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Research Paper

ON THE GENUS OF ANNIHILATOR INTERSECTION GRAPH OF COMMUTATIVE RINGS

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Abstract. Let R be a commutative ring with unity and $A(R)$ be the set of annihilating-ideals of R. The annihilator intersection graph of R, represented by $AIG(R)$, is an undirected graph with $A(R)^*$ as the vertex set and $\mathfrak{M} \sim \mathfrak{N}$ is an edge of $AIG(R)$ if and only if $Ann(\mathfrak{M}\mathfrak{N}) \neq Ann(\mathfrak{M}) \cap Ann(\mathfrak{N})$, for distinct vertices $\mathfrak{M}$ and $\mathfrak{N}$ of $AIG(R)$. In this paper, we first defined finite commutative rings whose annihilator intersection graph is isomorphic to various well-known graphs, and then all finite commutative rings with a planar or toroidal annihilator intersection graph were characterized.

1. Introduction

Throughout this paper all rings are commutative with unit element such that $1 \neq 0$. For a commutative ring R, we use $\mathbb{I}(R)$ to denote the set of ideals of R and $\mathbb{I}(R)^* = \mathbb{I}(R) \setminus \{0\}$. An ideal $\mathfrak{M}$ of R is said to be *annihilator ideal* if there is a nonzero ideal $\mathfrak{N}$ of R such that

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$\mathfrak{M}\mathfrak{N} = (0)$. For $\mathfrak{M} \in \mathbb{I}(R)$, we define *annihilator* of $\mathfrak{M}$ as $Ann(\mathfrak{M}) = \{\mathfrak{N} \in \mathbb{I}(R) : \mathfrak{M}\mathfrak{N} = (0)\}$. We use $A(R)$ to denote the set of annihilator ideals of R and $A(R)^* = A(R) \setminus \{0\}$. We denote the set of zero-divisors, nilpotent elements, minimal prime ideals and unit elements of R by $Z(R)$, $Nil(R)$, $Min(R)$ and $U(R)$, respectively. For any undefined notation or terminology in ring theory, we refer the reader to [3].

A connected graph G is said to be a *tree* if it does not contain any cycle. A graph G is said to be *unicycle* if it contains unique cycle. A graph G is a *split graph* if the vertex set can be partitioned into a clique and an independent set. A graph G is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends. A subdivision of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths. A remarkably simple characterization of a planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph G is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$. An undirected graph G is said to be *outerplanar* if it can be embedded in the plane in such a way that all the vertices lies on the unbounded face of the drawing. The *genus* of a graph G, denoted by $\gamma(G)$, is the minimum integer k such that the graph can be drawn without crossing itself on a sphere with k handles (i.e., an oriented surface of genus k). Thus, a planar graph has genus 0, because it can be drawn on a sphere without self-crossing. For more details on graph theory, we refer the reader to [11, 12].

Beck [4] established the concept of the zero-divisor graph of a commutative ring in 1988, where he was primarily concerned in colorings. Beck proposed that $\chi(R) = \omega(R)$ for any commutative ring R in [4]. For some types of rings, such as reduced rings and principal ideal rings, he established the supposition. However, this is not the case in general. This was established in 1993, when Anderson and Naseer presented a convincing counter example (see Theorem 2.1 in [2]) that proved Beck's conjecture for general rings to be false. Anderson and Naseer continued their research into the colorings of a commutative ring. They take the vertex set as the ring elements and define an edge between the vertices a and b if and only if $ab = 0$. In [1], Anderson and Livingston introduced the zero-divisor graph of R, denoted by $\Gamma(R)$, with vertex set $Z(R)^*$ and for distinct $a, b \in Z(R)^*$, the vertices a and b are adjacent if and only if $ab = 0$.

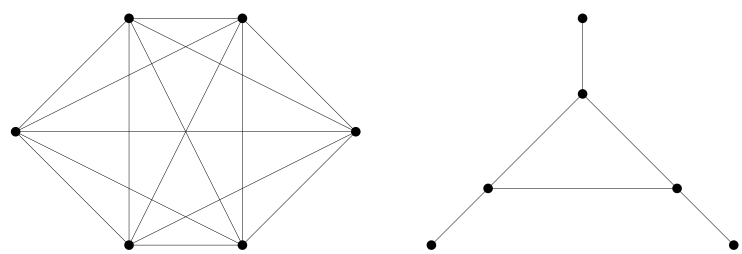
In 2011, Behboodi and Rakeei [5, 6] described a new graph, called it *annihilating-ideal graph* $AG(R)$ on R, with the vertex set $A(R)^*$ and two distinct vertices $\mathfrak{M}$ and $\mathfrak{N}$ are adjacent if and only if $\mathfrak{M}\mathfrak{N} = 0$ (see [7, 8, 9] for more details).

In [10], Vafaei et al. introduced and studied the *annihilator intersection graph* of R denoted by $AIG(R)$. It is an undirected graph with $A(R)^*$ as the vertex set and $\mathfrak{M} \sim \mathfrak{N}$ is an edge of $AIG(R)$ if and only if $Ann(\mathfrak{M}\mathfrak{N}) \neq Ann(\mathfrak{M}) \cap Ann(\mathfrak{N})$, for distinct vertices $\mathfrak{M}$ and $\mathfrak{N}$ of $AIG(R)$. In this paper, we first characterized the finite commutative rings whose annihilator intersection graph is a tree, a unicycle, a split graph or an outerplanar graph. Further, up to isomorphism, we classify the rings R whose annihilator intersection graph is planar or toroidal graph.

In the following examples, the annihilator intersection ideal graph of some commutative rings are given.

