implies that $AKS \subseteq BKS$ or $BKS \subseteq AKS$. Thus, it has proved that AKS and BKS are comparable. Conversely, we must show that each proper (R, S)-submodule N of M is a left weakly jointly prime (R, S)-submodule. To see this, let A and B be ideals of R and K be an (R, S)-submodule of M such that $ABKS \subseteq N$. By our hypothesis, we may assume that $AKS \subseteq BKS$. Then we obtain $AKS = A^2KS \subseteq ABKS \subseteq N$. Assuming that $BKS \subseteq AKS$ we get $BKS = B^2KS \subseteq BAKS = ABKS \subseteq N$. Hence, N is a left weakly jointly prime (R, S)-submodule. Thus, M is a fully left weakly jointly prime (R, S)-module. □

Next, we present the necessary and sufficient condition of fully left weakly jointly prime (R, S)-modules related to their cyclic (R, S)-submodules.

Proposition 3.10. *Let M be an (R, S)-module. Then M is a fully left weakly jointly prime if and only if each proper cyclic (R, S)-submodule of M is a left weakly jointly prime (R, S)-submodule.*

Proof. Let M be a fully left weakly jointly prime (R, S)-module. Thus, every proper (R, S)-submodule of M is a left weakly jointly prime (R, S)-submodule, and each proper cyclic (R, S)-submodule of M is a left weakly jointly prime. Conversely, assume that each proper cyclic (R, S)-submodule of M is a left weakly jointly prime. Let N be a proper (R, S)-submodule of M that is not cyclic. Let element $n \in N$, we can construct a proper cyclic (R, S)-submodule $\langle n \rangle$. Let $a, b \in R$ and (R, S)-submodule K of M with $abKS \subseteq \langle n \rangle$. Based on our hypothesis, $\langle n \rangle$ is a left weakly jointly prime (R, S)-submodule. Then we have $aKS \subseteq \langle n \rangle$ or $bKS \subseteq \langle n \rangle$. Since $\langle n \rangle$ is contained in N, then from $abKS \subseteq \langle n \rangle \subseteq N$ we have $aKS \subseteq N$ or $bKS \subseteq N$. Thus, it is proved that N a left weakly jointly prime (R, S)-submodule of M. Hence, M is a fully left weakly jointly prime (R, S)-module. □

We recall the following properties, a left weakly jointly prime (R, S)-submodule N of M is said to be minimal if it is minimal in the class of left weakly jointly prime (R, S)-submodules of M. Moreover, an (R, S)-module M satisfies the minimum condition if every non-empty family of (R, S)-submodules of M contains a minimal number.

According to [11], every prime submodule contains a minimal prime submodule. Based on this property, it is easy to show that every left weakly jointly prime (R, S)-submodule of M contains a minimal left weakly jointly prime (R, S)-submodule. Using this property, we present the last property of fully left weakly jointly prime (R, S)-modules as follows.

Proposition 3.11. *Let M be an (R,S)-module. If M is a fully left weakly jointly prime (R,S)-module, then M is a left weakly jointly prime (R,S)-module, and the set of proper cyclic (R,S)-submodules satisfies the minimum condition.*

Proof. Since M is a fully left weakly jointly prime (R,S)-module then (R,S)-submodule 0 is left weakly jointly prime. Therefore, M is a left weakly jointly prime (R,S)-module. Moreover, based on Proposition 3.10, we have that every proper cyclic (R,S)-submodule is left weakly jointly prime. Based on [Proposition 3.10, [16]], we have every cyclic (R,S)-submodule of M contains a minimal left weakly jointly prime (R,S)-submodule. Therefore, the set of proper cyclic (R,S)-submodules of M contains a minimal number. Hence, the proper cyclic (R,S)-submodules set satisfies the minimum condition. □

4. Conclusion

Further work on the properties of left weakly jointly prime (R,S)-submodules can be carried out. For example, research on the radical structure of the left weakly jointly prime (R,S)-modules and the dualization of the left weakly jointly prime (R,S)-modules.

5. Acknowledgments

The authors thank the referee for the valuable suggestions and comments. This work was supported by the Directorate General of Higher Education, Ministry of Research, Technology and Higher Education Indonesia through the 2018-2020 National Competitive Research Grant.

References

[1] H. Ansari-Toroghy and F. Farshadifar, *The dual notions of some generalizations of prime submodules*, Commun. Algebra, **39** No. 7 (2011) 2396-2416.

[2] S. E. Atani and F. Farzalipour, *On weakly prime submodules*, Tamkang J. Math., **38** No. 3 (2007) 247-252.

[3] A. Azizi, *Weakly prime submodules and prime modules*, Glasg. Math. J., **48** No. 2 (2006) 343-346.

[4] A. Azizi, *Radical formula and weakly prime submodules*, Glasg. Math. J., **51** No. 2 (2007) 405-412.

[5] A. Azizi, *On prime and weakly prime submodules*, Vietnam J. Math., **36** No. 3 (2008) 315-325.

[6] M. Behboodi, *On weakly prime radical of modules and semi-compatible modules*, Acta Math. Hung., **113** No. 3 (2006) 243-254.

[7] M. Behboodi and H. Koohy, *Weakly prime modules*, Vietnam J. Math., **32** No. 2 (2004) 185-195.

[8] W. D. Blair and H. Tsutsui, *Fully prime rings*, Comm. Algebra, **22** No. 13 (1994) 5389-5400.

[9] J. Dauns, *Prime modules*, J. fur Reine Angew. Math., **298** (1978) 156-181.

[10] T. Duraivel, S. Mangayarcarassy and K. Premkumar, *Prime extension dimension of a module*, J. Algebra Relat. Top., **6** No. 2 (2018) 97-106.

[11] K. R. Goodearl and R. B. Warfiled, *An Introduction to Noncommutative Notherian Rings*, Cambridge University Press, New York, 2004.

[12] A. Hajikarimi, *Modules whose nonzero finitely generated submodules are dense*, ASTA, **8** No. 1 (2021) 89-97.

[13] T. Khumprapussorn, S. Pianskool, and M. Hall, (R, S)*-modules and their fully and jointly prime submodules*, Int. Math. Forum, **7** No. 33 (2012) 163-1643.

[14] N. V. Sanh, A. V. Nguyen, K. F. U. Ahmed, S. Asawasamrit and L. P. Thao, *Primeness in module category*, Asian-Eur. J. Math., **3** No. 1 (2010) 145-154.

[15] T. Tsutsui, *Fully prime rings II*, Comm. Algebra, **24** No. 9 (1996) 2981-2989.

[16] D. A. Yuwaningsih, *Some properties of left weakly jointly prime* (R, S)*-submodules*, J. Indones. Math. Soc., **26** No. 2 (2020) 234-241.

[17] D. A. Yuwaningsih, I. E. Wijayanti and P. W. Prasetyo, *On* (R, S)*-module homomorphisms*, J. Phys. Conf. Ser., **1188** No. 1 (2019) 012114.

Dian Ariesta Yuwaningsih
Department of Mathematics Education,
Universitas Ahmad Dahlan,
Yogyakarta, Indonesia.
dian.ariesta@pmat.uad.ac.id

Indah Emilia Wijayanti
Department of Mathematics,
Universitas Gadjah Mada,
Yogyakarta, Indonesia.
ind_wijayanti@ugm.ac.id