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Research Paper

A CLASS OF ALMOST UNISERIAL RINGS

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ABSTRACT. An $R-$module M is called almost uniserial if any two non-isomorphic submodules of M are comparable. A ring R is an almost left uniserial ring if ${}_RR$ is almost uniserial. In this paper, we introduce a class of artinian almost uniserial rings. Also we give a classification of almost uniserial modules over principal ideal domains.

1. INTRODUCTION

In this paper, all rings have identity elements and all modules are unitary left modules. We call a left R-module M *uniserial* if every two submodules of M are comparable. A ring R is called *left uniserial* if ${}_RR$ is uniserial. An R-module M is called a *serial* module if it is direct sum of uniserial modules. Serial rings and modules are studied in many articles such as [4],[5],[6] and [8]. Serial rings and modules are used to study direct sum decomposition of a module and Krull-Schmidt theorem for such decompositions. A commutative uniserial

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rings is called a valuation rings. Valuation rings has many application in Number Theory and Algebraic Geometry. We say that a left R-module M is almost uniserial if any two non-isomorphic submodules are comparable. A ring R is called almost left uniserial if any two non-isomorphic left ideals of R are comparable. Also a direct sum of almost uniserial modules is called almost serial[2]. The class of *almost left uniserial rings* is defined in [1]. It generalizes both of left uniserial rings and left principal ideal domains. Trivially each left uniserial ring is almost left uniserial, but the converse is not true in general. For instance, $\mathbb{Z}$ is an almost left uniserial ring which is not a left uniserial ring. It is clear that a submodule of an almost uniserial module is almost uniserial. But the quotient of an almost uniserial modules is not almost uniserial in genral.($\mathbb{Z}$ is almost uniserial but $\mathbb{Z}_6$ is not almost uniserial). In [1, Theorem 3.5] some equivalent conditions are given for a commutative ring R to be an Artinian almost uniserial ring. Also [1, example 3.6] give an example of a commutative artinian almost uniserial ring. This example motivated us to give a class of commutative artinian almost uniserial rings. In [3] some examples of almost uniserial quotient of almost uniserial modules are given. Also some results of [1] are generalized to the non commutative case. In [3] it is proved that in a commutative almost left uniserial ring, the ideal $Nil(R)$ is a prime ideal. Also $I_n = \{a \in R : a^n = 0\}$ is an ideal of R and $I_n^n = 0$.

The notion of uniserial and almost uniserial can be studied in any category. We here define three levels of uniseriality in categories which coincide with the usual definition in the categroy of modules. An object M in a category C is called a uniserial object if for any two monomorphisms $f : N \longrightarrow M$ and $g : K \longrightarrow M$ one of them factors through another. An object M in a category C is called an almost uniserial object if for any two monomorphisms $f : N \longrightarrow M$ and $g : K \longrightarrow M$ one of them factors through another or $K \cong N$ An object M in a category C is called a weakly almost uniserial object if for any two monomorphisms $f : N \longrightarrow M$ and $g : K \longrightarrow M$ there is a monmorphism from N to K or there is a moomorphism from K to N. Also by replacing the word "monomorphism" by "epimorphism" and reversing the arrows we have dual notions.For example, in the category SET, a set X is uniserial iff $|X| \leq 1$ and X is almost uniserial iff $|X| \leq 2$. Also one of equivalent forms of axiom of choice is that for any two sets X, Y we have $|X| \leq |Y|$ or $|Y| \leq |X|$ which implies every set is a weakly almost uniserial set. If C is the category of vector spaces over a fixed field F then a vector space V is uniserial iff $dim(V) \leq 1$ and V is almost uniserial iff $dim(V) \leq 2$. Also every vector space is weakly almost uniserial. Since every subgroup of a free group is a free group by Nielsen-Schreier theorem, we conclude that all free groups are weakly almost uniserial in the category of groups. Another intersting example is the field $F(x)$ which is a weakly almost uniserial field in the categroy of field extensions of a fixed field F by Luroth theorem. Luroth theorem states that if $F \subsetneqq L \subseteq F(x)$ then $L \cong F(x)$. This categorical notion and its dual may be studied for its own sake

Here we give some notions and definitions. A ring R is called a left principal ideal ring(LPIR) if every left ideal of R is principal. A principal ideal domain(PID) is an integral domain which every ideal is principal. A ring with a unique maximal left ideal is called a local ring. A discrete valuation ring(DVR) is a local PID. We denote the Jacobson radical of a ring R by $J(R)$. We denote the length of a module M by $l(M)$. For a torsion module M over a PID R and a prime element $p \in R$ define $M(p) = \{x \in M : \exists n \in \mathbb{N} \;\; p^n x = 0\}$. If every finitely generated submodule of an $R-$module M is cyclic, we say M is a locally cyclic module.

2. Preliminaries

In this section we give the theorems which we need to prove main results.