Example 1.1.
If $R = F_1 \times F_2$, where F_1 and F_2 are fields, then $AG(R) = AIG(R) = K_2$.

Example 1.2.
If $R = F_1 \times F_2 \times F_3$, where F_i is a field for each $i = 1, 2, 3$. Then $AIG(R)$ and $AG(R)$ are given in Fig. 1.



$AIG(F_1 \times F_2 \times F_3)$ $AG(F_1 \times F_2 \times F_3)$

Fig. 1.

The following observation proved by Vafaei et al. [10] is used frequently in this article.

Lemma 1.3. [10, Lemma 2.1] *Let R be a commutative ring and $\mathfrak{M}, \mathfrak{N} \in A(R)^*$. Then the following statements hold:*

(1) *If $\mathfrak{M} \sim \mathfrak{N}$ is not an edge of $AIG(R)$, then $Ann(\mathfrak{M}) = Ann(\mathfrak{N})$.*
(2) *If $\mathfrak{M} \sim \mathfrak{N}$ is an edge of $AG(R)$, then $\mathfrak{M} \sim \mathfrak{N}$ is an edge of $AIG(R)$.*
(3) *If $\mathfrak{M} \sim \mathfrak{N}$ is not an edge of $AIG(R)$, then there exists a vertex $\mathfrak{N}_1 \in A(R)^*$ such that $\mathfrak{M} \sim \mathfrak{N}_1 \sim \mathfrak{N}$ is a path in $AIG(R)$.*

Lemma 1.4. [10, Lemma 2.2] *Let R be a non-reduced ring. Then every nonzero nilpotent ideal of R is adjacent to all other vertices of $AIG(R)$. In particular, the induced subgraph by nilpotent ideals is a complete subgraph of $AIG(R)$.*

Theorem 1.5. *Let R be a local commutative ring. Then $AIG(R)$ is a complete graph.*

2. Annihilator intersection graph as some special type of graph

In this section, we characterized the finite commutative rings for which the annihilator intersection graph is isomorphic to some well-know graph such as a tree, a unicycle or a split graph.

Theorem 2.1. *Let R be a finite commutative ring. Then $AIG(R)$ is unicycle if and only if R is local with $|\mathbb{I}(R)^*| = 3$.*

Proof. Suppose $AIG(R)$ is a unicycle graph. Since R is finite, $R \cong R_1 \times R_2 \times \cdots \times R_n$, where R_j is local for each $1 \leq j \leq n$.

Suppose that $n \geq 3$. Then $\mathfrak{M}_1 \sim \mathfrak{M}_2 \sim \mathfrak{M}_3 \sim \mathfrak{M}_1$ and $\mathfrak{N}_1 \sim \mathfrak{N}_2 \sim \mathfrak{N}_3 \sim \mathfrak{N}_1$, where $\mathfrak{M}_1 = R_1 \times (0) \times (0) \times \cdots \times (0)$, $\mathfrak{M}_2 = (0) \times R_2 \times (0) \times \cdots \times (0)$, $\mathfrak{M}_3 = (0) \times (0) \times R_3 \times (0) \times \cdots \times (0)$, $\mathfrak{N}_1 = R_1 \times R_2 \times (0) \times \cdots \times (0)$, $\mathfrak{N}_2 = R_1 \times (0) \times R_3 \times (0) \times \cdots \times (0)$, $\mathfrak{N}_3 = (0) \times R_2 \times R_3 \times (0) \times \cdots \times (0)$, are two distinct cycles in $AIG(R)$, a contradiction to our assumption that $AIG(R)$ is unicycle. Hence $n \leq 2$.

Suppose $n = 2$. If R_1 and R_2 both are fields, then $AIG(R) \cong K_2$, a contradiction. Thus, R_j is not a field for at least one $j = 1, 2$. Without compromising generality, we can suppose that R_1 is not a field with a maximum ideal $\mathrm{Im}_1 \neq (0)$. Consider $\mathfrak{K}_1 = R_1 \times (0)$, $\mathfrak{K}_2 = (0) \times R_2$, $\mathfrak{K}_3 = \mathrm{Im}_1 \times (0)$ and $\mathfrak{K}_4 = \mathrm{Im}_1 \times R_2$. It is easy to see that $\mathfrak{K}_1 \sim \mathfrak{K}_2 \sim \mathfrak{K}_3 \sim \mathfrak{K}_1$ as well as $\mathfrak{K}_1 \sim \mathfrak{K}_3 \sim \mathfrak{K}_4 \sim \mathfrak{K}_1$ are two distinct cycles in $AIG(R)$, a contradiction to our assumption that $AIG(R)$ is unicycle. Hence $n = 1$, which implies that R is a local ring. Thus, $AIG(R)$ is a complete graph by Theorem 1.5. Since $AIG(R)$ is unicycle, $|\mathbb{I}(R)^*| = 3$. $\square$

Theorem 2.2. *Let R be a finite commutative ring. Then $AIG(R)$ is a tree if and only if either R is local with $|\mathbb{I}(R)^*| \leq 2$ or $R \cong F_1 \times F_2$, where F_1 and F_2 are fields.*

Proof. Suppose $AIG(R)$ is a tree. Since R is finite, $R \cong R_1 \times R_2 \times \cdots \times R_n$, where R_j is local for each $1 \leq j \leq n$.

Suppose $n \geq 3$. Consider the nonzero proper ideals $\mathfrak{M}_1 = R_1 \times (0) \times (0) \times \cdots \times (0)$, $\mathfrak{M}_2 = (0) \times R_2 \times (0) \times \cdots \times (0)$ and $\mathfrak{M}_3 = (0) \times (0) \times R_3 \times (0) \times \cdots \times (0)$ in R. Since $Ann(\mathfrak{M}_j\mathfrak{M}_{\mathfrak{k}}) \neq Ann(\mathfrak{M}_j) \cap Ann(\mathfrak{M}_k)$ for each j, k. Then $\mathfrak{M}_1 \sim \mathfrak{M}_2 \sim \mathfrak{M}_3 \sim \mathfrak{M}_1$ is a cycle in $AIG(R)$, which contradict the assumption that $AIG(R)$ is tree. Hence $n \leq 2$.

