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HYBRID IDEALS ON A LATTICE

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Abstract. The fuzzy set is a fantastic tool for expressing hesitancy and dealing with uncertainty in real-world circumstances. Soft set theory has recently been developed to deal with practical problems. The soft and fuzzy sets were combined by Jun et al. to generate hybrid structures. The idea of hybrid ideals on a distributive lattice is discussed in this work. The relation between hybrid congruences and hybrid ideals on a distributive lattice is also examined. In addition, the product of hybrid ideals and its numerous results are discussed.

1. Introduction

Zadeh [25] defined the notion of a fuzzy set in 1965, and it opened a different way of thinking for many engineers, mathematicians, physicists, chemists, and others, due to the fact that it has several applications in multiple fields. In 1971, Rosenfeld [22] defined a fuzzy subgroup of a group based on the idea of a fuzzy subset of a set.

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In 1990, Yuan et al., [24] pioneered the idea of fuzzy ideals on a distributive lattice. The relation between fuzzy congruences and fuzzy ideals on a distributive lattice is explored. On a generalised Boolean algebra, they proved that the lattice of fuzzy congruence and fuzzy ideals are isomorphic. In [1], the idea of fuzzy lattices was established in 1994 by Ajmal et al. Several characterizations of fuzzy sublattices, fuzzy prime ideals, fuzzy ideals are also established. Swamy et al. [23] proposed the idea of fuzzy ideals and congruences of lattices in 1998. Chon [3] pioneered the notion of fuzzy lattices and fuzzy partial order relations, and also developed several basic aspects of fuzzy lattices, characterised a fuzzy lattice as a fuzzy relation, and defined a fuzzy partial order relation using its level set. In 2011, Ramarao et al. [21] introduced the notion of fuzzy ideals and filters of lattice and also some of their results are discussed.

Soft set theory, which was created by Molodtsov [14] in 1999 as a generalisation of fuzzy set theory, has been successfully tested in many different fields. In [12], soft sets were first applied to decision-making issues by Maji et al. It is based on the knowledge reduction theory of rough sets.

Jun et al. [11] created hybrid structures by the fusion of fuzzy and soft sets. In a set of parameters over an initial universe set, they looked into a variety of characteristics of hybrid structure. Based on this approach, they developed the notions of hybrid subalgebra, hybrid field, and hybrid linear space. Hybrid structures have been utilised to solve a number of algebraic systems with varying results (see [2], [4], [5], [6], [7], [8], [9], [13], [15], [16], [17], [18], [19] and [20]).

In this work, we discuss the relations between hybrid congruences and hybrid ideals on a distributive lattice $\mathscr{F}$. In a generalized Boolean algebra, we obtain some results related to hybrid congruence and hybrid ideals. Furthermore, the product of hybrid ideals is also introduced, and a hybrid ideal is obtained on the direct sum of lattices to be representable as a direct sum of hybrid ideals on each lattice.

2. Preliminaries

In this section, we will collect some definitions and observations to support us with our main findings. Throughout this paper, $\mathscr{F} = (\mathscr{F}, +, \cdot)$ represents a lattice, where $t + k = t \vee k$ and $t \cdot k = t \wedge k$, $\mathcal{P}(\mathcal{A})$ represents the power set of a non-empty set $\mathcal{A}$.

Definition 2.1. [11] Let $\mathcal{Q}$ be an universal set, a hybrid structure in a non-empty set $\mathscr{F}$ over $\mathcal{Q}$ is $\widetilde{j}_\vartheta := (\widetilde{j}, \vartheta) : \mathscr{F} \to \mathcal{P}(\mathcal{Q}) \times [0,1],\ v_0 \mapsto (\widetilde{j}(v_0), \vartheta(v_0))$, where $\widetilde{j} : \mathscr{F} \to \mathcal{P}(\mathcal{Q})$ and $\vartheta : \mathscr{F} \to [0,1]$ are mappings.

A relation $\ll$ is defined on the family of all hybrid structures, represented by $\mathcal{H}(\mathscr{F})$, in $\mathscr{F}$ over $\mathcal{Q}$ as follows:

$$(\forall\ \widetilde{a}_\varsigma, \widetilde{z}_\varrho \in \mathcal{H}(\mathscr{F}))\left(\widetilde{a}_\varsigma \ll \widetilde{z}_\varrho \Leftrightarrow \widetilde{a} \mathrel{\widetilde{\subseteq}} \widetilde{z}, \varsigma \succeq \varrho\right),$$

where $\widetilde{a} \mathrel{\widetilde{\subseteq}} \widetilde{z}$ means: $\widetilde{a}(q) \subseteq \widetilde{z}(q)$ and $\varsigma \succeq \varrho$ means: $\varsigma(q) \geq \varrho(q)\ \forall q \in \mathscr{F}$. Then $(\mathcal{H}(\mathscr{F}), \ll)$ is a poset.

Definition 2.2. [10] A subset $\mathscr{I}(\neq \emptyset)$ of $\mathscr{F}$ is said to be an ideal of $\mathscr{F}$ if for any $q \in \mathscr{I}, b \in \mathscr{F}$, $q \wedge b \in \mathscr{I}$ and $q, b \in \mathscr{I}, q \vee b \in \mathscr{I}$.

Definition 2.3. [10] A subset $\mathcal{J}(\neq \emptyset)$ of $\mathscr{F}$ is described as a filter of $\mathscr{F}$ if for any $q \in \mathcal{J}, b \in \mathscr{F}, q \vee b \in \mathcal{J}$ and $q, b \in \mathcal{J}, q \wedge b \in \mathcal{J}$.

Definition 2.4. [24] An element $z \in \mathscr{F}$ is relatively complemented if z is complemented in every $[j, w]$ with $j \leq z \leq w$ (i.e., $z + u = w$ and $zu = j$ for some $u \in [j, w]$). The lattice $\mathscr{F}$ is relatively complemented if every $z \in \mathscr{F}$ is relatively complemented.

Definition 2.5. [24] A relatively complemented distributive lattice with smallest element 0 is a generalized Boolean algebra.

In a generalized Boolean algebra $\mathscr{F}$, $c_0, h_0 \in \mathscr{F}$, the difference is defined by $c_0 - h_0$, to be the relative complement of c_0h_0 in $[0, c_0]$, and the symmetric difference is described as $c_0 \oplus h_0 = (c_0 - h_0) + (h_0 - c_0)$.

It is easy to show that

(i) $c_0 - h_0 \leq c_0$,

(ii) $h_0(c_0 - h_0) = 0$,

(iii) $h_0 + (c_0 - h_0) = c_0 + h_0$ for any $c_0, h_0 \in \mathscr{F}$.

Lemma 2.6. [24] *If $\mathscr{F}$ is a generalized Boolean algebra, then $t_0 + a_0 = t_0 \oplus a_0 \oplus t_0a_0$ holds for all $t_0, a_0 \in \mathscr{F}$.*

3. Hybrid ideals and hybrid filters on lattices

We define hybrid ideals and hybrid filters on a distributive lattice in this section. We prove some equivalent conditions related to hybrid ideals and hybrid filters.

Definition 3.1. Let $\widetilde{k}_\varpi \in \mathcal{H}(\mathscr{F})$. Then $\widetilde{k}_\varpi$ is said to be a hybrid sublattice of $\mathscr{F}$ if

(i) $(\forall x_0, b_0 \in \mathscr{F})\begin{pmatrix} \widetilde{k}(x_0 + b_0) \supseteq \widetilde{k}(x_0) \cap \widetilde{k}(b_0) \\ \varpi(x_0 + b_0) \leq \varpi(x_0) \vee \varpi(b_0) \end{pmatrix}$.

(ii) $(\forall x_0, b_0 \in \mathscr{F})\begin{pmatrix} \widetilde{k}(x_0 \cdot b_0) \supseteq \widetilde{k}(x_0) \cap \widetilde{k}(b_0) \\ \varpi(x_0 \cdot b_0) \leq \varpi(x_0) \vee \varpi(b_0) \end{pmatrix}$.

Definition 3.2. A hybrid sublattice $\widetilde{k}_\varpi$ of $\mathscr{F}$ is called

(i) a hybrid filter if $v \leq b$ in $\mathscr{F}$ implies $\begin{pmatrix} \widetilde{k}(v) \subseteq \widetilde{k}(b) \\ \varpi(v) \geq \varpi(b) \end{pmatrix}$ for $v, b \in \mathscr{F}$.

(ii) a hybrid ideal if $v \leq b$ in $\mathscr{F}$ implies $\begin{pmatrix} \widetilde{k}(v) \supseteq \widetilde{k}(b) \\ \varpi(v) \leq \varpi(b) \end{pmatrix}$ for $v, b \in \mathscr{F}$.

Definition 3.3. Let $\widetilde{k}_\varpi$ in $\mathscr{F}$ be a hybrid ideal. Then $\widetilde{k}_\varpi$ is defined as a hybrid prime ideal if $(\forall v_1, i_1 \in \mathscr{F}) \begin{pmatrix} \widetilde{k}(v_1 \cdot i_1) \subseteq \widetilde{k}(v_1) \cup \widetilde{k}(i_1) \\ \varpi(v_1 \cdot i_1) \geq \varpi(v_1) \wedge \varpi(i_1) \end{pmatrix}$.