Proposition 2.1. [1, Proposition 2.1.] *Let M be an almost uniserial module. Then either M is an indecomposable module or M is a direct sum of two isomorphic simple modules. Moreover, every finitely generated submodule of M is at most two-generated and the set of all non-cyclic submodules of M is a chain.*

Theorem 2.2. [3, Theorem 3.7] *Let (R, m) be a local left Artinian ring. Assume that the set of two-generated non-cyclic ideals of R is $S = \{m^i : 1 \leq i < k\}$, where k is the nilpotency index of m. Then R is an almost left uniserial ring.*

The following theorem is well known, but we give here a proof for the sake of completeness.

Theorem 2.3. *Let M be a torsion module over a principal ideal domain R. Let P be a set of prime elements of R such that for each prime element $q \in R$ there is an element $p \in P$ such that $Rq = Rp$. Then $M = \bigoplus_{p \in P} M(p)$*

Proof. Let $0 \neq m \in M$. Then $ann(m) = Ra$ where $a \neq 0$ is not a unit in R. So $a = up_1^{n_1} \cdots p_k^{n_k}$ where $p_i \in P$ and u is a unit. If $a_i = \frac{a}{p_i^{n_i}}$ then $R = \langle a_1, \cdots, a_k \rangle$. Hence there exist $r_1, \cdots, r_n \in R$ such that $1 = r_1 a_1 + \cdots + r_n a_n$. Then $m = r_1 a_1 m + \cdots + r_n a_n m$. If $m_i = r a_i m$ then $p_i^{n_i} m_i = r_i (p_i^{n_i} a_i) m = r_i a m = 0$. Hence $m_i \in M(p_i)$. So $M = \sum_{p \in P} M(p)$. Assume $m_1 + \cdots + m_n = 0$ where $m_i \in M(p_i)$. Hence there exists $n_i \in \mathbb{N}$ such that $p_i^{n_i} m_i = 0$. Set $a = p_1^{n_1} \cdots p_k^{n_k}$ and $a_i = \frac{a}{p_i^{n_i}}$. Then $a_i m_j = 0$ for any $i \neq j$. Hence $a_i m_i = 0$. Since $R = \langle p_i^{n_i}, a_i \rangle$, so $m_i = 0$. Thus $M = \bigoplus_{p \in P} M(p)$ □

Theorem 2.4. *Let R be a PID with K as a field of fractions and M be an $R-$module. Then M is a locally cyclic module if and only if M is isomorphic to a submodule of K or K/R. In particular, every quotient of K/R is isomorphic to a submodule of K/R.*

Proof. Since R is a PID, every submodule of a cyclic module is cyclic. If $m_i = \frac{a_i}{b_i} \in K$ then $Rm_1 + \cdots + Rm_n \subseteq Rm$ where $m = \frac{1}{b_1 \cdots b_n}$. Hence it is a cyclic module. So K is a locally cyclic module. Also its quotient K/R is locally cyclic. Conversely, assume M is a locally cyclic module. Then M is a torsion module or it is torsion free.

(1) M is torsion free: In this case the map $M \to M \bigotimes_R K, m \longrightarrow m \otimes 1$ is injective. Also $M \bigotimes_R K$ is also a a locally cyclic $R-$module, since K is a locally cyclic $R-$module. But $M \bigotimes_R K$ is a nonzero vector space over K. Hence $dim_K M \bigotimes_R K = 1$. Thus $M \bigotimes_R K \cong K$ and M is isomorphic to a submodule of K.

(2) M is a torsion $R-$module: Let $E = K/R$. It suffices to prove that $N = M(p)$ is isomorphic to a submodule $E(p)$ by Theorem 2.3. Set $N_n = \{x \in N : p^n x = 0\}$. Then $N_n \subseteq N_{n+1}$ and $N = \bigcup N_n$. Note that $R/p^n R$ is an artinian ring and N_n is an $R/p^n R-$module. It is clear that for any cyclic $R/p^n R-$module Q we have $l(Q) \leq l(R/p^n R) = n$. Let $x \in N_n$ be such that $l(Rx) = k$ be maximum. Then for any $y \in N_n, Rx + Ry$ is a cyclic module. So $k = l(Rx) \leq l(Rx + Ry) \leq k$. Hence $Rx + Ry = Rx$ which implies $y \in Rx$. Thus $N_n = Rx$. Also if $x \in N_n - N_{n-1}$ then $ann(x) = p^n R$ and $l(Rx) = l(R/ann(x)) = l(R/p^n R) = n$. So $Rx = N_n$. Now we complete the proof in two cases.

  (a) There is an $n \in \mathbb{N}$ such that $N_n = N_{n+1}$: Let $t = min\{n \in \mathbb{N} : N_n = N_{n+1}\}$. Then $N = N_t \cong R/p^t R$ which is isomorphic to the submodule generated by $\frac{1}{p^t} + R \in E(p)$.