First, suppose that $n = 2$. Assume that R_1 is not a field with maximal ideal $\mathrm{Im}_1 \neq (0)$. Consider the nonzero proper ideals $\mathfrak{N}_1 = R_1 \times (0)$, $\mathfrak{N}_2 = \mathrm{Im}_1 \times (0)$ and $\mathfrak{N}_3 = (0) \times R_2$ in R. One can see that $\mathfrak{N}_1 \sim \mathfrak{N}_2 \sim \mathfrak{N}_3 \sim \mathfrak{N}_1$ is a cycle in $AIG(R)$, which contradict the assumption that $AIG(R)$ is tree. Hence R_1 is a field. Similarly, one can prove that R_2 is a field.

Now, suppose $n = 1$. Then R is local and thus, $AIG(R)$ is a complete graph by Theorem 1.5. Since $AIG(R)$ is tree, $|\mathbb{I}(R)^*| \leq 2$. $\square$

Theorem 2.3. [11] *Let G be a connected graph. Then G is a split graph if and only if G contains no induced subgraph isomorphic to $2K_2$, C_4 or C_5.*

Theorem 2.4. *Let R be a finite commutative ring. Then $AIG(R)$ is a split graph if and only if either R is local with $|\mathbb{I}(R)^*| \leq 3$ or $R \cong F_1 \times F_2$, where F_1 and F_2 are fields.*

Proof. Suppose that $AIG(R)$ is a split graph. Since R is finite, $R \cong R_1 \times R_2 \times \cdots \times R_n$, where R_j is local for each $1 \leq j \leq n$.

Suppose $n \geq 3$. Consider the nonzero proper ideals $\mathfrak{M}_1 = R_1 \times (0) \times (0) \times \cdots \times (0)$, $\mathfrak{M}_2 = (0) \times R_2 \times (0) \times \cdots \times (0)$, $\mathfrak{M}_3 = (0) \times (0) \times R_3 \times (0) \times \cdots \times (0)$ and $\mathfrak{M}_4 = R_1 \times R_2 \times (0) \times \cdots \times (0)$. Since $Ann(\mathfrak{M}_1) \neq Ann(\mathfrak{M}_2), Ann(\mathfrak{M}_2) \neq Ann(\mathfrak{M}_3), Ann(\mathfrak{M}_3) \neq Ann(\mathfrak{M}_4)$ and $Ann(\mathfrak{M}_4) \neq Ann(\mathfrak{M}_1)$, then $\mathfrak{M}_1 \sim \mathfrak{M}_2 \sim \mathfrak{M}_3 \sim \mathfrak{M}_4 \sim \mathfrak{M}_1$ is C_4 in $AIG(R)$, which contradict the assumption that $AIG(R)$ is a split graph. Hence $n \leq 2$.

First, suppose $n = 2$. Assume that R_2 is not a field with maximal ideal $\mathrm{Im}_2 \neq (0)$. Then $\mathfrak{N}_1 \sim \mathfrak{N}_2 \sim \mathfrak{N}_3 \sim \mathfrak{N}_4 \sim \mathfrak{N}_1$, where $\mathfrak{N}_1 = (0) \times R_2$, $\mathfrak{N}_2 = (0) \times \mathrm{Im}_2$, $\mathfrak{N}_3 = R_1 \times (0)$, $\mathfrak{N}_4 = R_1 \times \mathrm{Im}_2$, is C_4 in $AIG(R)$, which contradict the assumption that $AIG(R)$ is a split graph. Hence R_2 is a field. Similarly, one can prove that R_1 is a field.

Now, suppose $n = 1$. Then R is a local ring and thus $AIG(R)$ is complete by Theorem 1.5. Since $AIG(R)$ is split graph, $|\mathbb{I}(R)^*| \leq 3$. □

3. Planarity of annihilator intersection graph

In this section, we classify all the finite commutative rings for which the annihilator intersection graph is a planar graph or an outerplanar graph.

Theorem 3.1. [12] (*Kuratowski's Theorem*) *A graph G is planar if and only if it does not contain subdivision of K_5 or $K_{3,3}$.*

Theorem 3.2. *Let R be a finite local commutative ring. Then $AIG(R)$ is a planar graph if and only if $|\mathbb{I}(R)^*| \leq 4$.*

Proof. Since R is local, $AIG(R)$ is complete by Theorem 1.5. Hence the result follows from Theorem 3.1. □

We can now classify finite reduced non-local rings whose annihilator intersection graph is a planar graph.

Theorem 3.3. *Let R be a finite reduced ring. Then $AIG(R)$ is a planar graph if and only if R is the direct product of two fields.*

Proof. Suppose $AIG(R)$ is a planar graph. Since R is finite reduced ring, $R = F_1 \times \cdots \times F_n$, where F_j is a field for each j and $n \geq 2$.