Definition 3.4. Let $\widetilde{k}_\varpi$ be a hybrid filter in $\mathscr{F}$. Then $\widetilde{k}_\varpi$ is defined as a hybrid prime filter if $(\forall y_0, b_0 \in \mathscr{F}) \begin{pmatrix} \widetilde{k}(y_0 + b_0) \subseteq \widetilde{k}(y_0) \cup \widetilde{k}(b_0) \\ \varpi(y_0 + b_0) \geq \varpi(y_0) \wedge \varpi(b_0) \end{pmatrix}$.

Definition 3.5. Let $\mathscr{F}$ and $\mathscr{F}'$ be two lattices. Let $\Omega : \mathscr{F} \to \mathscr{F}'$, and let $\widetilde{k}_\varpi$ and $\widetilde{j}_\lambda$ be hybrid structures in $\mathscr{F}$ and $\mathscr{F}'$, respectively.

(i) The image of $\widetilde{k}_\varpi$ under Ω, denoted by $\Omega(\widetilde{k}_\varpi) := (\Omega(\widetilde{k}), \Omega(\varpi))$, is a hybrid structure of $\mathscr{F}'$ defined as follows: For each $b \in \mathscr{F}'$,

$$\Omega(\widetilde{k})(b) = \begin{cases} \bigcup\limits_{q \in \Omega^{-1}(b)} \widetilde{k}(q), & if\ \Omega^{-1}(b) \neq \emptyset, \\ \emptyset, & otherwise, \end{cases} \quad ; \quad \Omega(\varpi)(b) = \begin{cases} \bigwedge\limits_{q \in \Omega^{-1}(b)} \varpi(q), & if\ \Omega^{-1}(b) \neq \emptyset, \\ 1, & otherwise. \end{cases}$$

(ii) The preimage of $\widetilde{j}_\lambda$ under Ω, represented by $\Omega^{-1}(\widetilde{j}_\lambda) := (\Omega^{-1}(\widetilde{j}), \Omega^{-1}(\lambda))$, is a hybrid structure of $\mathscr{F}$ described as $\begin{pmatrix} \Omega^{-1}(\widetilde{j})(q) = \widetilde{j}(\Omega(q)) \\ \Omega^{-1}(\lambda)(q) = \lambda(\Omega(q)) \end{pmatrix}$ for all $q \in \mathscr{F}$.

Theorem 3.6. *Let $\widetilde{k}_\varpi \in \mathcal{H}(\mathscr{F})$. Then the conditions mentioned below are equivalent:*

(i) $(\forall h_1, b_1 \in \mathscr{F}) \begin{pmatrix} \widetilde{k}(h_1 + b_1) = \widetilde{k}(h_1) \cap \widetilde{k}(b_1) \\ \varpi(h_1 + b_1) = \varpi(h_1) \vee \varpi(b_1) \end{pmatrix}$,

(ii) $\widetilde{k}_\varpi$ is a hybrid ideal.

Proof. $(i) \Rightarrow (ii)$ Assume $\widetilde{k}(h_1 + b_1) = \widetilde{k}(h_1) \cap \widetilde{k}(b_1)$ and $\varpi(h_1 + b_1) = \varpi(h_1) \vee \varpi(b_1)$ for all $h_1, b_1 \in \mathscr{F}$. If $h_1 \leq b_1$, then $\widetilde{k}(b_1) = \widetilde{k}(h_1) \cap \widetilde{k}(b_1)$ and $\varpi(b_1) = \varpi(h_1) \vee \varpi(b_1)$ which imply $\widetilde{k}(b_1) \subseteq \widetilde{k}(h_1)$ and $\varpi(b_1) \geq \varpi(h_1)$. As $h_1 \cdot b_1 \leq h_1$ and $h_1 \cdot b_1 \leq b_1$, we get $\widetilde{k}(h_1 \cdot b_1) \supseteq \widetilde{k}(h_1) \cap \widetilde{k}(b_1)$ and $\varpi(h_1 \cdot b_1) \leq \varpi(h_1) \vee \varpi(b_1)$. Therefore $\widetilde{k}_\varpi$ is a hybrid ideal.

$(ii) \Rightarrow (i)$ For $h_1, b_1 \in \mathscr{F}$. Clearly, $\widetilde{k}(h_1 + b_1) \supseteq \widetilde{k}(h_1) \cap \widetilde{k}(b_1)$ and $\varpi(h_1 + b_1) \leq \varpi(h_1) \vee \varpi(b_1)$. For $h_1 \leq h_1 + b_1$ and $b_1 \leq h_1 + b_1$, we have $\widetilde{k}(h_1) \supseteq \widetilde{k}(h_1 + b_1)$; $\varpi(h_1) \leq \varpi(h_1 + b_1)$ and $\widetilde{k}(b_1) \supseteq \widetilde{k}(h_1 + b_1)$; $\varpi(b_1) \leq \varpi(h_1 + b_1)$. Thus $\widetilde{k}(h_1 + b_1) \subseteq \widetilde{k}(h_1) \cap \widetilde{k}(b_1)$ and $\varpi(h_1 + b_1) \geq \varpi(h_1) \vee \varpi(b_1)$, hence $\widetilde{k}(h_1 + b_1) = \widetilde{k}(h_1) \cap \widetilde{k}(b_1)$ and $\varpi(h_1 + b_1) = \varpi(h_1) \vee \varpi(b_1)$. □

Theorem 3.7. *Let $\widetilde{k}_{\varpi} \in \mathcal{H}(\mathscr{F})$. Then the following criteria are equivalent:*

(ii) $(\forall q_0, b_0 \in \mathscr{F}) \begin{pmatrix} \widetilde{k}(q_0 \cdot b_0) = \widetilde{k}(q_0) \cap \widetilde{k}(b_0) \\ \varpi(q_0 \cdot b_0) = \varpi(q_0) \vee \varpi(b_0) \end{pmatrix}$,

(ii) $\widetilde{k}_{\varpi}$ *is a hybrid filter.*

Proof. This theorem's proof is similar to that of Theorem 3.6. □

Theorem 3.8. *Let $\Omega : \mathscr{F} \to \mathscr{F}'$ be a homomorphism from a lattice $\mathscr{F}$ onto a lattice $\mathscr{F}'$. Then the below criteria are hold:*

(i) If $\widetilde{k}_{\varpi}$ is a hybrid sublattice (filter, ideal) of $\mathscr{F}$, then $\Omega(\widetilde{k}_{\varpi})$ is a hybrid sublattice (filter, ideal) of $\mathscr{F}'$.

(ii) If $\widetilde{j}_{\lambda}$ is a hybrid sublattice (filter, ideal, prime filter, prime ideal) of $\mathscr{F}'$, then $\Omega^{-1}(\widetilde{j}_{\lambda})$ is a hybrid sublattice (filter, ideal, prime filter, prime ideal) of $\mathscr{F}$.

Proof. (i) Let $\widetilde{k}_{\varpi}$ be a hybrid sublattice of $\mathscr{F}$ and let $c', s' \in \mathscr{F}'$. Then $\Omega(\widetilde{k})(c'+s') = \bigcup_{v \in \Omega^{-1}(c'+s')} \widetilde{k}(v) = \bigcup_{v \in \Omega^{-1}(c')+\Omega^{-1}(s')} \widetilde{k}(v) \supseteq \left(\bigcup_{v \in \Omega^{-1}(c')} \widetilde{k}(v) \right) \cap \left(\bigcup_{v \in \Omega^{-1}(s')} \widetilde{k}(v) \right) = \Omega(\widetilde{k})(c') \cap \Omega(\widetilde{k})(s')$; $\Omega(\varpi)(c'+s') = \bigwedge_{v \in \Omega^{-1}(c'+s')} \varpi(v) = \bigwedge_{v \in \Omega^{-1}(c')+\Omega^{-1}(s')} \varpi(v) \leq \left(\bigwedge_{v \in \Omega^{-1}(c')} \varpi(v) \right) \vee \left(\bigwedge_{v \in \Omega^{-1}(s')} \varpi(v) \right) = \Omega(\varpi)(c') \vee \Omega(\varpi)(s')$.

Also, $\Omega(\widetilde{k})(c' \cdot s') = \bigcup_{v \in \Omega^{-1}(c' \cdot s')} \widetilde{k}(v) = \bigcup_{v \in \Omega^{-1}(c') \cdot \Omega^{-1}(s')} \widetilde{k}(v) \supseteq \left(\bigcup_{v \in \Omega^{-1}(c')} \widetilde{k}(v) \right) \cap \left(\bigcup_{v \in \Omega^{-1}(s')} \widetilde{k}(v) \right) = \Omega(\widetilde{k})(c') \cap \Omega(\widetilde{k})(s')$; $\Omega(\varpi)(c' \cdot s') = \bigwedge_{v \in \Omega^{-1}(c' \cdot s')} \varpi(v) = \bigwedge_{v \in \Omega^{-1}(c') \cdot \Omega^{-1}(s')} \varpi(v) \leq \left(\bigwedge_{v \in \Omega^{-1}(c')} \varpi(v) \right) \vee \left(\bigwedge_{v \in \Omega^{-1}(s')} \varpi(v) \right) = \Omega(\varpi)(c') \vee \Omega(\varpi)(s')$.

