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LOCAL AUTOMORPHISMS OF n-DIMENSIONAL NATURALLY GRADED QUASI-FILIFORM LEIBNIZ ALGEBRA OF TYPE I

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Abstract. The notions of a local automorphism for Lie algebras are defined as similar to the associative case. Every automorphism of a Lie algebra $\mathcal{L}$ is a local automorphism. For a given Lie algebra $\mathcal{L}$, the main problem concerning these notions is to prove that they automatically become an automorphism or to give examples of local automorphisms of $\mathcal{L}$, which are not automorphisms. In this paper, we study local automorphisms on quasi-filiform Leibniz algebras. It is proved that quasi-filiform Leibniz algebras of type I, as a rule, admit local automorphisms which are not automorphisms.

1. Introduction

In recent years non-associative analogues of classical constructions become of interest in connection with their applications in many branches of mathematics and physics. The notions

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of local and 2-local derivations (automorphisms) are also popular for some non-associative algebras such as Lie and Leibniz algebras.

R.Kadison [17] introduced the concept of a local derivation and proved that each continuous local derivation from a von Neumann algebra into its dual Banach bimodule is a derivation. B. Johnson [16] extended the above result by proving that every local derivation from a C*-algebra into its Banach bimodule is a derivation. In particular, Johnson gave an automatic continuity result by proving that local derivations of a C*-algebra A into a Banach A-bimodule X are continuous even if not assumed a priori to be so (cf. [16, Theorem 7.5]). Based on these results, many authors have studied local derivations on operator algebras.

A similar notion, which characterizes non-linear generalizations of automorphisms, was introduced by P.Šemrl in [21] as 2-local automorphisms. He described such maps on the algebra $B(H)$ of all bounded linear operators on an infinite dimensional separable Hilbert space H. After P.Šemrl's work, numerous new results related to the description of local and 2-local automorphisms of some varieties have been appeared (see, for example, [6, 9, 12, 13, 19, 20]).

The first results concerning to local and 2-local derivations and automorphisms on finite-dimensional Lie algebras over algebraically closed field of zero characteristic were obtained in [6, 7, 10] and [12]. Namely, in [10] it is proved that every 2-local derivation on a semi-simple Lie algebra $\mathcal{L}$ is a derivation and that each finite-dimensional nilpotent Lie algebra of dimension larger than two admits 2-local derivation which is not a derivation. In [7] the authors have proved that every local derivation on semi-simple Lie algebras is a derivation and gave examples of finite-dimensional nilpotent Lie algebras with local derivations which are not derivations. Concerning 2-local automorphism, Z.Chen and D.Wang in [12] proved that if $\mathcal{L}$ is a simple Lie algebra of type A_l, D_l or $E_k, (k = 6, 7, 8)$ over an algebraically closed field of characteristic zero, then every 2-local automorphism of $\mathcal{L}$ is an automorphism. Finally, in [6] Sh.Ayupov and K.Kudaybergenov generalized the result of [12] and proved that every 2-local automorphism of a finite-dimensional semi-simple Lie algebra over an algebraically closed field of characteristic zero is an automorphism. Moreover, they showed also that every nilpotent Lie algebra with finite dimension larger than two admits 2-local automorphisms which are not an automorphism. Local automorphisms of certain finite-dimensional simple Lie and Leibniz algebras are investigated in [8]. Concerning local automorphism, T.Becker, J.Escobar, C.Salas and R.Turdibaev in [11] established that the set of local automorphisms $LAut(sl_2)$ coincides with the group $Aut^{\pm}(sl_2)$ of all automorphisms and anti-automorphisms. Later in [13] M.Costantini proved that a linear map on a simple Lie algebra is a local automorphism if and only if it is either an automorphism or an anti-automorphism. Similar results concerning local and 2-local derivations and automorphisms on Lie superalgebras were obtained in [14, 22] and [23]. The first example of a simple (ternary) algebra with nontrivial local derivations is constructed by B.Ferreira, I.Kaygorodov and K.Kudaybergenov in [15]. Moreover, in [18] I.Kaygorodov, K.Kudaybergenov and I.Yuldashev proved that every local derivation on a complex semi-simple finite-dimensional Leibniz algebra is a derivation. After that, in [3] Sh.Ayupov, A.Elduque and K.Kudaybergenov constructed an example for a simple (binary) algebra with non-trivial local derivations(automorphisms). In [2] the authors proved that direct sum null-filiform nilpotent Leibniz algebras as a rule admit local automorphisms which are not automorphisms.