Assume that $n \geq 3$. Consider the nonzero proper ideals $\mathfrak{M}_1 = F_1 \times (0) \times (0) \times \cdots \times (0)$, $\mathfrak{M}_2 = (0) \times F_2 \times (0) \times \cdots \times (0)$, $\mathfrak{M}_3 = (0) \times (0) \times F_3 \times (0) \times \cdots \times (0)$, $\mathfrak{N}_1 = F_1 \times F_2 \times (0) \times \cdots \times (0)$, $\mathfrak{N}_2 = (0) \times F_2 \times F_3 \times \cdots \times (0)$ and $\mathfrak{N}_3 = F_1 \times (0) \times F_3 \times \cdots \times (0)$ in R. Since $Ann(\mathfrak{M}_j\mathfrak{N}_k) \neq Ann(\mathfrak{M}_j) \cap Ann(\mathfrak{N}_k)$ for each j, k, then $AIG(R)$ contains a copy of $K_{3,3}$, which contradict our assumption. $R \cong F_1 \times F_2$, where F_1 and F_2 are fields.

Conversely, if $R \cong F_1 \times F_2$, where F_1 and F_2 are fields, then $AIG(R) \cong K_2$ is planar. □

Theorem 3.4. *Let $R = R_1 \times R_2$ be a finite commutative ring, where (R_j, Im_j) is local ring with $\mathrm{Im}_j \neq (0)$ for each $j = 1, 2$. Then $AIG(R)$ is not a planar graph.*

Proof. Consider the nonzero proper ideals $\mathfrak{M}_1 = \mathrm{Im}_1 \times (0), \mathfrak{M}_2 = (0) \times \mathrm{Im}_2, \mathfrak{M}_3 = \mathrm{Im}_1 \times \mathrm{Im}_2, \mathfrak{N}_1 = R_1 \times (0), \mathfrak{N}_2 = (0) \times R_2$ and $\mathfrak{N}_3 = R_1 \times \mathrm{Im}_2$ in R. Since $Ann(\mathfrak{M}_j\mathfrak{N}_k) \neq Ann(\mathfrak{M}_j) \cap Ann(\mathfrak{N}_k)$, then $AIG(R)$ contains a copy of $K_{3,3}$. Hence by Theorem 3.1, $AIG(R)$ is not planar. □

Finally, we classify finite non-reduced non-local rings with a planar annihilator intersection graph.

Theorem 3.5. *Let $R \cong F_1 \times F_2 \times \cdots \times F_n \times R_1 \times R_2 \times \cdots \times R_m$ be a finite commutative ring, where F_j is a field for each j, (R_k, Im_k) is a local ring with $\mathrm{Im}_k \neq (0)$ for each k and $n, m \geq 1$. Then $AIG(R)$ is a planar graph if and only if $n = m = 1$ and R_1 has unique nonzero proper ideal.*

Proof. Suppose $R \cong F_1 \times R_1$, where F_1 is a field and R_1 is a local ring with unique nonzero proper ideal Im_1. Then the vertex set of $AIG(R)$ is given by the set $\{F_1 \times (0), F_1 \times \mathrm{Im}_1, (0) \times R_1, (0) \times \mathrm{Im}_1\}$ and graph $AIG(R)$ is illustrated in Fig. 2.

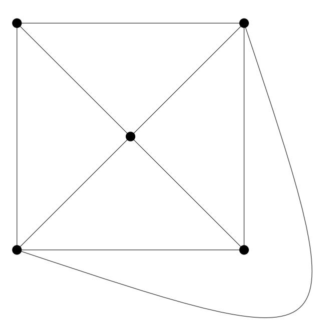


Fig. 2. planar embadding of $AIG(F_1 \times R_1)$,
where R_1 has unique nonzero proper ideal.

Conversely, suppose that $AIG(R)$ is a planar graph. If $m \geq 2$, then by Theorem 3.4, $AIG(R)$ is non-planar, a contradiction. Hence $m = 1$.

Suppose $n \geq 2$. Consider $\mathfrak{M}_1 = F_1 \times (0) \times (0) \times \cdots \times (0), \mathfrak{M}_2 = (0) \times F_2 \times (0) \times \cdots \times (0), \mathfrak{M}_3 = (0) \times (0) \times \cdots \times (0) \times R_1, \mathfrak{N}_1 = F_1 \times (0) \times \cdots \times (0) \times R_1, \mathfrak{N}_2 = (0) \times F_2 \times \cdots \times (0) \times R_1$ and $\mathfrak{N}_3 = F_1 \times F_2 \times (0) \times \cdots \times (0)$. Since $Ann(\mathfrak{M}_j\mathfrak{N}_k) \neq Ann(\mathfrak{M}_j) \cap Ann(\mathfrak{N}_k)$ for each j, k. Then the set $\{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3, \mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{N}_3\}$ induces a subdivision of $K_{3,3}$ in $AIG(R)$, which contradict our assumption. Hence $n = 1$ and so $R \cong F_1 \times R_1$.

Suppose that $\mathfrak{m}$ is a nonzero proper ideal of R_1 with $\mathfrak{m} \neq \mathrm{Im}_1$. One can see that the induced subgraph by the set $\{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3, \mathfrak{M}_3, \mathfrak{M}_5\}$, where $\mathfrak{M}_1 = F_1 \times (0), \mathfrak{M}_2 = (0) \times R_1, \mathfrak{M}_3 = (0) \times \mathrm{Im}_1, \mathfrak{M}_4 = (0) \times \mathfrak{m}, \mathfrak{M}_5 = F_1 \times \mathrm{Im}_1$, contains K_5 as a subgraph of $AIG(R)$, which contradict our assumption. Hence R_1 has exactly one nonzero proper ideal. □

Theorem 3.6. [12] *A graph G is outerplanar if and only if it does not contains a subdivision of K_4 or $K_{2,3}$.*

Theorem 3.7. *Let R be a finite commutative ring. Then $AIG(R)$ is an outerplanar graph if and only if either R is local with $|\mathbb{I}(R)^*| \leq 3$ or $R \cong F_1 \times F_2$, where F_1 and F_2 are fields.*

Proof. Suppose $AIG(R)$ is outerplanar. Since R is finite, $R \cong R_1 \times R_2 \times \cdots \times R_n$, where (R_j, Im_j) is local for each $1 \leq j \leq n$ and $n \geq 1$.