Therefore $\Omega(\widetilde{k}_{\varpi})$ is a hybrid sublattice of $\mathscr{F}'$.

Assume that $\widetilde{k}_{\varpi}$ is a hybrid ideal of $\mathscr{F}$. Let $c', s' \in \mathscr{F}'$ and $c' \leq s'$. Then $\Omega(\widetilde{k})(c') = \bigcup_{v \in \Omega^{-1}(c')} \widetilde{k}(v) = \bigcup_{v \in \Omega^{-1}(c') \leq \Omega^{-1}(s')} \widetilde{k}(v) \supseteq \bigcup_{v \in \Omega^{-1}(s')} \widetilde{k}(v) = \Omega(\widetilde{k})(s')$; $\Omega(\varpi)(c') = \bigwedge_{v \in \Omega^{-1}(c')} \varpi(v) = \bigwedge_{v \in \Omega^{-1}(c') \leq \Omega^{-1}(s')} \varpi(v) \leq \bigwedge_{v \in \Omega^{-1}(s')} \varpi(v) = \Omega(\varpi)(s')$. Therefore $\Omega(\widetilde{k}_{\varpi})$ is a hybrid ideal of $\mathscr{F}'$. Similarly, we can prove that, if $\widetilde{k}_{\varpi}$ is a hybrid filter of $\mathscr{F}$, then $\Omega(\widetilde{k}_{\varpi})$ is a hybrid filter of $\mathscr{F}'$.

(ii) Assume that $\widetilde{j}_{\lambda}$ is a hybrid sublattice of $\mathscr{F}'$ and let $x, b \in \mathscr{F}$. Then $\Omega^{-1}(\widetilde{j})(x+b) = \widetilde{j}(\Omega(x+b)) = \widetilde{j}(\Omega(x)+\Omega(b)) \supseteq \widetilde{j}(\Omega(x)) \cap \widetilde{j}(\Omega(b)) = \Omega^{-1}(\widetilde{j})(x) \cap \Omega^{-1}(\widetilde{j})(b)$; $\Omega^{-1}(\lambda)(x+b) = \lambda(\Omega(x+b)) = \lambda(\Omega(x)+\Omega(b)) \leq \lambda(\Omega(x)) \vee \lambda(\Omega(b)) = \Omega^{-1}(\lambda)(x) \vee \Omega^{-1}(\lambda)(b)$.

Also, $\Omega^{-1}(\widetilde{j})(x \cdot b) = \widetilde{j}(\Omega(x \cdot b)) = \widetilde{j}(\Omega(x) \cdot \Omega(b)) \supseteq \widetilde{j}(\Omega(x)) \cap \widetilde{j}(\Omega(b)) = \Omega^{-1}(\widetilde{j})(x) \cap \Omega^{-1}(\widetilde{j})(b)$;

$\Omega^{-1}(\lambda)(x\cdot b)=\lambda(\Omega(x\cdot b))=\lambda(\Omega(x)\cdot\Omega(b))\leq\lambda(\Omega(x))\vee\lambda(\Omega(b))=\Omega^{-1}(\lambda)(x)\vee\Omega^{-1}(\lambda)(b)$.

Hence $\Omega^{-1}(\widetilde{j}_\lambda)$ is a hybrid sublattice of $\mathscr{F}$.

Assume that $\widetilde{j}_\lambda$ is a hybrid ideal of $\mathscr{F}'$. Let $x,b\in\mathscr{F}$ and $x\leq b$. Then $\Omega^{-1}(\widetilde{j})(x)=\widetilde{j}(\Omega(x))\supseteq\widetilde{j}(\Omega(b))=\Omega^{-1}(\widetilde{j})(b)$; $\Omega^{-1}(\lambda)(x)=\lambda(\Omega(x))\leq\lambda(\Omega(b))=\Omega^{-1}(\lambda)(b)$. Hence $\Omega^{-1}(\widetilde{j}_\lambda)$ is a hybrid ideal of $\mathscr{F}$. In a similar way, we can prove the cases of hybrid filters, hybrid prime filters, and hybrid prime ideals. □

4. Relation between hybrid ideals and hybrid congruences

In this section, we define a hybrid congruence as well as a hybrid relation induced by hybrid congruence. We define a hybrid congruence induced by a hybrid structure.

Definition 4.1. A hybrid equivalence relation Λ_λ on $\mathscr{F}$ is a hybrid structure of $\mathscr{F}\times\mathscr{F}$ that fulfills the below conditions; $\forall\ g_0,v_0,b_0\in\mathscr{F}$,

(i) $\begin{pmatrix}\Lambda(g_0,g_0)=\bigcup\limits_{v_0,b_0\in\mathscr{F}}\Lambda(v_0,b_0)\\ \lambda(g_0,g_0)=\bigwedge\limits_{v_0,b_0\in\mathscr{F}}\lambda(v_0,b_0)\end{pmatrix}$ (hybrid reflexive).

(ii) $\begin{pmatrix}\Lambda(g_0,v_0)=\Lambda(v_0,g_0)\\ \lambda(g_0,v_0)=\lambda(v_0,g_0)\end{pmatrix}$ (hybrid symmetric).

(iii) $\begin{pmatrix}\Lambda(g_0,v_0)\supseteq\Lambda(g_0,b_0)\cap\Lambda(b_0,v_0)\\ \lambda(g_0,v_0)\leq\lambda(g_0,b_0)\vee\lambda(b_0,v_0)\end{pmatrix}$ (hybrid transitive).

A hybrid equivalence relation Λ_λ on $\mathscr{F}$ is a hybrid congruence of $\mathscr{F}$ if $\forall\ z_0,z_1,f_0,f_1\in\mathscr{F}$,

(iv) $\begin{pmatrix}\Lambda(f_0+f_1,z_0+z_1)\supseteq\Lambda(f_0,z_0)\cap\Lambda(f_1,z_1)\\ \lambda(f_0+f_1,z_0+z_1)\leq\lambda(f_0,z_0)\vee\lambda(f_1,z_1)\end{pmatrix}$.

(v) $\begin{pmatrix}\Lambda(f_0f_1,z_0z_1)\supseteq\Lambda(f_0,z_0)\cap\Lambda(f_1,z_1)\\ \lambda(f_0f_1,z_0z_1)\leq\lambda(f_0,z_0)\vee\lambda(f_1,z_1)\end{pmatrix}$.

Example 4.2. Let $\mathscr{B}_0$ be the set of all non-negative integers. Then $(\mathscr{B}_0,\vee,\wedge)$ is a lattice, where $a\vee b=\max\{a,b\}$ and $a\wedge b=\min\{a,b\}$.

(i) For $s_0,j_0\in\mathscr{B}_0$; $E\in\mathcal{P}(\mathcal{Q})\setminus\{\emptyset\}$ and $e\in[0,1)$. Define a hybrid relation Λ_λ on $\mathscr{B}_0$ as follows:

$$\Lambda(s_0,j_0)=\begin{cases}\emptyset & if\ s_0=j_0\\ E, & if\ s_0\neq j_0 \text{ and}\\ & \text{both } s_0,j_0 \text{ are even or}\\ & \text{both } s_0,j_0 \text{ are odd}\\ \mathcal{Q}, & otherwise,\end{cases}\ ;\ \lambda(s_0,j_0)=\begin{cases}1, & if\ s_0=j_0\\ e, & if\ s_0\neq j_0 \text{ and}\\ & \text{both } s_0,j_0 \text{ are even or}\\ & \text{both } s_0,j_0 \text{ are odd,}\\ 0, & otherwise.\end{cases}$$

Then Λ_λ is a hybrid symmetric, but it is not a hybrid reflexive as $\Lambda(1,1) \neq \bigcup_{v_0,b_0\in\mathscr{B}_0} \Lambda(v_0,b_0)$ and $\lambda(1,1) \neq \bigwedge_{v_0,b_0\in\mathscr{B}_0} \lambda(v_0,b_0)$.

(ii) For $s_0, j_0 \in \mathscr{B}_0$; $M, W \in \mathcal{P}(\mathcal{Q}) \setminus \{\emptyset\}$ and $m, w \in [0,1)$ with $M \subset W$ and $m > w$. Define a hybrid relation Λ_λ on $\mathscr{B}_0$ as follows:

$$\Lambda(s_0,j_0) = \begin{cases} \mathcal{Q}, & if \ s_0 = j_0, \\ M, & if \ s_0 \text{ is even}, \\ W, & if \ s_0 \text{ is odd}, \\ \emptyset, & if \ s_0 = 0, \end{cases} \quad ; \quad \lambda(s_0,j_0) = \begin{cases} 0, & if \ s_0 = j_0, \\ m, & if \ s_0 \text{ is even}, \\ w, & if \ s_0 \text{ is odd}, \\ 1, & if \ s_0 = 0. \end{cases}$$

Then Λ_λ is a hybrid reflexive, but it is not a hybrid symmetric as $\Lambda(1,2) \neq \Lambda(2,1)$ and $\lambda(1,2) \neq \lambda(2,1)$.