In [4] local derivations of solvable Lie algebras are investigated and it is shown that in the class of solvable Lie algebras there exist algebras which admit local derivations which are not derivation and also algebras for which every local derivation is a derivation. Moreover, it is proved that every local derivation on a finite-dimensional solvable Lie algebra with model nilradical and maximal dimension of complementary space is a derivation. Sh.Ayupov, A.Khudoyberdiyev and B.Yusupov proved similar results concerning local derivations on solvable Leibniz algebras in their recent papers [5, 24].

In the present paper we describe local derivations of quasi-filiform Leibniz algebra of type I and show the existence of a local automorphism on quasi-filiform Leibniz algebra of type I which is not an automorphism.

2. Preliminaries

In this section we recall some basic notions and concepts used throughout the paper.

Definition 2.1. A vector space with a bilinear bracket $(\mathcal{L}, [\cdot,\cdot])$ is called a Leibniz algebra if for any $x, y, z \in \mathcal{L}$ the so-called Leibniz identity

$$\big[x,[y,z]\big] = \big[[x,y],z\big] - \big[[x,z],y\big],$$

holds.

For a given Leibniz algebra $(\mathcal{L}, [\cdot,\cdot])$, the sequence of two-sided ideals is defined recursively as follows:

$$\mathcal{L}^1 = \mathcal{L},\ \mathcal{L}^{k+1} = [\mathcal{L}^k, \mathcal{L}],\ k \geq 1.$$

This sequence is said to be the lower central series of L.

Definition 2.2. A Leibniz algebra $\mathcal{L}$ is said to be nilpotent, if there exists $n \in \mathbb{N}$ such that $\mathcal{L}^n = \{0\}$.

It is easy to see that the sum of two nilpotent ideals is also nilpotent. Therefore, the maximal nilpotent ideal always exists. The maximal nilpotent ideal of a Leibniz algebra is said to be the nilradical of the algebra.

Now we give the definitions of automorphisms and local automorphisms.

Definition 2.3. A linear bijective map $\varphi : \mathcal{L} \to \mathcal{L}$ is called an automorphism (resp. an anti-automorphism) if it satisfies $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ (resp. $\varphi([x, y]) = [\varphi(y), \varphi(x)]$) for all $x, y \in \mathcal{L}$.

Definition 2.4. Let $\mathcal{L}$ be an algebra. A linear map $\Delta : \mathcal{L} \to \mathcal{L}$ is called a local automorphism if for any element $x \in A$ there exists an automorphism $\varphi_x : \mathcal{L} \to \mathcal{L}$ such that $\Delta(x) = \varphi_x(x)$.

Below we define the notion of a quasi-filiform Leibniz algebra.

Definition 2.5. A Leibniz algebra $\mathcal{L}$ is called quasi-filiform if $\mathcal{L}^{n-2} \neq \{0\}$ and $\mathcal{L}^{n-1} = \{0\}$, where $n = \dim \mathcal{L}$.

Given an n-dimensional nilpotent Leibniz algebra $\mathcal{L}$ such that $\mathcal{L}^{s-1} \neq \{0\}$ and $\mathcal{L}^{s} = \{0\}$, put $\mathcal{L}_i = \mathcal{L}^i/\mathcal{L}^{i+1},\ 1 \leq i \leq s-1$, and $gr(\mathcal{L}) = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_{s-1}$. Due to $[\mathcal{L}_i, \mathcal{L}_j] \subseteq \mathcal{L}_{i+j}$ we obtain the graded algebra $gr(\mathcal{L})$. If $gr(\mathcal{L})$ and $\mathcal{L}$ are isomorphic, $gr(\mathcal{L}) \cong \mathcal{L}$, we say that $\mathcal{L}$ is *naturally graded*.

Let x be a nilpotent element of the set $\mathcal{L} \setminus \mathcal{L}^2$. For the nilpotent operator of right multiplication $\mathcal{R}_x$ we define a decreasing sequence $C(x) = (n_1, n_2, \ldots, n_k)$, where $n = n_1 + n_2 + \cdots + n_k$, which consists of the dimensions of Jordan blocks of the operator $\mathcal{R}_x$. On the set of such sequences we consider the lexicographic order, that is, $C(x) = (n_1, n_2, \ldots, n_k) \leq C(y) = (m_1, m_2, \ldots, m_t)$ iff there exists $i \in \mathbb{N}$ such that $n_j = m_j$ for any $j < i$ and $n_i < m_i$.

Definition 2.6. The sequence $C(\mathcal{L}) = \max\limits_{x \in \mathcal{L} \setminus \mathcal{L}^2} C(x)$ is called the characteristic sequence of the algebra $\mathcal{L}$.