Suppose that $n \geq 3$. Consider the nonzero proper ideals $\mathfrak{M}_1 = R_1 \times (0) \times (0) \times \cdots \times (0), \mathfrak{M}_2 = (0) \times R_2 \times (0) \times \cdots \times (0), \mathfrak{M}_3 = (0) \times (0) \times R_3 \times (0) \times \cdots \times (0)$ and $\mathfrak{M}_4 = R_1 \times R_2 \times (0) \times \cdots \times (0)$ in R. Since $Ann(\mathfrak{M}_j\mathfrak{M}_k) \neq Ann(\mathfrak{M}_j) \cap Ann(\mathfrak{M}_k)$ for each j, k. Then the subgraph induced by the set $\{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3, \mathfrak{M}_4\}$ is K_4, which is a contradiction by Theorem 3.6. Hence $n \leq 2$.

First, suppose that $n = 2$. If $\mathrm{Im}_1 \neq (0)$, then the set $\{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3, \mathfrak{M}_4\}$, where $\mathfrak{M}_1 = R_1 \times (0), \mathfrak{M}_2 = (0) \times R_2, \mathfrak{M}_3 = \mathrm{Im}_1 \times (0), \mathfrak{M}_4 = \mathrm{Im}_1 \times R_2$, induces a subdivision of K_4, which is a contradiction by Theorem 3.6. Hence $\mathrm{Im}_j = (0)$ for all $j = 1, 2$ and so each R_j is a field.

Now, suppose that $n = 1$. Then R is a local ring. Thus, $AIG(R)$ is complete by Theorem 1.5. Since $AIG(R)$ is outerplanar, $|\mathbb{I}(R)^*| \leq 3$. □

4. Annihilator intersection graph with genus one

In this section, we classify all finite commutative rings for which annihilator intersection graph is a toroidal graph.

The following results deal with genus features of complete graph and complete bipartite graphs, which help us to characterize the rings with genus one annihilator intersection graph.

Theorem 4.1. [12] *If $m \geq 3$, then*

$$\gamma(K_m) = \left\lceil \frac{(m-3)(m-4)}{12} \right\rceil.$$

Theorem 4.2. [12] *If $m, n \geq 2$, then*

$$\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil .$$

Let us begin by classifying the finite commutative local rings whose annihilator intersection graph is a toroidal graph.

Theorem 4.3. *Let R be a finite local commutative ring. Then $\gamma(AIG(R)) = 1$ if and only if $5 \leq |\mathbb{I}(R)^*| \leq 7$.*

Proof. Since R is local, $AIG(R)$ is complete by Theorem 1.5. Hence the result follows from Theorem 4.1. □

We can now characterize the finite commutative reduced non-local ring whose annihilator intersection graph is toroidal graph.

Theorem 4.4. *Let R be a finite commutative reduced non-local ring. Then $\gamma(AIG(R)) = 1$ if and only if R is the direct product of three fields.*

Proof. Suppose $\gamma(AIG(R)) = 1$. Since R is a finite reduced ring, $R = F_1 \times F_2 \times \cdots \times F_n$, where F_j is a field for each j and $n \geq 2$. Assume that $n \geq 4$. Consider the nonzero proper ideals $\mathfrak{N}_1 = F_1 \times (0) \times (0) \times \cdots \times (0)$, $\mathfrak{N}_2 = (0) \times F_2 \times (0) \times \cdots \times (0)$, $\mathfrak{N}_3 = (0) \times (0) \times F_3 \times (0) \times \cdots \times (0)$, $\mathfrak{N}_4 = (0) \times (0) \times (0) \times F_4 \times \cdots \times (0)$, $\mathfrak{K}_1 = F_1 \times F_2 \times (0) \times \cdots \times (0)$, $\mathfrak{K}_2 = F_1 \times (0) \times F_3 \times (0) \times \cdots \times (0)$, $\mathfrak{K}_3 = F_1 \times (0) \times (0) \times F_4 \times (0) \times \cdots \times (0)$, $\mathfrak{K}_4 = (0) \times F_2 \times F_3 \times (0) \cdots \times (0)$, $\mathfrak{K}_5 = (0) \times F_2 \times (0) \times F_4 \times (0) \times \cdots \times (0)$ in R. Since $Ann(\mathfrak{N}_j \mathfrak{K}_l) \neq Ann(\mathfrak{N}_j) \cap Ann(\mathfrak{K}_l)$ for each j, l, then $AIG(R)$ contains $K_{4,5}$ as a induced subgraph, a contradiction. Hence $n = 2$ or 3. If $n = 2$, then by Theorem 3.3, $AIG(R)$ is planar and so $\gamma(AIG(R)) = 0$, again a contradiction. Hence $n = 3$.