(iii) For $s_0, j_0 \in \mathscr{B}_0$; $E \in \mathcal{P}(\mathcal{Q}) \setminus \{\emptyset\}$ and $e \in [0,1)$. Define a hybrid relation Λ_λ on $\mathscr{B}_0$ as follows:

$$\Lambda(s_0,j_0) = \begin{cases} \mathcal{Q}, & if \ s_0 = j_0, \\ E, & if \ s_0 \neq j_0 \text{ and} \\ & \text{both } s_0, j_0 \text{ are even or} \\ & \text{both } s_0, j_0 \text{ are odd}, \\ \emptyset, & otherwise, \end{cases} \quad ; \quad \lambda(s_0,j_0) = \begin{cases} 0, & if \ s_0 = j_0, \\ e, & if \ s_0 \neq j_0 \text{ and} \\ & \text{both } s_0, j_0 \text{ are even or} \\ & \text{both } s_0, j_0 \text{ are odd}, \\ 1, & otherwise. \end{cases}$$

Then Λ_λ is a hybrid congruence on $\mathscr{B}_0$.

(iv) For $s_0, j_0 \in \mathscr{B}_0$; $M, W \in \mathcal{P}(\mathcal{Q}) \setminus \{\emptyset\}$ and $m, w \in [0,1)$ with $M \subset W$ and $m > w$. Define a hybrid relation Λ_λ on $\mathscr{B}_0$ as follows:

$$\Lambda(s_0,j_0) = \begin{cases} \mathcal{Q}, & if \ s_0 = j_0, \\ M, & if \text{ both } s_0, j_0 \text{ are even}, \\ W, & if \text{ both } s_0, j_0 \text{ are odd}, \\ \emptyset, & otherwise, \end{cases} \quad ; \quad \lambda(s_0,j_0) = \begin{cases} 0, & if \ s_0 = j_0, \\ m, & if \text{ both } s_0, j_0 \text{ are even}, \\ w, & if \text{ both } s_0, j_0 \text{ are odd}, \\ 1, & otherwise. \end{cases}$$

Then Λ_λ is a hybrid equivalence relation, but it is not a hybrid congruence of $\mathscr{B}_0$ as $\Lambda(4,12) \not\supseteq \Lambda(3,5) \cap \Lambda(1,7)$ and $\lambda(4,12) \not\leq \lambda(3,5) \vee \lambda(1,7)$.

Definition 4.3. Let $\widetilde{k}_{\varpi}$ of $\mathscr{F}$ be a hybrid ideal. A hybrid relation $\Lambda_{\lambda} = \mathcal{C}(\widetilde{k}_{\varpi})$ on $\mathscr{F}$ is described as below:

$$(\forall j, d \in \mathscr{F}) \qquad \left(\begin{array}{l} \Lambda(j,d) = \bigcup\limits_{\substack{s+j=s+d; \\ s\in\mathscr{F}}} \widetilde{k}(s) \\ \lambda(j,d) = \bigwedge\limits_{\substack{s+j=s+d; \\ s\in\mathscr{F}}} \varpi(s) \end{array} \right),$$

is known as the hybrid relation induced by $\widetilde{k}_{\varpi}$.

Theorem 4.4. *If $\widetilde{k}_{\varpi}$ of a distributive lattice $\mathscr{F}$ is a hybrid ideal, then $\mathcal{C}(\widetilde{k}_{\varpi})$ of $\mathscr{F}$ is a hybrid congruence.*

Proof. Consider $\Lambda_{\lambda} = \mathcal{C}(\widetilde{k}_{\varpi})$.

(i) Let $q, m, a \in \mathscr{F}$. Then

$$\Lambda(q,q) = \bigcup_{\substack{t+q=t+q; \\ t\in\mathscr{F}}} \widetilde{k}(t) = \bigcup_{t\in\mathscr{F}} \widetilde{k}(t) \text{ and } \lambda(q,q) = \bigwedge_{\substack{t+q=t+q; \\ t\in\mathscr{F}}} \varpi(t) = \bigwedge_{t\in\mathscr{F}} \varpi(t).$$

Now, $\Lambda(m,a) = \bigcup\limits_{\substack{v+m=v+a; \\ v\in\mathscr{F}}} \widetilde{k}(v) \subseteq \bigcup\limits_{v\in\mathscr{F}} \widetilde{k}(v) = \Lambda(q,q)$ implies that $\Lambda(q,q) = \bigcup\limits_{m,a\in\mathscr{F}} \Lambda(m,a)$ and $\lambda(m,a) = \bigwedge\limits_{\substack{v+m=v+a; \\ v\in\mathscr{F}}} \varpi(v) \geq \bigwedge\limits_{v\in\mathscr{F}} \varpi(v) = \lambda(q,q)$ implies that $\lambda(q,q) = \bigwedge\limits_{m,a\in\mathscr{F}} \lambda(m,a)$. So Λ_{λ} is a hybrid reflexive.

(ii) Let $q, m \in \mathscr{F}$. Then $\Lambda(q,m) = \bigcup\limits_{\substack{t+q=t+m; \\ t\in\mathscr{F}}} \widetilde{k}(t) = \bigcup\limits_{\substack{t+m=t+q; \\ t\in\mathscr{F}}} \widetilde{k}(t) = \Lambda(m,q)$; $\lambda(q,m) = \bigwedge\limits_{\substack{t+q=t+m; \\ t\in\mathscr{F}}} \varpi(t) = \bigwedge\limits_{\substack{t+m=t+q; \\ t\in\mathscr{F}}} \varpi(t) = \lambda(m,q)$. So Λ_{λ} is a hybrid symmetric.

(iii) Let $t, q, a, m, v, g \in \mathscr{F}$. If $t+q=t+a$, $v+a=v+m$, setting $g=t+v$, we get $g+q=g+m$, so

$$\Lambda(q,a) \cap \Lambda(a,m) = \bigcup_{t+q=t+a} \widetilde{k}(t) \cap \bigcup_{v+a=v+m} \widetilde{k}(v) = \bigcup_{\substack{t+q=t+a; \\ v+a=v+m}} \widetilde{k}(t+v) \subseteq \bigcup_{g+q=g+m} \widetilde{k}(g) = \Lambda(q,m),$$

$$\lambda(q,a) \vee \lambda(a,m) = \bigwedge_{t+q=t+a} \varpi(t) \vee \bigwedge_{v+a=v+m} \varpi(v) = \bigwedge_{\substack{t+q=t+a; \\ v+a=v+m}} \varpi(t+v) \geq \bigwedge_{g+q=g+m} \varpi(g) = \lambda(q,m).$$

So Λ_{λ} is a hybrid transitive.

(iv) For any $q_1, q_2, m_1, m_2 \in \mathscr{F}$, we have

$$\begin{aligned}
\Lambda(q_1,m_1)\cap\Lambda(q_2,m_2) &= \bigcup_{\substack{v+q_1=v+m_1;\\ v\in\mathscr{F}}} \widetilde{k}(v) \cap \bigcup_{\substack{g+q_2=g+m_2;\\ g\in\mathscr{F}}} \widetilde{k}(g)\\
&= \bigcup_{\substack{v+q_1=v+m_1;\\ g+q_2=g+m_2;\\ v,g\in\mathscr{F}}} \widetilde{k}(v+g)\\
&\subseteq \bigcup_{\substack{t+(q_1+q_2)=t+(m_1+m_2);\\ t\in\mathscr{F}}} \widetilde{k}(t) = \Lambda(q_1+q_2, m_1+m_2),\\
\lambda(q_1,m_1)\vee\lambda(q_2,m_2) &= \bigwedge_{\substack{v+q_1=v+m_1;\\ v\in\mathscr{F}}} \varpi(v) \vee \bigwedge_{\substack{g+q_2=g+m_2;\\ g\in\mathscr{F}}} \varpi(g)\\
&= \bigwedge_{\substack{v+q_1=v+m_1;\\ g+q_2=g+m_2;\\ v,g\in\mathscr{F}}} \varpi(v+g)\\
&\geq \bigwedge_{\substack{t+(q_1+q_2)=t+(m_1+m_2);\\ t\in\mathscr{F}}} \varpi(t) = \lambda(q_1+q_2, m_1+m_2).
\end{aligned}$$

(v) Let $v, g, q_1, q_2, m_1, m_2 \in \mathscr{F}$. If $v+q_1 = v+m_1$ and $g+q_2 = g+m_2$, then $(v+q_1)(g+q_2) = (v+m_1)(g+m_2)$ which implies $(vg+q_1g+q_2v)+q_1q_2 = (vg+m_1g+m_2v)+m_1m_2$.