Definition 2.7. A quasi-filiform non Lie Leibniz algebra L is called an algebra of the type I (respectively, type II) if there exists an element $x \in L \setminus L^2$ such that the operator $\mathcal{R}_x$ has the form $\begin{pmatrix} J_{n-2} & 0 \\ 0 & J_2 \end{pmatrix}$ (respectively, $\begin{pmatrix} J_2 & 0 \\ 0 & J_{n-2} \end{pmatrix}$).

The following theorem obtained in [1] gives the classification of naturally graded quasi-filiform Leibniz algebras.

Theorem 2.8. *An arbitrary n-dimensional naturally graded quasi-filiform Leibniz algebra of type I is isomorphic to one of the following pairwise non-isomorphic algebras of the families:*

$$\mathcal{L}_n^{1,\beta}:\begin{cases}[e_i,e_1]=e_{i+1},\ 1\leq i\leq n-3,\\ [e_{n-1},e_1]=e_n,\\ [e_1,e_{n-1}]=\beta e_n,\ \beta\in\mathbb{C},\end{cases}\qquad \mathcal{L}_n^{2,\beta}:\begin{cases}[e_i,e_1]=e_{i+1},\ 1\leq i\leq n-3,\\ [e_{n-1},e_1]=e_n,\\ [e_1,e_{n-1}]=\beta e_n,\ \beta\in\{0,1\},\\ [e_{n-1},e_{n-1}]=e_n,\end{cases}$$

$$\mathcal{L}_n^{3,\beta}:\begin{cases}[e_i,e_1]=e_{i+1},\ 1\leq i\leq n-3,\\ [e_{n-1},e_1]=e_n+e_2,\\ [e_1,e_{n-1}]=\beta e_n,\ \beta\in\{-1,0,1\},\end{cases}\qquad \mathcal{L}_n^{4,\gamma}:\begin{cases}[e_i,e_1]=e_{i+1},\ 1\leq i\leq n-3,\\ [e_{n-1},e_1]=e_n+e_2,\\ [e_{n-1},e_{n-1}]=\gamma e_n,\ \gamma\neq 0,\end{cases}$$

$$\mathcal{L}_n^{5,\beta,\gamma}:\begin{cases}[e_i,e_1]=e_{i+1},\ 1\leq i\leq n-3\\ [e_{n-1},e_1]=e_n+e_2,\\ [e_1,e_{n-1}]=\beta e_n,\ (\beta,\gamma)=(1,1)\ or\ (2,4),\\ [e_{n-1},e_{n-1}]=\gamma e_n,\end{cases}$$

where $\{e_1,e_2,\dots,e_n\}$ *is a basis of the algebra.*

3. Automorphisms of naturally graded quasi-filiform Leibniz algebra of type I

In the section we consider the main result concerning automorphisms of naturally graded quasi-filiform Leibniz algebra of type I.

Proposition 3.1. *A linear map* $\varphi:\mathcal{L}\to\mathcal{L}$ *is an automorphism if and only if* φ *has the following form:*

$$\varphi(\mathcal{L}_n^{1,\beta}):\begin{cases}\varphi(e_1) &= \sum\limits_{i=1}^{n} a_ie_i,\\ \varphi(e_2) &= a_1\left(\sum\limits_{i=2}^{n-2} a_{i-1}e_i+a_{n-1}(1+\beta)e_n\right),\\ \varphi(e_j) &= a_1^{j-1}\sum\limits_{i=j}^{n-2} a_{i-j+1}e_i,\qquad 3\leq j\leq n-2,\\ \varphi(e_{n-1}) &= \sum\limits_{i=n-3}^{n} b_ie_i,\\ \varphi(e_n) &= a_1\left(b_{n-3}e_{n-2}+b_{n-1}e_n\right),\end{cases}$$

where $a_1b_{n-1}\neq 0$;

$$\varphi(\mathcal{L}_n^{2,0}):\begin{cases}\varphi(e_1) &= \sum\limits_{i=1}^{n} a_ie_i,\\ \varphi(e_2) &= a_1\sum\limits_{i=2}^{n-2} a_{i-1}e_i+a_{n-1}(a_1+a_{n-1})e_n,\\ \varphi(e_j) &= a_1^{j-1}\sum\limits_{i=j}^{n-2} a_{i-j+1}e_i,\qquad 3\leq j\leq n-2,\\ \varphi(e_{n-1}) &= b_{n-2}e_{n-2}+b_{n-1}e_{n-1}+b_ne_n,\\ \varphi(e_n) &= (a_1+a_{n-1})b_{n-1}e_n,\end{cases}$$