Conversely, suppose $n = 3$. The vertex set of $AIG(R)$ is given by $\{\mathfrak{M}_1 = F_1 \times (0) \times (0),\ \mathfrak{M}_2 = (0) \times F_2 \times (0),\ \mathfrak{M}_3 = (0) \times (0) \times F_3,\ \mathfrak{M}_4 = F_1 \times F_2 \times (0),\ \mathfrak{M}_5 = F_1 \times (0) \times F_3,\ \mathfrak{M}_6 = (0) \times F_2 \times F_3\}$ and the graph $AIG(R)$ is illustrated in the following Fig. 3. □

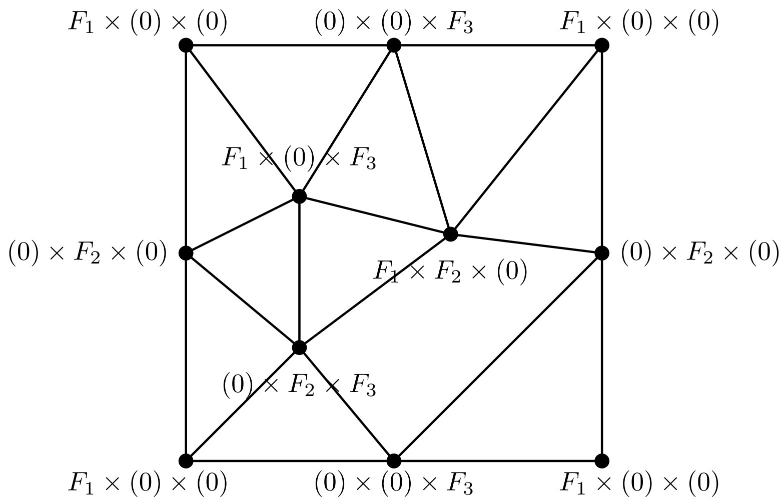


Fig. 3. toroidal embadding of $AIG(F_1 \times F_2 \times F_3)$.

Now, we classify finite commutative non-reduced non-local rings for which annihilator intersection graph is a toroidal graph.

Theorem 4.5. *Let $R \cong R_1 \times R_2 \times \cdots \times R_n$ be a finite commutative ring, where (R_i, Im_i) is local ring with $\mathrm{Im}_i \neq (0)$ for each $1 \leq i \leq n$ and $n \geq 2$. Then $\gamma(AIG(R)) = 1$ if and only if $n = 2$ and R_i has unique nonzero proper ideal for each $i = 1, 2$.*

Proof. Suppose that $\gamma(AIG(R)) = 1$. Suppose $n \geq 3$. Consider the nonzero proper ideals $\mathfrak{M}_1 = \mathrm{Im}_1 \times (0) \times (0) \times (0) \times \cdots \times (0)$, $\mathfrak{M}_2 = (0) \times \mathrm{Im}_2 \times (0) \times \cdots \times (0)$, $\mathfrak{M}_3 = (0) \times (0) \times \mathrm{Im}_3 \times (0) \times \cdots \times (0)$, $\mathfrak{M}_4 = \mathrm{Im}_1 \times \mathrm{Im}_2 \times (0) \times \cdots \times (0)$, $\mathfrak{N}_1 = R_1 \times (0) \times (0) \times \cdots \times (0)$, $\mathfrak{N}_2 = (0) \times R_2 \times (0) \times \cdots \times (0)$, $\mathfrak{N}_3 = (0) \times (0) \times R_3 \times (0) \times \cdots \times (0)$, $\mathfrak{N}_4 = R_1 \times R_2 \times (0) \times \cdots \times (0)$, $\mathfrak{N}_5 = R_1 \times (0) \times R_3 \times (0) \cdots \times (0)$ in R. Since $Ann(\mathfrak{M}_j\mathfrak{N}_k) \neq Ann(\mathfrak{M}_j) \cap Ann(\mathfrak{N}_k)$ for each j, k, then $AIG(R)$ contains $K_{4,5}$ as a induced subgraph, a contradiction. Hence $n = 2$.

Let n_j be the nipotency index of R_j for $j = 1, 2$. Suppose that $n_1 \geq 3$ and $n_2 \geq 3$. Consider the nonzero proper ideals $\mathfrak{K}_1 = \mathrm{Im}_1 \times (0)$, $\mathfrak{K}_2 = (0) \times \mathrm{Im}_2$, $\mathfrak{K}_3 = \mathrm{Im}_1 \times \mathrm{Im}_2$, $\mathfrak{K}_4 = \mathrm{Im}_1^{n_1-1}$, $\mathfrak{L}_1 = R_1 \times (0)$, $\mathfrak{L}_2 = (0) \times R_2$, $\mathfrak{L}_3 = \mathrm{Im}_1 \times R_2$, $\mathfrak{L}_4 = R_1 \times \mathrm{Im}_2$, $\mathfrak{L}_5 = \mathrm{Im}_1^{n_1-1} \times R_2$ in R. Since $Ann(\mathfrak{K}_j\mathfrak{L}_{\mathfrak{k}}) \neq Ann(\mathfrak{K}_j) \cap Ann(\mathfrak{L}_k)$ for each j, k, then $K_{4,5}$ is a induced subgraph of $AIG(R)$, a contradiction. Hence $n_1 = 2$ or $n_2 = 2$. Assume, without sacrificing generality, that $n_2 = 2$.

Suppose $n_1 \geq 3$. Consider the nonzero proper ideals $\mathfrak{P}_1 = \mathrm{Im}_1 \times (0)$, $\mathfrak{P}_2 = (0) \times \mathrm{Im}_1$, $\mathfrak{P}_3 = \mathrm{Im}_1 \times \mathrm{Im}_2$, $\mathfrak{P}_4 = \mathrm{Im}_1^{n_1-1} \times (0)$, $\mathfrak{S}_1 = R_1 \times (0)$, $\mathfrak{S}_2 = (0) \times R_2$, $\mathfrak{S}_3 = \mathrm{Im}_1 \times R_2$, $\mathfrak{S}_4 = R_1 \times \mathrm{Im}_2$, $\mathfrak{S}_5 = \mathrm{Im}_1^{n_1-1} \times R_2$ in R. Since $Ann(\mathfrak{P}_j\mathfrak{S}_k) \neq Ann(\mathfrak{P}_j) \cap Ann(\mathfrak{S}_k)$ for each j, k, $K_{4,5}$ is a induced subgraph of $AIG(R)$, a contradiction. Hence $n_1 = 2$.