Since $(v+q_1)g = (v+m_1)g$ and $(g+q_2)v = (g+m_2)v$, we have $vg+q_1g+q_2v = vg+m_1g+m_2v$. As $\widetilde{k}_\varpi$ is a hybrid ideal of $\mathscr{F}$, we get $\widetilde{k}(vg+q_1g+q_2v) = \widetilde{k}(vg)\cap\widetilde{k}(q_1g)\cap\widetilde{k}(q_2v) \supseteq \widetilde{k}(v)\cap\widetilde{k}(g)$ and $\varpi(vg+q_1g+q_2v) = \varpi(vg)\vee\varpi(q_1g)\vee\varpi(q_2v) \leq \varpi(v)\vee\varpi(g)$.Now,

$$\begin{aligned}
\Lambda(q_1,m_1)\cap\Lambda(q_2,m_2) &= \bigcup_{\substack{v+q_1=v+m_1;\\ v\in\mathscr{F}}} \widetilde{k}(v) \cap \bigcup_{\substack{g+q_2=g+m_2;\\ g\in\mathscr{F}}} \widetilde{k}(g)\\
&= \bigcup_{\substack{v+q_1=v+m_1;\\ g+q_2=g+m_2;\\ v,g\in\mathscr{F}}} \widetilde{k}(v)\cap\widetilde{k}(g)\\
&\subseteq \bigcup_{\substack{v+q_1=v+m_1;\\ g+q_2=g+m_2;\\ v,g\in\mathscr{F}}} \widetilde{k}(vg+q_1g+q_2v)\\
&\subseteq \bigcup_{\substack{(vg+q_1g+q_2v)+q_1q_2=(vg+m_1g+m_2v)+m_1m_2;\\ v,g\in\mathscr{F}}} \widetilde{k}(vg+q_1g+q_2v)\\
&\subseteq \bigcup_{\substack{t+q_1q_2=t+m_1m_2;\\ t\in\mathscr{F}}} \widetilde{k}(t) = \Lambda(q_1q_2, m_1m_2),
\end{aligned}$$

$$\begin{aligned}
\lambda(q_1,m_1)\vee\lambda(q_2,m_2) &= \bigwedge_{\substack{v+q_1=v+m_1;\\ v\in\mathscr{F}}}\varpi(v)\vee\bigwedge_{\substack{g+q_2=g+m_2;\\ g\in\mathscr{F}}}\varpi(g)\\
&= \bigwedge_{\substack{v+q_1=v+m_1;\\ g+q_2=g+m_2;\\ v,g\in\mathscr{F}}}\varpi(v)\vee\varpi(g)\\
&\geq \bigwedge_{\substack{v+q_1=v+m_1;\\ g+q_2=g+m_2;\\ v,g\in\mathscr{F}}}\varpi(vg+q_1g+q_2v)\\
&\geq \bigwedge_{\substack{(vg+q_1g+q_2v)+q_1q_2=(vg+m_1g+m_2v)+m_1m_2;\\ v,g\in\mathscr{F}}}\varpi(vg+q_1g+q_2v)\\
&\geq \bigwedge_{\substack{t+q_1q_2=t+m_1m_2;\\ t\in\mathscr{F}}}\varpi(t)=\lambda(q_1q_2,m_1m_2).
\end{aligned}$$

So $\Lambda_\lambda=\mathcal{C}(\widetilde{k}_\varpi)$ of $\mathscr{F}$ is a hybrid congruence. □

Definition 4.5. For a hybrid equivalence relation Λ_λ on $\mathscr{F}$, the hybrid structure $\widetilde{k}_\varpi=\mathcal{I}(\Lambda_\lambda)$ of $\mathscr{F}$ is defined as below:

$$(\forall z_1\in\mathscr{F})\qquad \begin{pmatrix} \widetilde{k}(z_1)=\bigcap\limits_{d_1\in\mathscr{F}}\Lambda(z_1d_1,z_1)\\ \varpi(z_1)=\bigvee\limits_{d_1\in\mathscr{F}}\lambda(z_1d_1,z_1)\end{pmatrix},$$

is known as the hybrid structure induced by Λ_λ.

Theorem 4.6. *If Λ_λ of a distributive lattice $\mathscr{F}$ is a hybrid congruence, then $\mathcal{I}(\Lambda_\lambda)$ of $\mathscr{F}$ is a hybrid ideal.*

Proof. Put $\widetilde{k}_\varpi=\mathcal{I}(\Lambda_\lambda)$, and let $z,m\in\mathscr{F}$. Then

(i) $\widetilde{k}(m+z)=\bigcap\limits_{d\in\mathscr{F}}\Lambda((m+z)d,m+z)=\bigcap\limits_{d\in\mathscr{F}}\Lambda(md+zd,m+z)\supseteq\bigcap\limits_{d\in\mathscr{F}}(\Lambda(md,m)\cap\Lambda(zd,z))=\left\{\bigcap\limits_{d\in\mathscr{F}}\Lambda(md,m)\right\}\cap\left\{\bigcap\limits_{d\in\mathscr{F}}\Lambda(zd,z)\right\}=\widetilde{k}(m)\cap\widetilde{k}(z)$; $\varpi(m+z)=\bigvee\limits_{d\in\mathscr{F}}\lambda((m+z)d,m+z)=\bigvee\limits_{d\in\mathscr{F}}\lambda(md+zd,m+z)\leq\bigvee\limits_{d\in\mathscr{F}}(\lambda(md,m)\vee\lambda(zd,z))=\left\{\bigvee\limits_{d\in\mathscr{F}}\lambda(md,m)\right\}\vee\left\{\bigvee\limits_{d\in\mathscr{F}}\lambda(zd,z)\right\}=\varpi(m)\vee\varpi(z)$.

(ii) If $m\leq z$, then $\widetilde{k}(m)=\bigcap\limits_{d\in\mathscr{F}}\Lambda(md,m)=\bigcap\limits_{d\in\mathscr{F}}\Lambda(mzd,mz)\supseteq\bigcap\limits_{d\in\mathscr{F}}(\Lambda(m,m)\cap\Lambda(zd,z))=\bigcap\limits_{d\in\mathscr{F}}\Lambda(zd,z)=\widetilde{k}(z)$; $\varpi(m)=\bigvee\limits_{d\in\mathscr{F}}\lambda(md,m)=\bigvee\limits_{d\in\mathscr{F}}\lambda(mzd,mz)\leq\bigvee\limits_{d\in\mathscr{F}}(\lambda(m,m)\vee\lambda(zd,z))=\bigvee\limits_{d\in\mathscr{F}}\lambda(zd,z)=\varpi(z)$.

(iii) By (ii), we have $\begin{pmatrix} \widetilde{k}(mz) \supseteq \widetilde{k}(m) \\ \varpi(mz) \leq \varpi(m) \end{pmatrix}$ and $\begin{pmatrix} \widetilde{k}(mz) \supseteq \widetilde{k}(z) \\ \varpi(mz) \leq \varpi(z) \end{pmatrix}$. So $\begin{pmatrix} \widetilde{k}(mz) \supseteq \widetilde{k}(m) \cap \widetilde{k}(z) \\ \varpi(mz) \leq \varpi(m) \vee \varpi(z) \end{pmatrix}$. Therefore $\widetilde{k}_\varpi = \mathcal{I}(\Lambda_\lambda)$ of $\mathscr{F}$ is a hybrid ideal. □

Theorem 4.7. *If $\widetilde{k}_\varpi$ of a distributive lattice $\mathscr{F}$ with smallest element 0 is a hybrid ideal, then $\widetilde{k}_\varpi = \mathcal{I}(\mathcal{C}(\widetilde{k}_\varpi))$.*

Proof. Take $\Lambda_\lambda = \mathcal{C}(\widetilde{k}_\varpi)$ and $\widetilde{a}_\varphi = \mathcal{I}(\Lambda_\lambda)$. Then $\forall v \in \mathscr{F}$, we get $\widetilde{a}(v) = \bigcap\limits_{p \in \mathscr{F}} \Lambda(vp, v)$ and $\varphi(v) = \bigvee\limits_{p \in \mathscr{F}} \lambda(vp, v)$. Since $v + vp = v + v$, we get $\Lambda(vp, v) = \bigcup\limits_{\substack{z+vp=z+v;\\ z \in \mathscr{F}}} \widetilde{k}(z) \supseteq \widetilde{k}(v); \lambda(vp, v) = \bigwedge\limits_{\substack{z+vp=z+v;\\ z \in \mathscr{F}}} \varpi(z) \leq \varpi(v)$. So, $\widetilde{a}(v) \supseteq \bigcap\limits_{p \in \mathscr{F}} \widetilde{k}(v) = \widetilde{k}(v)$ and $\varphi(v) \leq \bigvee\limits_{p \in \mathscr{F}} \varpi(v) = \varpi(v)$. Thus $\widetilde{k}_\varpi \ll \widetilde{a}_\varphi$.

Also, $\widetilde{a}(v) = \bigcap\limits_{p \in \mathscr{F}} \Lambda(vp, v) \subseteq \Lambda(v0, v) = \bigcup\limits_{\substack{z+v0=z+v;\\ z \in \mathscr{F}}} \widetilde{k}(z) = \bigcup\limits_{\substack{z=z+v;\\ z \in \mathscr{F}}} \widetilde{k}(z) = \bigcup\limits_{\substack{z \geq v;\\ z \in \mathscr{F}}} \widetilde{k}(z) \subseteq \widetilde{k}(v)$;

$\varphi(v) = \bigvee\limits_{p \in \mathscr{F}} \lambda(vp, v) \geq \lambda(v0, v) = \bigwedge\limits_{\substack{z+v0=z+v;\\ z \in \mathscr{F}}} \varpi(z) = \bigwedge\limits_{\substack{z=z+v;\\ z \in \mathscr{F}}} \varpi(z) = \bigwedge\limits_{\substack{z \geq v;\\ z \in \mathscr{F}}} \varpi(z) \geq \varpi(v)$.