where $a_1b_{n-1}\neq 0,\ b_{n-1}=a_1+a_{n-1}$;

$$\varphi(\mathcal{L}_n^{2,1}):\begin{cases}\varphi(e_1) &= \sum\limits_{i=1}^{n} a_i e_i,\\ \varphi(e_2) &= a_1\sum\limits_{i=2}^{n-2} a_{i-1}e_i + a_{n-1}(2a_1+a_{n-1})e_n,\\ \varphi(e_j) &= a_1^{j-1}\sum\limits_{i=j}^{n-2} a_{i-j+1}e_i, \qquad 3\leq j\leq n-2,\\ \varphi(e_{n-1}) &= b_{n-2}e_{n-2}+b_{n-1}e_{n-1}+b_ne_n,\\ \varphi(e_n) &= (a_1+a_{n-1})b_{n-1}e_n,\end{cases}$$

where $a_1b_{n-1}\neq 0,\ b_{n-1}=a_1+a_{n-1};$

$$\varphi(\mathcal{L}_n^{3,-1}):\begin{cases}\varphi(e_1) &= \sum\limits_{i=1}^{n} a_i e_i,\\ \varphi(e_j) &= a_1^{j-1}(a_1+a_{n-1})e_j + a_1^{n-1}\sum\limits_{i=j+1}^{n-2} a_{i-j+1}e_i, \qquad 2\leq j\leq n-2,\\ \varphi(e_{n-1}) &= \sum\limits_{i=2}^{n-3} a_ie_i + b_{n-2}e_{n-2}+(a_1+a_{n-1})e_{n-1}+b_ne_n,\\ \varphi(e_n) &= a_1(a_1+a_{n-1})e_n,\end{cases}$$

where $a_1(a_1+a_{n-1})\neq 0;$

$$\varphi(\mathcal{L}_n^{3,0}):\begin{cases}\varphi(e_1) &= \sum\limits_{i=1}^{n} a_i e_i,\\ \varphi(e_2) &= a_1(a_1+a_{n-1})e_2 + a_1\sum\limits_{i=3}^{n-2} a_{i-1}e_i + a_1a_{n-1}e_n,\\ \varphi(e_j) &= a_1^{j-1}(a_1+a_{n-1})e_j + a_1^{j-1}\sum\limits_{i=j+1}^{n-2} a_{i-j+1}e_i, \qquad 2\leq j\leq n-2,\\ \varphi(e_{n-1}) &= \sum\limits_{i=2}^{n-4} a_ie_i + b_{n-3}e_{n-3}+b_{n-2}e_{n-2}+(a_1+a_{n-1})e_{n-1}+b_ne_n,\\ \varphi(e_n) &= (b_{n-3}-a_{n-3})a_1e_{n-2}+a_1^2e_n,\end{cases}$$

where $a_1(a_1+a_{n-1})\neq 0;$

for the algebras $\mathcal{L}_n^{3,1},\ \mathcal{L}_n^{4,\gamma},\ \mathcal{L}_n^{5,\beta,\gamma}$

$$\begin{cases}\varphi(e_1) &= \sum\limits_{i=1}^{n-2} a_i e_i + a_ne_n,\\ \varphi(e_j) &= a_1^{i-1}\sum\limits_{i=j}^{n-2} a_{i-j+1}e_i, \qquad 2\leq j\leq n-2,\\ \varphi(e_{n-1}) &= b_{n-2}e_{n-2}+a_1e_{n-1}+b_ne_n,\\ \varphi(e_n) &= 2a_1^2e_n,\end{cases}$$

where $a_1\neq 0.$

Proof. Let φ be an automorphism of the algebra. We set

$$\varphi(e_1) = \sum_{i=1}^{n} a_i e_i, \quad \varphi(e_{n-1}) = \sum_{i=1}^{n} b_i e_i.$$

Consider the equalities

$$\varphi(e_2) = \varphi([e_1, e_1]) = [\varphi(e_1), \varphi(e_1)] = \left[\sum_{i=1}^{n} a_i e_i, \sum_{i=1}^{n} a_i e_i\right] =$$
$$= a_1 \sum_{i=1}^{n-3} a_i e_{i+1} + a_1 a_{n-1} e_n + a_1 a_{n-1} \beta e_n = a_1 \left(\sum_{i=2}^{n-2} a_{i-1} e_i + a_{n-1}(1+\beta) e_n\right),$$

$$\varphi(e_3) = \varphi([e_2, e_1]) = [\varphi(e_2), \varphi(e_1)] = \left[a_1 \left(\sum_{i=2}^{n-2} a_{i-1} e_i + a_{n-1}(1+\beta) e_n\right), \sum_{i=1}^{n} a_i e_i\right] =$$
$$= a_1^2 \left(\sum_{i=3}^{n-3} a_{i-2} e_i\right).$$