Let $\mathfrak{m}$ be a nonzero proper ideal of R_1 such that $\mathfrak{m} \neq \mathrm{Im}_1$. Consider the nonzero proper ideals $\mathfrak{B}_1 = \mathrm{Im}_1 \times (0)$, $\mathfrak{B}_2 = (0) \times \mathrm{Im}_2$, $\mathfrak{B}_3 = \mathfrak{m} \times (0)$, $\mathfrak{B}_4 = \mathrm{Im}_1 \times \mathrm{Im}_2$, $\mathfrak{D}_1 = R_1 \times (0)$, $\mathfrak{D}_2 = (0) \times R_2$, $\mathfrak{D}_3 = \mathrm{Im}_1 \times R_2$, $\mathfrak{D}_4 = R_1 \times \mathrm{Im}_2$, $\mathfrak{D}_5 = \mathfrak{m} \times R_2$ in R. Since $Ann(\mathfrak{B}_j\mathfrak{D}_k) \neq Ann(\mathfrak{B}_j) \cap Ann(\mathfrak{B}_k)$

for each j, k, then $K_{4,5}$ is a induced subgraph of $AIG(R)$, a contradiction. Hence R_1 has unique nonzero proper ideals which is Im_1.
Similarly, we can show that R_2 has unique nonzero proper ideals which is Im_2.

Conversely, suppose $R \cong R_1 \times R_2$, where Im_1 and Im_2 are the only nonzero proper ideals of R_1 and R_2 respectively. Then the vertex set of $AIG(R)$ is $\{R_1 \times (0), (0) \times R_2, \mathrm{Im}_1 \times (0), (0) \times \mathrm{Im}_2, \mathrm{Im}_1 \times \mathrm{Im}_2, R_1 \times \mathrm{Im}_2, \mathrm{Im}_1 \times R_2\}$ and the graph $AIG(R)$ is illustrated in the following Fig. 4. □

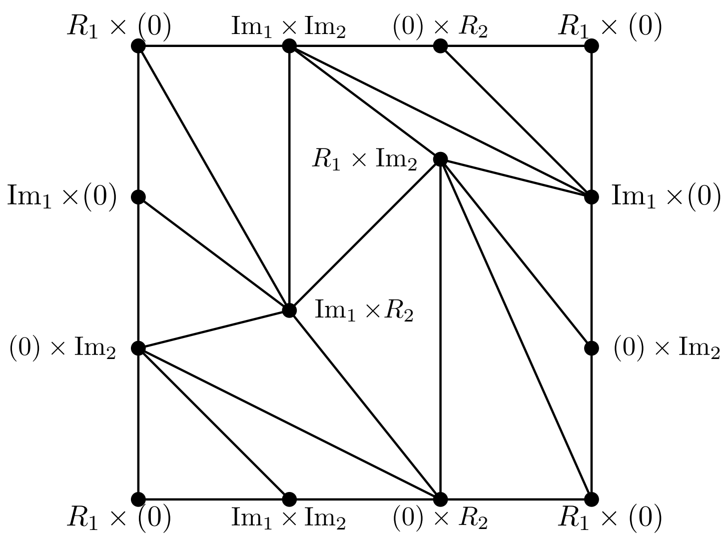


Fig. 4. toroidal embadding of $AIG(R_1 \times R_2)$.

Theorem 4.6. *Let $R \cong F_1 \times F_2 \times \cdots \times F_n \times R_1 \times R_2 \times \cdots \times R_m$ be a finite commutative ring, where each F_i is a field, (R_j, Im_j) is a local ring with $\mathrm{Im}_j \neq 0$ for each j and $n, m \geq 1$. Then $\gamma(AIG(R)) = 1$ if and only if $n = m = 1$ and Im_1, Im_1^2 are only ideals of R_1 and nilpotency index of Im_1 is 3.*

Proof. Assume that $\gamma(AIG(R)) = 1$. Suppose $n \geq 2$. Consider the nonzero proper ideals $\mathfrak{M}_1 = F_1 \times (0) \times (0) \times \cdots \times (0)$, $\mathfrak{M}_2 = (0) \times F_2 \times (0) \times \cdots \times (0)$, $\mathfrak{M}_3 = F_1 \times F_2 \times (0) \times \cdots \times (0)$, $\mathfrak{M}_4 = (0) \times (0) \times \cdots \times (0) \times \mathrm{Im}_1 \times (0) \times \cdots \times (0)$, $\mathfrak{N}_1 = (0) \times (0) \times \cdots \times (0) \times R_1 \times (0) \times \cdots \times (0)$, $\mathfrak{N}_2 = F_1 \times (0) \times \cdots \times (0) \times \mathrm{Im}_1 \times (0) \times \cdots \times (0)$, $\mathfrak{N}_3 = (0) \times F_2 \times (0) \times \cdots \times (0) \times \mathrm{Im}_1 \times (0) \times \cdots \times (0)$, $\mathfrak{N}_4 = F_1 \times (0) \times \cdots \times (0) \times R_1 \times (0) \times \cdots \times (0)$, $\mathfrak{N}_5 = (0) \times F_2 \times (0) \times \cdots \times (0) \times R_1 \times (0) \times \cdots \times (0)$ in R. Since $Ann(\mathfrak{M}_j \mathfrak{N}_k) \neq Ann(\mathfrak{M}_j) \cap Ann(\mathfrak{N}_k)$ for each j, k, then $AIG(R)$ contains $K_{4,5}$ as a induced subgraph, a contradiction. Hence $n = 1$.