Thus $\widetilde{a}_\varphi \ll \widetilde{k}_\varpi$ and hence $\widetilde{k}_\varpi = \widetilde{a}_\varphi = \mathcal{I}(\mathcal{C}(\widetilde{k}_\varpi))$. □

Theorem 4.8. *Let $\widetilde{k}_\varpi$ be a hybrid ideal of a generalized Boolean algebra $\mathscr{F}$ and $\Lambda_\lambda = \mathcal{C}(\widetilde{k}_\varpi)$. Then $\widetilde{k}_\varpi(p_0 \oplus b_0) = \Lambda_\lambda(p_0, b_0)$ for all $p_0, b_0 \in \mathscr{F}$.*

Proof. By Lemma 2.6, $p_0 + (p_0 \oplus b_0) = p_0 \oplus (p_0 \oplus b_0) \oplus p_0(p_0 \oplus b_0) = b_0 \oplus p_0 \oplus p_0 b_0 = p_0 + b_0$.

By symmetry, $b_0 + (p_0 \oplus b_0) = p_0 + b_0$ and so $p_0 + (p_0 \oplus b_0) = b_0 + (p_0 \oplus b_0)$. Therefore $\Lambda(p_0, b_0) = \bigcup\limits_{\substack{z_0+p_0=z_0+b_0;\\ z_0 \in \mathscr{F}}} \widetilde{k}(z_0) \supseteq \widetilde{k}(p_0 \oplus b_0)$ and $\lambda(p_0, b_0) = \bigwedge\limits_{\substack{z_0+p_0=z_0+b_0;\\ z_0 \in \mathscr{F}}} \varpi(z_0) \leq \varpi(p_0 \oplus b_0)$.

Thus $\Lambda(p_0, b_0) \supseteq \widetilde{k}(p_0 \oplus b_0)$ and $\lambda(p_0, b_0) \leq \varpi(p_0 \oplus b_0)$.

Conversely, if $z_0 \in \mathscr{F}$ satisfies $z_0 + p_0 = z_0 + b_0$, then $p_0 - b_0 = (p_0 - b_0)z_0 + (p_0 - b_0) = (p_0 - b_0)z_0 + (p_0 - b_0)p_0 = (p_0 - b_0)(z_0 + p_0) = (p_0 - b_0)(z_0 + b_0) = (p_0 - b_0)z_0 + (p_0 - b_0)b_0 = (p_0 - b_0)z_0$ shows that $p_0 - b_0 \leq z_0$ and by symmetry $b_0 - p_0 \leq z_0$.

Thus $p_0 \oplus b_0 = (p_0 - b_0) + (b_0 - p_0) \leq z_0$ and thus $\widetilde{k}(z_0) \subseteq \widetilde{k}(p_0 \oplus b_0)$ and $\varpi(z_0) \geq \varpi(p_0 \oplus b_0)$.

$$\text{So, } \Lambda(p_0, b_0) = \bigcup_{\substack{z_0+p_0=z_0+b_0;\\ z_0 \in \mathscr{F}}} \widetilde{k}(z_0) \subseteq \bigcup_{\substack{z_0+p_0=z_0+b_0;\\ z_0 \in \mathscr{F}}} \widetilde{k}(p_0 \oplus b_0) = \widetilde{k}(p_0 \oplus b_0);$$

$$\lambda(p_0, b_0) = \bigwedge_{\substack{z_0+p_0=z_0+b_0;\\ z_0 \in \mathscr{F}}} \varpi(z_0) \geq \bigwedge_{\substack{z_0+p_0=z_0+b_0;\\ z_0 \in \mathscr{F}}} \varpi(p_0 \oplus b_0) = \varpi(p_0 \oplus b_0).$$

Thus $\Lambda(p_0, b_0) \subseteq \widetilde{k}(p_0 \oplus b_0)$ and $\lambda(p_0, b_0) \geq \varpi(p_0 \oplus b_0)$ and hence $\widetilde{k}(p_0 \oplus b_0) = \Lambda(p_0, b_0)$; $\varpi(p_0 \oplus b_0) = \lambda(p_0, b_0)$. □

Theorem 4.9. *If Θ_ϑ of $\mathscr{F}$ with smallest element 0 is a hybrid congruence, then $\mathcal{I}(\Theta_\vartheta)(d_1) = \Theta_\vartheta(d_1, 0)$ holds for all $d_1 \in \mathscr{F}$.*

Proof. Let $\widetilde{k}_\varpi = \mathcal{I}(\Theta_\vartheta)$. Then $\widetilde{k}(d_1) = \bigcap_{j_1 \in \mathscr{F}} \Theta(d_1 j_1, d_1) \subseteq \Theta(d_1 0, d_1) = \Theta(0, d_1) = \Theta(d_1, 0)$; $\varpi(d_1) = \bigvee_{j_1 \in \mathscr{F}} \vartheta(d_1 j_1, d_1) \geq \vartheta(d_1 0, d_1) = \vartheta(0, d_1) = \vartheta(d_1, 0)$.

Since $\Theta(d_1 j_1, d_1) = \Theta(d_1 j_1 + 0, d_1 j_1 + d_1) \supseteq \Theta(d_1 j_1, d_1 j_1) \cap \Theta(0, d_1) = \Theta(d_1, 0)$ and $\vartheta(d_1 j_1, d_1) = \vartheta(d_1 j_1 + 0, d_1 j_1 + d_1) \leq \vartheta(d_1 j_1, d_1 j_1) \vee \vartheta(0, d_1) = \vartheta(d_1, 0)$, we get, $\bigcap_{j_1 \in \mathscr{F}} \Theta(d_1 j_1, d_1) \supseteq \bigcap_{j_1 \in \mathscr{F}} \Theta(d_1, 0) = \Theta(d_1, 0)$ and $\bigvee_{j_1 \in \mathscr{F}} \vartheta(d_1 j_1, d_1) \leq \bigvee_{j_1 \in \mathscr{F}} \vartheta(d_1, 0) = \vartheta(d_1, 0)$.

Thus $\widetilde{k}(d_1) \supseteq \Theta(d_1, 0)$ and $\varpi(d_1) \leq \vartheta(d_1, 0)$ and hence $\widetilde{k}(d_1) = \Theta(d_1, 0)$ and $\varpi(d_1) = \vartheta(d_1, 0)$. □

Lemma 4.10. *Let $\widetilde{k}_\varpi$ and $\widetilde{v}_\varrho$ be two hybrid ideals of $\mathscr{F}$, and Λ_λ and Δ_δ be two hybrid congruences of $\mathscr{F}$. Then the following conditions are hold:*

(i) If $\widetilde{k}_\varpi \ll \widetilde{v}_\varrho$, then $\mathcal{C}(\widetilde{k}_\varpi) \ll \mathcal{C}(\widetilde{v}_\varrho)$.

(ii) If $\Lambda_\lambda \ll \Delta_\delta$, then $\mathcal{I}(\Lambda_\lambda) \ll \mathcal{I}(\Delta_\delta)$.

Proof. The proofs are trivial. □

Theorem 4.11. *If Λ_λ is a hybrid congruence on a generalized Boolean algebra $\mathscr{F}$, then $\mathcal{C}(\mathcal{I}(\Lambda_\lambda)) = \Lambda_\lambda$.*

Proof. Let $\widetilde{k}_\varpi = \mathcal{I}(\Lambda_\lambda)$. By Theorem 4.9, we have $\widetilde{k}(p_0) = \Lambda(p_0, 0)$ and $\varpi(p_0) = \lambda(p_0, 0)$. Suppose $p_0 + v_0 = p_0 + w_0$. Then

$$\Lambda(p_0 + v_0, v_0) = \Lambda(p_0 + v_0, 0 + v_0) \supseteq \Lambda(p_0, 0) \cap \Lambda(v_0, v_0) = \Lambda(p_0, 0);$$

$$\lambda(p_0 + v_0, v_0) = \lambda(p_0 + v_0, 0 + v_0) \leq \lambda(p_0, 0) \vee \lambda(v_0, v_0) = \lambda(p_0, 0).$$

By symmetry, $\Lambda(p_0 + w_0, w_0) \supseteq \Lambda(p_0, 0)$ and $\lambda(p_0 + w_0, w_0) \leq \lambda(p_0, 0)$. Now,

$$\begin{aligned} \Lambda(v_0, w_0) &\supseteq \Lambda(v_0, p_0 + v_0) \cap \Lambda(p_0 + v_0, w_0) \\ &= \Lambda(v_0, p_0 + v_0) \cap \Lambda(p_0 + w_0, w_0) \supseteq \Lambda(p_0, 0) = \widetilde{k}(p_0); \\ \lambda(v_0, w_0) &\leq \lambda(v_0, p_0 + v_0) \vee \lambda(p_0 + v_0, w_0) \\ &= \lambda(v_0, p_0 + v_0) \vee \lambda(p_0 + w_0, w_0) \leq \lambda(p_0, 0) = \varpi(p_0). \end{aligned}$$

We have,

$$\mathcal{C}(\widetilde{k})(v_0,w_0)=\bigcup_{\substack{p_0+v_0=p_0+w_0;\\ p_0\in\mathscr{F}}}\widetilde{k}(p_0)\subseteq\bigcup_{\substack{p_0+v_0=p_0+w_0;\\ p_0\in\mathscr{F}}}\Lambda(v_0,w_0)=\Lambda(v_0,w_0);$$

$$\mathcal{C}(\varpi)(v_0,w_0)=\bigwedge_{\substack{p_0+v_0=p_0+w_0;\\ p_0\in\mathscr{F}}}\varpi(p_0)\geq\bigwedge_{\substack{p_0+v_0=p_0+w_0;\\ p_0\in\mathscr{F}}}\lambda(v_0,w_0)=\lambda(v_0,w_0).$$

Thus $\mathcal{C}(\widetilde{k}_\varpi)\ll\Lambda_\lambda$.