Now we shall prove the following equalities by an induction on j

$$\varphi(e_j) = a_1^{j-1} \left(\sum_{i=j}^{n-2} a_{i-j+1} e_i\right), \quad 4 \le j \le n-2.$$

Consider the equalities

$$(1) \qquad \beta \varphi(e_n) = \varphi([e_1, e_{n-1}]) = [\varphi(e_1), \varphi(e_{n-1})] = \left[\sum_{i=1}^{n} a_i e_i, \sum_{i=1}^{n} b_i e_i\right] = a_1 b_1 e_2 + a_{n-1} b_1 e_n.$$

$$(2) \qquad \varphi(e_n) = \varphi([e_{n-1}, e_1]) = [\sum_{i=2}^{n} b_i e_i, \sum_{i=1}^{n} a_i e_i] = a_1 \left(\sum_{n=2}^{n-3} b_i e_{i+1} + b_{n-1} e_n\right).$$

Comparing (1) and (2) we obtain that $b_1 = 0$.

From the equality

$$0 = \varphi([e_n, e_1]) = \left[a_1 \left(\sum_{n=2}^{n-3} b_i e_{i+1} + b_{n-1} e_n\right), \sum_{i=1}^{n} a_i e_i\right] = a_1^2 \left(\sum_{n=2}^{n-4} b_i e_{i+2}\right),$$

we get $b_j = 0,\ 2 \le j \le n-4$.

Now, we shall prove the sufficiency condition of the theorem.

Let us take $u, v \in \mathcal{L}_n^{1,\beta}$ and denote them by

$$u = \sum_{i=1}^{n} \lambda_i e_i, \quad v = \sum_{i=1}^{n} \mu_i e_i.$$

Consider

$$\varphi([u,v]) = \varphi\left(\left[\sum_{i=1}^{n}\lambda_i e_i, \sum_{i=1}^{n}\mu_i e_i\right]\right) = \varphi\left(\sum_{i=2}^{n-2}\lambda_{i-1}\mu_1 e_i + (\lambda_{n-1}\mu_1 + \beta\lambda_1\mu_{n-1})e_n\right) =$$
$$= \sum_{i=2}^{n-2}\lambda_{i-1}\mu_1\varphi(e_i) + (\lambda_{n-1}\mu_1 + \beta\lambda_1\mu_{n-1})\varphi(e_n) = \lambda_1\mu_1 a_1\left(\sum_{i=2}^{n-2}a_{i-1}e_i + a_{n-1}(1+\beta)e_n\right) +$$
$$+ \sum_{i=3}^{n-2}\lambda_{i-1}\mu_1 a^{i-1}\sum_{j=i}^{n-2}a_{i-j+1}e_j + (\lambda_{n-1}\mu_1 + \beta\lambda_1\mu_{n-1})\left(b_{n-3}e_{n-2} + b_{n-1}e_n\right).$$

On the other hand,

$$[\varphi(u),\varphi(v)] = \left[\varphi\left(\sum_{i=1}^{n}\lambda_i e_i\right), \varphi\left(\sum_{i=1}^{n}\mu_i e_i\right)\right] =$$
$$= [\lambda_1\sum_{j=1}^{n}a_j e_j + \lambda_2 a_1\left(\sum_{j=2}^{n-2}a_{j-1}e_j + a_{n-1}(1+\beta)e_n\right) + \sum_{i=3}^{n-2}\lambda_i a^{i-1}\sum_{j=i}^{n-2}a_{i-j+1}e_j +$$
$$+ \lambda_{n-1}\sum_{j=n-3}^{n}b_j e_j + \lambda_n a_1\left(b_{n-3}e_{n-2} + b_{n-1}e_n\right), \mu_1\sum_{j=1}^{n}a_j e_j +$$
$$\mu_2 a_1\left(\sum_{j=2}^{n-2}a_{j-1}e_j + a_{n-1}(1+\beta)e_n\right) + \sum_{i=3}^{n-2}\mu_i a^{i-1}\sum_{j=i}^{n-2}a_{i-j+1}e_j +$$
$$+ \mu_{n-1}\sum_{j=n-3}^{n}b_j e_j + +\mu_n a_1\left(b_{n-3}e_{n-2} + b_{n-1}e_n\right)] =$$
$$= \lambda_1\mu_1 a_1\left(\sum_{i=2}^{n-2}a_{i-1}e_i + a_{n-1}(1+\beta)e_n\right) + + \sum_{i=3}^{n-2}\lambda_{i-1}\mu_1 a^{i-1}\sum_{j=i}^{n-2}a_{i-j+1}e_j +$$
$$+ (\lambda_{n-1}\mu_1 + \beta\lambda_1\mu_{n-1})\left(b_{n-3}e_{n-2} + b_{n-1}e_n\right).$$