Suppose $m \geq 3$. Consider the nonzero proper ideals $\mathfrak{K}_1 = (0) \times \mathrm{Im}_1 \times (0) \times \cdots \times (0)$, $\mathfrak{K}_2 = (0) \times (0) \times \mathrm{Im}_2 \times (0) \times \cdots \times (0)$, $\mathfrak{K}_3 = (0) \times \mathrm{Im}_1 \times \mathrm{Im}_2 \times (0) \times \cdots \times (0)$, $\mathfrak{K}_4 = F_1 \times (0) \times \cdots \times (0)$, $\mathfrak{L}_1 = (0) \times R_1 \times (0) \times \cdots \times (0)$, $\mathfrak{L}_2 = (0) \times (0) \times R_2 \times (0) \times \cdots \times (0)$, $\mathfrak{L}_3 = (0) \times R_1 \times R_2 \times (0) \cdots \times (0)$, $\mathfrak{L}_4 = (0) \times \mathrm{Im}_1 \times R_2 \times (0) \times \cdots \times (0)$, $\mathfrak{L}_5 = (0) \times R_1 \times \mathrm{Im}_2 \times (0) \times \cdots \times (0)$ in R. Since

$Ann(\mathfrak{K}_j\mathfrak{L}_k) \neq Ann(\mathfrak{K}_j) \cap Ann(\mathfrak{L}_k)$ for each j, k, then $AIG(R)$ contains $K_{4,5}$ as a induced subgraph, a contradiction. Hence $m = 1$.

Let n_1 be the nilpotency index of m_1. Suppose $n_1 \geq 4$. Consider the nonzero proper ideals $\mathfrak{P}_1 = F_1 \times (0)$, $\mathfrak{P}_2 = (0) \times \mathrm{Im}_1$, $\mathfrak{P}_3 = (0) \times \mathrm{Im}_1^{n_1-1}$, $\mathfrak{P}_4 = (0) \times \mathrm{Im}_1^{n_1-2}$, $\mathfrak{P}_5 = (0) \times R_1$, $\mathfrak{P}_6 = F_1 \times \mathrm{Im}_1$, $\mathfrak{P}_7 = F_1 \times \mathrm{Im}_1^{n_1-1}$, $\mathfrak{P}_8 = F_1 \times \mathrm{Im}_1^{n_1-2}$ in R. Since $Ann(\mathfrak{P}_j\mathfrak{P}_k) \neq Ann(\mathfrak{P}_j) \cap Ann(\mathfrak{P}_k)$ for each j, k, then $K_{4,5}$ is a induced subgraph of $AIG(R)$, a contradiction. Hence $n_1 = 3$.

Let $\mathfrak{m}$ be a nonzero proper ideal of R_1 such that $\mathfrak{m} \neq \mathrm{Im}_1, \mathrm{Im}_1^2$. Consider the nonzero proper ideals $\mathfrak{S}_1 = F_1 \times (0)$, $\mathfrak{S}_2 = (0) \times R_2$, $\mathfrak{S}_3 = (0) \times \mathrm{Im}_1$, $\mathfrak{S}_4 = (0) \times \mathrm{Im}_1^2$, $\mathfrak{S}_5 = (0) \times \mathfrak{m}$, $\mathfrak{S}_6 = F_1 \times \mathrm{Im}_1$, $\mathfrak{S}_7 = F_1 \times \mathrm{Im}_1^2$, $\mathfrak{S}_8 = F_1 \times \mathfrak{m}$ in R. Since $Ann(\mathfrak{S}_j\mathfrak{S}_k) \neq Ann(\mathfrak{S}_j) \cap Ann(\mathfrak{S}_k)$ for each j, k, then $K_{4,5}$ is a induced subgraph of $AIG(R)$, a contradiction. Hence Im_1 and Im_1^2 are the only nonzero proper ideals of R_1.

Conversely, suppose $R \cong F_1 \times R_1$ and the only nonzero proper ideals of R_1 are Im_1 and Im_1^2. Then the vertex set of $AIG(R)$ is given by the set $\{F_1 \times (0), (0) \times \mathrm{Im}_1, (0) \times R_1, (0) \times \mathrm{Im}_1^2, F_1 \times \mathrm{Im}_1, F_1 \times \mathrm{Im}_1^2\}$ and the graph $AIG(R)$ is illustrated in the following Fig. 5. □

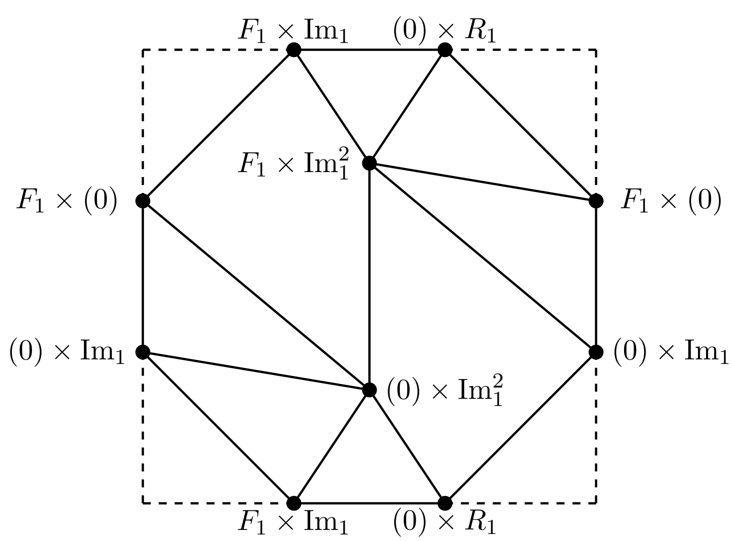


Fig. 5. toroidal embadding of $AIG(F_1 \times R_1)$.

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