Conversely, by Theorem 4.8 and Theorem 4.9, $\mathcal{C}(\widetilde{k})(f_0,c_0)=\widetilde{k}(f_0\oplus c_0)=\Lambda(f_0\oplus c_0,0)=\Lambda((f_0-c_0)+(c_0-f_0),0)\supseteq\Lambda(f_0-c_0,0)\cap\Lambda(c_0-f_0,0)=\Lambda(f_0(f_0-c_0),c_0(f_0-c_0))\cap\Lambda(c_0(c_0-f_0),f_0(c_0-f_0))\supseteq\Lambda(f_0,c_0)$; $\mathcal{C}(\varpi)(f_0,c_0)=\varpi(f_0\oplus c_0)=\lambda(f_0\oplus c_0,0)=\lambda((f_0-c_0)+(c_0-f_0),0)\leq\lambda(f_0-c_0,0)\vee\lambda(c_0-f_0,0)=\lambda(f_0(f_0-c_0),c_0(f_0-c_0))\vee\lambda(c_0(c_0-f_0),f_0(c_0-f_0))\leq\lambda(f_0,c_0)$.

Thus $\Lambda_\lambda\ll\mathcal{C}(\widetilde{k}_\varpi)$. Therefore $\Lambda_\lambda=\mathcal{C}(\widetilde{k}_\varpi)=\mathcal{C}(\mathcal{I}(\Lambda_\lambda))$. □

5. Products of hybrid ideals

This section illustrates the idea of product of hybrid ideals and also define the projections of hybrid structure. Throughout this section, $\mathscr{F}_1$ and $\mathscr{F}_2$ represent lattices.

Definition 5.1. Let $\widetilde{k}_\varpi\in\mathcal{H}(\mathscr{F}_1)$ and $\widetilde{s}_\varphi\in\mathcal{H}(\mathscr{F}_2)$. Then the product $\widetilde{k}_\varpi\otimes\widetilde{s}_\varphi:=(\widetilde{k}\times\widetilde{s},\varpi\times\varphi)$ of $\widetilde{k}_\varpi$ and $\widetilde{s}_\varphi$ is the element of $\mathcal{H}(\mathscr{F}_1\times\mathscr{F}_2)$ is defined by setting

$$(\forall(b,u)\in\mathscr{F}_1\times\mathscr{F}_2)\left(\begin{array}{c}(\widetilde{k}\times\widetilde{s})(b,u)=\widetilde{k}(b)\cap\widetilde{s}(u)\\ (\varpi\times\varphi)(b,u)=\varpi(b)\vee\varphi(u)\end{array}\right).$$

Definition 5.2. For a hybrid sublattice $\widetilde{k}_\varpi$ of $\mathscr{F}_1\times\mathscr{F}_2$,

$$(\forall j\in\mathscr{F}_1)\quad\left(\begin{array}{c}pr_1(\widetilde{k})(j)=\bigcup\limits_{c\in\mathscr{F}_2}\widetilde{k}(j,c)\\ pr_1(\varpi)(j)=\bigwedge\limits_{c\in\mathscr{F}_2}\varpi(j,c)\end{array}\right),\ \text{and}\ (\forall c\in\mathscr{F}_2)\quad\left(\begin{array}{c}pr_2(\widetilde{k})(c)=\bigcup\limits_{j\in\mathscr{F}_1}\widetilde{k}(j,c)\\ pr_2(\varpi)(c)=\bigwedge\limits_{j\in\mathscr{F}_1}\varpi(j,c)\end{array}\right),$$

are called the hybrid projections of $\widetilde{k}_\varpi$ on $\mathscr{F}_1$ and $\mathscr{F}_2$, respectively.

Proposition 5.3. *If $(\widetilde{k}_\varpi)_i$ are hybrid sublattices (ideals) of $\mathscr{F}_i$ $(i=1,2)$, then $(\widetilde{k}_\varpi)_1\otimes(\widetilde{k}_\varpi)_2$ is a hybrid sublattice (ideal) of $\mathscr{F}_1\times\mathscr{F}_2$. If $\widetilde{k}_\varpi$ is a hybrid sublattice of $\mathscr{F}_1\times\mathscr{F}_2$, then $pr_i(\widetilde{k}_\varpi)$ $(i=1,2)$ are hybrid sublattices of $\mathscr{F}_i$ $(i=1,2)$, respectively.*

Proof. The first assertion is simple to prove. Next, let $\widetilde{k}_\varpi$ be a hybrid sublattice of $\mathscr{F}_1\times\mathscr{F}_2$. For all $c,t\in\mathscr{F}_1$, we get $pr_1(\widetilde{k})(c+t)=\bigcup\limits_{a\in\mathscr{F}_2}\widetilde{k}(c+t,a)=\bigcup\limits_{\substack{a_i\in\mathscr{F}_2;\\ i=1,2}}\widetilde{k}(c+t,a_1+a_2)\supseteq\bigcup\limits_{\substack{a_i\in\mathscr{F}_2;\\ i=1,2}}\{\widetilde{k}(c,a_1)\cap\widetilde{k}(t,a_2)\}=\bigcup\limits_{a_1\in\mathscr{F}_2}\widetilde{k}(c,a_1)\cap\bigcup\limits_{a_2\in\mathscr{F}_2}\widetilde{k}(t,a_2)=pr_1(\widetilde{k})(c)\cap pr_1(\widetilde{k})(t)$; $pr_1(\varpi)(c+t)=\bigwedge\limits_{a\in\mathscr{F}_2}\varpi(c+$

$t,a) = \bigwedge_{\substack{a_i\in\mathscr{F}_2;\\ i=1,2}} \varpi(c+t,a_1+a_2) \leq \bigwedge_{\substack{a_i\in\mathscr{F}_2;\\ i=1,2}} \{\varpi(c,a_1)\vee\varpi(t,a_2)\} = \bigwedge_{a_1\in\mathscr{F}_2} \varpi(c,a_1) \vee \bigwedge_{a_2\in\mathscr{F}_2} \varpi(t,a_2) = pr_1(\varpi)(c)\vee pr_1(\varpi)(t)$.

Also, $pr_1(\widetilde{k})(ct) = \bigcup_{a\in\mathscr{F}_2} \widetilde{k}(ct,a) = \bigcup_{\substack{a_i\in\mathscr{F}_2;\\ i=1,2}} \widetilde{k}(ct,a_1a_2) \supseteq \bigcup_{\substack{a_i\in\mathscr{F}_2;\\ i=1,2}} \{\widetilde{k}(c,a_1)\cap\widetilde{k}(t,a_2)\} = \bigcup_{a_1\in\mathscr{F}_2} \widetilde{k}(c,a_1) \cap \bigcup_{a_2\in\mathscr{F}_2} \widetilde{k}(t,a_2) = pr_1(\widetilde{k})(c)\cap pr_1(\widetilde{k})(t)$; $pr_1(\varpi)(ct) = \bigwedge_{a\in\mathscr{F}_2} \varpi(ct,a) = \bigwedge_{\substack{a_i\in\mathscr{F}_2;\\ i=1,2}} \varpi(ct,a_1a_2) \leq \bigwedge_{\substack{a_i\in\mathscr{F}_2;\\ i=1,2}} \{\varpi(c,a_1)\vee\varpi(t,a_2)\} = \bigwedge_{a_1\in\mathscr{F}_2} \varpi(c,a_1) \vee \bigwedge_{a_2\in\mathscr{F}_2} \varpi(t,a_2) = pr_1(\varpi)(c)\vee pr_1(\varpi)(t)$.

Thus $pr_1(\widetilde{k}_\varpi)$ is a hybrid sublattice of $\mathscr{F}_1$. Similarly, $pr_2(\widetilde{k}_\varpi)$ is a hybrid sublattice of $\mathscr{F}_2$.