Comparing coefficients at the basis elements we obtain

$$\varphi([u,v]) = [\varphi(u),\varphi(v)],$$

and we complete the proof of theorem. $\square$

4. Local automorphisms of naturally graded quasi-filiform Leibniz algebra of type I

Now we shall give the main result concerning local automorphisms of a naturally graded quasi-filiform Leibniz algebra of type I.

Theorem 4.1. *Let Δ be a linear operator on a naturally graded quasi-filiform Leibniz algebra of type I. Then Δ is a local automorphism if and only if:*

for the algebras $\mathcal{L}_n^{1,\beta}$

$$LocAut : \begin{cases} \Delta(e_1) & = \sum\limits_{j=1}^{n} c_{j,1}e_j, \\ \Delta(e_2) & = \sum\limits_{j=2}^{n-2} c_{j,2}e_j + c_{n,2}e_n, \\ \Delta(e_i) & = \sum\limits_{j=i}^{n-2} c_{j,i}e_j, \quad 3 \leq i \leq n-2 \\ \Delta(e_{n-1}) & = \sum\limits_{i=n-3}^{n} c_{i,n-1}e_i, \\ \Delta(e_n) & = c_{n-2,n}e_{n-2} + c_{n,n}e_n, \end{cases} \tag{3}$$

for the algebras $\mathcal{L}_n^{2,\beta}$

$$LocAut : \begin{cases} \Delta(e_1) & = \sum\limits_{j=1}^{n} c_{j,1}e_j, \\ \Delta(e_2) & = \sum\limits_{j=2}^{n-2} c_{j,2}e_j + c_{n,2}e_n, \\ \Delta(e_i) & = \sum\limits_{j=i}^{n-2} c_{j,i}e_j, \quad 3 \leq i \leq n-2 \\ \Delta(e_{n-1}) & = c_{n-2,n-1}e_{n-2} + c_{n-1,n-1}e_{n-1} + c_{n,n-1}e_n, \\ \Delta(e_n) & = c_{n-2,n}e_{n-2} + c_{n,n}e_n, \end{cases}$$

for the algebra $\mathcal{L}_n^{3,-1}$

$$LocAut : \begin{cases} \Delta(e_1) & = \sum\limits_{j=1}^{n} c_{j,1}e_j, \\ \Delta(e_i) & = \sum\limits_{j=i}^{n-2} c_{j,i}e_j, \quad 2 \leq i \leq n-2 \\ \Delta(e_{n-1}) & = \sum\limits_{j=2}^{n} c_{j,n-1}e_j, \\ \Delta(e_n) & = c_{n-2,n}e_{n-2} + c_{n,n}e_n, \end{cases}$$

for the algebra $\mathcal{L}_n^{3,0}$

$$LocAut : \begin{cases} \Delta(e_1) & = \sum\limits_{j=1}^{n} c_{j,1}e_j, \\ \Delta(e_2) & = \sum\limits_{j=2}^{n-2} c_{j,2}e_j + c_{n,2}e_n, \\ \Delta(e_i) & = \sum\limits_{j=i}^{n-2} c_{j,i}e_j, \quad 3 \leq i \leq n-2 \\ \Delta(e_{n-1}) & = \sum\limits_{j=2}^{n} c_{j,n-1}e_j, \\ \Delta(e_n) & = c_{n-2,n}e_{n-2} + c_{n,n}e_n, \end{cases}$$

for the algebras $\mathcal{L}_n^{3,1},\ \mathcal{L}_n^{4,\gamma},\ \mathcal{L}_n^{5,\beta,\gamma}$

$$LocAut:\begin{cases}\Delta(e_1) & = \sum\limits_{j=1}^{n-2} c_{j,1}e_j + c_{n,1}e_n,\\ \Delta(e_2) & = \sum\limits_{j=2}^{n-2} c_{j,2}e_j,\\ \Delta(e_i) & = \sum\limits_{j=i}^{n-2} c_{j,i}e_j, \quad 3\leq i\leq n-2\\ \Delta(e_{n-1}) & = c_{n-2,n-1}e_{n-2}+c_{n-1,n-1}e_{n-1}+c_{n,n-1}e_n,\\ \Delta(e_n) & = c_{n-2,n}e_{n-2}+c_{n,n}e_n,\end{cases}$$

where $\prod\limits_{i=1}^{n} c_{i,i}\neq 0.$

Proof. Since the proof repeats the same arguments that were presented earlier for each case, a detailed proof will be given only for the algebra $\mathcal{L}_n^{1,\beta}$, the rest of the cases are similar.

($\Rightarrow$) Let Δ be a local automorphisms of the algebras $\mathcal{L}_n^{1,\beta}$ and let

$$\Delta(e_i)=\sum_{j=1}^{n} c_{j,i}e_j, \quad 1\leq i\leq n.$$

Step 1. Take an automorphism φ_{e_2} such that $\Delta(e_2)=\varphi_{e_2}(e_2)$. Then

$$\Delta(e_2) = \sum_{j=1}^{n} c_{j,2}e_j,$$

$$\varphi_{e_2}(e_2) = a_1\left(\sum_{i=2}^{n-2} a_{i-1}e_i + a_{n-1}(1+\beta)e_n\right).$$

Comparing the coefficients at the basis elements, we get $c_{1,2}=c_{n-1,2}=0$.

Step 2. Take an automorphism φ_{e_i} such that $\Delta(e_i)=\varphi_{e_i}(e_i)$, where $3\leq i\leq n-2$. Then

$$\Delta(e_i) = \sum_{j=1}^{n} c_{j,i}e_j,$$

$$\varphi_{e_i}(e_i) = a_1^{i-1}\sum_{j=i}^{n-2} a_{i-j+1}e_j.$$

Comparing the coefficients at the basis elements for $\Delta(e_i)$ and $\varphi_{e_i}(e_i)$, we obtain $c_{1,i}=c_{2,i}=\cdots=c_{i-1,i}=c_{n-1,i}=c_{n,i}=0$.

Step 3. Take an automorphism $\varphi_{e_{n-1}}$ such that $\Delta(e_{n-1})=\varphi_{e_{n-1}}(e_{n-1})$. Then

$$\Delta(e_1) = \sum_{j=1}^{n-2} c_{j,n-1}e_j,$$

$$\varphi_{e_{n-1}}(e_{n-1}) = \sum_{i=n-3}^{n} b_ie_i,$$

which implies $c_{1,n-1} = c_{2,n-1} = c_{3,n-1} = \cdots = c_{n-4,n-1} = 0$.

Step 4. Take an automorphism φ_{e_n} such that $\Delta(e_n) = \varphi_{e_n}(e_n)$. Then

$$\begin{aligned} \Delta(e_n) = & \sum_{j=1}^{n} c_{j,n} e_j, \\ \varphi_{e_n}(e_n) = & a_1 \left(b_{n-3} e_{n-2} + b_{n-1} e_n \right). \end{aligned}$$

From this, we get that $c_{1,n} = c_{2,n} = \cdots = c_{n-3,n} = c_{n-1,n} = 0$. Thus, the operator Δ has the form (3).

($\Leftarrow$) Assume that the operator Δ has the form (3). Take an arbitrary element $x = \sum\limits_{j=1}^{n} x_j e_j$.

The coordinates $\Delta(x)$ of the basis $\{e_1, e_2, \ldots, e_n\}$ are

$$\begin{aligned} \Delta(x)_i &= \sum_{j=1}^{i} c_{i,j} x_j, \quad 1 \le i \le n-4, \\ \Delta(x)_{n-3} &= \sum_{j=1}^{n-3} c_{n-3,j} x_j + c_{n-3,n-1} x_{n-1}, \\ \Delta(x)_{n-2} &= \sum_{j=1}^{n-2} c_{n-2,j} x_j + c_{n-2,n-1} x_{n-1} + c_{n-2,n} x_n, \\ \Delta(x)_{n-1} &= c_{n-1,1} x_1 + c_{n-1,n-1} x_{n-1}, \\ \Delta(x)_n &= c_{n,1} x_1 + c_{n,2} x_2 + c_{n,n-1} x_{n-1} + c_{n,n} x_n. \end{aligned}$$