It is similar to prove that if $\widetilde{k}_\varpi$ is a hybrid ideal (filter) of $\mathscr{F}_1\times\mathscr{F}_2$, then $pr_i(\widetilde{k}_\varpi)$ $(i=1,2)$ are hybrid ideals (filters) of $\mathscr{F}_i$ $(i=1,2)$, respectively. □

Definition 5.4. Let $\widetilde{k}_\varpi \in \mathcal{H}(\mathscr{F}_1\times\mathscr{F}_2)$ and $d\in\mathscr{F}_1$, $j\in\mathscr{F}_2$. The marginal hybrid structures of $\widetilde{k}_\varpi$ (with respect to j and d) are $(\widetilde{k}_\varpi)_1^{(j)}\in\mathcal{H}(\mathscr{F}_1)$ and $(\widetilde{k}_\varpi)_2^{(d)}\in\mathcal{H}(\mathscr{F}_2)$ defined as below:

$$(v\in\mathscr{F}_1, w\in\mathscr{F}_2)\ \begin{pmatrix} \widetilde{k}_1^{(j)}(v)=\widetilde{k}(v,j) \\ \varpi_1^{(j)}(v)=\varpi(v,j) \end{pmatrix} \text{ and } \begin{pmatrix} \widetilde{k}_2^{(d)}(w)=\widetilde{k}(d,w) \\ \varpi_2^{(d)}(w)=\varpi(d,w) \end{pmatrix}.$$

Lemma 5.5. *If $\widetilde{k}_\varpi$ is a hybrid sublattice (ideal, filter) of $\mathscr{F}_1\times\mathscr{F}_2$, then for all $j\in\mathscr{F}_2$ and $d\in\mathscr{F}_1$, the hybrid structures $(\widetilde{k}_\varpi)_1^{(j)}$ and $(\widetilde{k}_\varpi)_2^{(d)}$ are hybrid sublattices (ideals, filters) of $\mathscr{F}_1$ and $\mathscr{F}_2$.*

Proof. The proof is obvious. □

Theorem 5.6. *If $\widetilde{k}_\varpi$ is a hybrid ideal of $\mathscr{F}_1\times\mathscr{F}_2$, then for all $j\in\mathscr{F}_2$ and $d\in\mathscr{F}_1$,*

$$(\forall(v,w)\in\mathscr{F}_1\times\mathscr{F}_2)\ \begin{pmatrix} (\widetilde{k}_1^{(j)}\times\widetilde{k}_2^{(d)})(v,w)\subseteq\widetilde{k}(v,w)\subseteq pr_1(\widetilde{k})\times pr_2(\widetilde{k})(v,w) \\ (\varpi_1^{(j)}\times\varpi_2^{(d)})(v,w)\geq\varpi(v,w)\geq pr_1(\varpi)\times pr_2(\varpi)(v,w) \end{pmatrix}.$$

Proof. Let $v\in\mathscr{F}_1$ and $w\in\mathscr{F}_2$. Then $(\widetilde{k}_1^{(j)}\times\widetilde{k}_2^{(d)})(v,w)=\widetilde{k}_1^{(j)}(v)\cap\widetilde{k}_2^{(d)}(w)=\widetilde{k}(v,j)\cap\widetilde{k}(d,w)=\widetilde{k}(v+d,j+w)=\widetilde{k}(v,w)\cap\widetilde{k}(d,j)\subseteq\widetilde{k}(v,w)$. Thus $(\widetilde{k}_1^{(j)}\times\widetilde{k}_2^{(d)})(v,w)\subseteq\widetilde{k}(v,w)$.

Also $(\varpi_1^{(j)}\times\varpi_2^{(d)})(v,w)=\varpi_1^{(j)}(v)\vee\varpi_2^{(d)}(w)=\varpi(v,j)\vee\varpi(d,w)=\varpi(v+d,j+w)=\varpi(v,w)\vee\varpi(d,j)\geq\varpi(v,w)$. Thus $(\varpi_1^{(j)}\times\varpi_2^{(d)})(v,w)\geq\varpi(v,w)$.

Next, $\widetilde{k}(v,w)\subseteq\bigcup_{a\in\mathscr{F}}\widetilde{k}(v,a)=pr_1(\widetilde{k})(v)$ and $\varpi(v,w)\geq\bigwedge_{a\in\mathscr{F}}\varpi(v,a)=pr_1(\varpi)(v)$. By symmetry, $\widetilde{k}(v,w)\subseteq pr_2(\widetilde{k})(w)$ and $\varpi(v,w)\geq pr_2(\varpi)(w)$, and thus $\widetilde{k}(v,w)\subseteq pr_1(\widetilde{k})(v)\cap pr_2(\widetilde{k})(w)=pr_1(\widetilde{k})\times pr_2(\widetilde{k})(v,w)$ and $\varpi(v,w)\geq pr_1(\varpi)(v)\vee pr_2(\varpi)(w)=pr_1(\varpi)\times pr_2(\varpi)(v,w)$.

Hence $(\widetilde{k}_1^{(j)} \times \widetilde{k}_2^{(d)})(v,w) \subseteq \widetilde{k}(v,w) \subseteq pr_1(\widetilde{k}) \times pr_2(\widetilde{k})(v,w)$ and $(\varpi_1^{(j)} \times \varpi_2^{(d)})(v,w) \geq \varpi(v,w) \geq pr_1(\varpi) \times pr_2(\varpi)(v,w)$. □

Theorem 5.7. *Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two lattices with smallest element 0 and $\widetilde{k}_\varpi$ a hybrid ideal of $\mathscr{F}_1 \times \mathscr{F}_2$. Then the criteria listed below are equivalent:*

(i) $\widetilde{k}_\varpi$ is the product of a hybrid ideal of $\mathscr{F}_1$ and of a hybrid ideal of $\mathscr{F}_2$,

(ii) $\begin{pmatrix} \widetilde{k}_1^{(0)} \times \widetilde{k}_2^{(0)} = pr_1(\widetilde{k}) \times pr_2(\widetilde{k}) \\ \varpi_1^{(0)} \times \varpi_2^{(0)} = pr_1(\varpi) \times pr_2(\varpi) \end{pmatrix}$.

Proof. $(i) \Rightarrow (ii)$ The proof follows from Theorem 5.6.

$(ii) \Rightarrow (i)$ Let $\widetilde{k}_\varpi = (\widetilde{k}_\varpi)'_1 \otimes (\widetilde{k}_\varpi)'_2$, where $(\widetilde{k}_\varpi)'_i$ is a hybrid ideal of $\mathscr{F}_i$ $(i = 1,2)$. Clearly $\widetilde{k}'_i(r) \subseteq \widetilde{k}'_i(0)$ and $\varpi'_i(r) \geq \varpi'_i(0)$ $(i = 1,2)$. Then $\bigcup\limits_{r\in\mathscr{F}_i} \widetilde{k}'_i(r) = \widetilde{k}'_i(0)$ and $\bigwedge\limits_{r\in\mathscr{F}_i} \varpi'_i(r) = \varpi'_i(0)$ $(i = 1,2)$ and so $\widetilde{k}_1^{(0)}(r) = \widetilde{k}(r,0) = \widetilde{k}'_1(r) \cap \widetilde{k}'_2(0) = \widetilde{k}'_1(r) \cap \bigcup\limits_{t\in\mathscr{F}_2} \widetilde{k}'_2(t) = \bigcup\limits_{t\in\mathscr{F}_2} \widetilde{k}'_1(r) \cap \widetilde{k}'_2(t) = \bigcup\limits_{t\in\mathscr{F}_2} \widetilde{k}(r,t) = pr_1(\widetilde{k})(r)$, $\varpi_1^{(0)}(r) = \varpi(r,0) = \varpi'_1(r) \vee \varpi'_2(0) = \varpi'_1(r) \vee \bigwedge\limits_{t\in\mathscr{F}_2} \varpi'_2(t) = \bigwedge\limits_{t\in\mathscr{F}_2} \varpi'_1(r) \vee \varpi'_2(t) = \bigwedge\limits_{t\in\mathscr{F}_2} \varpi(r,t) = pr_1(\varpi)(r)$.

Thus $\widetilde{k}_1^{(0)} = pr_1(\widetilde{k})$ and $\varpi_1^{(0)} = pr_1(\varpi)$. Similarly, $\widetilde{k}_2^{(0)} = pr_2(\widetilde{k})$ and $\varpi_2^{(0)} = pr_2(\varpi)$. Hence $\widetilde{k}_1^{(0)} \times \widetilde{k}_2^{(0)} = pr_1(\widetilde{k}) \times pr_2(\widetilde{k})$ and $\varpi_1^{(0)} \times \varpi_2^{(0)} = pr_1(\varpi) \times pr_2(\varpi)$. □

6. Conclusion

In a distributive lattice $\mathscr{F}$, we examined the relations between hybrid congruences and hybrid ideals. The product of hybrid ideals was also introduced. In addition, we obtained the necessary and sufficient condition for a hybrid ideal on the direct sum of lattices to be representable as a direct sum of hybrid ideals on each lattice. In the future, we propose to investigate the numerous kinds of concepts in hybrid structures over hybrid prime ideals in a distributive lattice.

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