The coordinates of $\varphi(x)$ are

$$\begin{aligned} \varphi(x)_i &= \sum_{j=1}^{i} a_1^{j-1} a_{i+1-j} x_j \quad 1 \le i \le n-4, \\ \varphi(x)_{n-3} &= \sum_{j=1}^{i} a_1^{j-1} a_{i+1-j} x_j + b_{n-3} x_{n-1}, \\ \varphi(x)_{n-2} &= \sum_{j=1}^{i} a_1^{j-1} a_{i+1-j} x_j + b_{n-2} x_{n-1} + a_1 b_{n-3} x_n, \\ \varphi(x)_{n-1} &= a_{n-1} x_1 + b_{n-1} x_{n-1}, \\ \varphi(x)_n &= a_n x_1 + a_1 a_{n-1}(1+\beta) x_2 + b_n x_{n-1} + a_1 b_{n-1} x_n. \end{aligned}$$

Comparing the coordinates of $\Delta(x)$ and $\varphi(x)$, we obtain

(4)
$$\begin{cases} \sum\limits_{j=1}^{i} a_1^{j-1} a_{i+1-j} x_j = \sum\limits_{j=1}^{i} c_{i,j} x_j, \quad 1 \leq i \leq n-4, \\ \sum\limits_{j=1}^{i} a_1^{j-1} a_{i+1-j} x_j + b_{n-3} x_{n-1} = \sum\limits_{j=1}^{n-3} c_{n-3,j} x_j + c_{n-3,n-1} x_{n-1}, \\ \sum\limits_{j=1}^{i} a_1^{j-1} a_{i+1-j} x_j + b_{n-2} x_{n-1} + a_1 b_{n-3} x_n = \sum\limits_{j=1}^{n-2} c_{n-2,j} x_j + c_{n-2,n-1} x_{n-1} + c_{n-2,n} x_n, \\ a_{n-1} x_1 + b_{n-1} x_{n-1} = c_{n-1,1} x_1 + c_{n-1,n-1} x_{n-1}, \\ a_n x_1 + a_1 a_{n-1} (1+\beta) x_2 + b_n x_{n-1} + a_1 b_{n-1} x_n = c_{n,1} x_1 + c_{n,2} x_2 + c_{n,n-1} x_{n-1} + c_{n,n} x_n. \end{cases}$$

Now we consider the following possible cases.

Case 1. Let $x_1 \neq 0$. In this case, putting $b_{n-3} = b_{n-2} = 0$, from (4) we uniquely determine $a_i,\ 1 \leq i \leq n$.

Case 2. Let $x_1 = 0$, $x_2 \neq 0$. Then putting $b_{n-3} = b_{n-2} = 0$ we determine $a_1, a_2, \ldots, a_{n-3}, a_{n-1}$.

Case 3. Let $x_1 = x_2 = \cdots = x_{k-1} = 0$, $x_k \neq 0$, $3 \leq k \leq n-2$. Putting $b_{n-3} = b_{n-2} = 0$ we determine remaining a_{n-i-1} form $k \leq i \leq n-2$.

Case 4. Let $x_1 = x_2 = \cdots = x_{n-2} = 0$ and $x_{n-1} \neq 0$. We determine $b_j,\ \ n-3 \leq j \leq n$.

Case 5. Let $x_1 = x_2 = \ldots = x_{n-1} = 0$ and $x_n \neq 0$. In this case, it sufficient to determine b_{n-3} and b_{n-1} from (4). Theorem 3.1 is proved. □

Example 4.2. Let us consider the linear operator Δ on $\mathcal{L}_n^{1,\beta}$, such that

$$\Delta(x) = x + x_2 e_{n-2}, \quad x = \sum_{i=1}^{n} x_i e_i.$$

By Proposition 3.1, it is not difficult to see that Δ is not an automorphism. We show that, Δ is a local automorphism on $\mathcal{L}_n^{1,\beta}$.

Consider the automorphism φ_1 and φ_2 on the algebra $\mathcal{L}_n^{1,\beta}$, defined as:

$$\varphi_1(x) = x + x_1 e_{n-3} + x_2 e_{n-2},$$
$$\varphi_2(x) = x + \sigma x_1 e_n, \quad x = \sum_{i=1}^{n} x_i e_i.$$

Now, for any $x = \sum\limits_{i=1}^{n} x_i e_i$, we shall find an automorphism φ, such that $\Delta(x) = \varphi(x)$.

If $x_1 = 0$, then

$$\varphi_1(x) = x + x_2 e_{n-2} = \Delta(x).$$

If $x_1 \neq 0$, then set $\sigma = \frac{x_2}{x_1}$, we obtain that

$$\varphi_2(x) = x + \sigma x_1 e_n = x + \frac{x_2}{x_1} x_1 e_n = x + x_2 e_n = \Delta(x).$$

Hence, Δ is a local automorphism.

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