  (b) For any $n \in \mathbb{N}, N_n \subsetneqq N_{n+1}$: We prove that there exists $x_n \in N_n$ such that $N_n = Rx_n$ and $px_{n+1} = x_n$. Let $y_{n+1} \in N_{n+1} \backslash N_n$ then $N_{n+1} = Ry_{n+1}$. Let $x_1 = y_1$ and assume we have chosen x_n. So $x_n = ry_{n+1}$. Since $p^n x_n = 0$ we conclude $p^{n+1} | p^n r$ or equivalently $p|r$. If $r = pa$ set $x_{n+1} = ay_{n+1}$. Then $px_{n+1} = pay_{n+1} = ry_{n+1} = x_n$. Now define $f : N = \bigcup N_n \longrightarrow E(p)$ such that $f(rx_n) = \frac{r}{p^n} + R$. We prove f is a well defined injective homomorphism. If $rx_n = sx_m$ where $m \geq n$ then $rp^{m-n} x_m = sx_m$. Hence $p^m | rp^{m-n} - s$ which implies $\frac{r}{p^n} - \frac{s}{p^m} \in R$ Hence $f(rx_n) = \frac{r}{p^n} + R = \frac{s}{p^m} + R = f(sx_m)$. So f is well defined.If $f(rx_n) = 0$ then $\frac{r}{p^n} \in R$. Hence $p^n | r$. So $rx_n = 0$. Thus f is injective. Hence N is isomorphic to a submodule of $E(p)$.

□

3. Main results

Theorem 3.1. *Let F be a field and $n \geq 3$ be an integer. Set $R = \frac{F[X,Y]}{\langle X^n, X^{n-1}Y, X^2 - Y^2 \rangle}$. Let x, y be the images of X, Y in R and $m = \langle x, y \rangle$. Then the following holds:*

(1) *(R, m) is an artinian local ring.*
(2) *$\{1, x^i, x^{i-1}y : 1 \leq i < n\}$ is a basis of ${}_FR$.*
(3) *For each $1 \leq k < n$ the ideal m^k is not principal and $m^k = \langle x^k, x^{k-1}y\rangle$.*
(4) *For any $x \in m^k \backslash m^{k+1}$ we have $ann(x) = m^{n-k}$.*

Proof. First note that $x^2 = y^2$ implies $x^k y^{i-k} \in \langle x^i, x^{i-1}y\rangle$. So $\{1, x^i, x^{i-1}y : 1 \leq i < n\}$ is generating set for ${}_FR$.

(1) Since $dim_F R$ is finite, so R is an artinian ring. Also $R/m \cong F$ implies that m is a maximal ideal. Since m is a nil ideal, it is the unique maximal ideal of R.
(2) Assume that

$$a_0 + \sum_{i=1}^{n-1}(a_i x^i + b_i x^{i-1}y) = 0, \tag{1}$$

where $a_i, b_i \in F$. Since $\sum_{i=1}^{n-1}(a_i x^i + b_i x^{i-1}y) \in Nil(R)$, so $a_0 = 0$. First we prove x^{n-1} and $x^{n-2}y$ are independent. If $ax^{n-1} + bx^{n-2}y = 0$ then

$$aX^{n-1} + bX^{n-2}Y = g(X,Y)X^n + h(X,Y)X^{n-1}Y + k(X,Y)(X^2 - Y^2),$$

for some $g(X,Y), h(X,Y), k(X,Y) \in F[X,Y]$ which implies $k(X,Y) = X^{n-2}k_1(X,Y)$. Hence

$$aX + bY = g(X,Y)X^2 + h(X,Y)XY + k_1(X,Y)(X^2 - Y^2).$$

If we substitute $X = 0$ we get $Y^2|bY$ which implies $b = 0$. Also substitution $Y = 0$ implies $X^2|aX$. Hence $a = 0$. If we multiply both sides of equation (1) by x^{n-2} we get $a_1 x^{n-1} + b_1 x^{n-2}y = 0$. So $a_1 = b_1 = 0$. Assume $i < n-1$ and for all $j \leq i,\ a_j = b_j = 0$. Then multiply both sides of equation (1) by x^{n-2-i}. Hence $a_{i+1}x^{n-1} + b_{i+1}x^{n-2}y = 0$ which implies $a_{i+1} = b_{i+1} = 0$. So the set $\{1, x^i, x^{i-1}y : 1 \leq i < n\}$ is a basis for ${}_FR$.
(3) Note that $\{x^i, x^{i-1}y : k \leq i < n\}$ is basis of ${}_Fm^k$. So m^k/m^{k+1} is generated by $x^k + m^{k+1}$ and $x^{k-1}y + m^{k+1}$. Hence $m^k = \langle x^k, x^{k-1}y\rangle$ by Nakayama's lemma. If m^k is cyclic, then it is generated by $x^k(x^{k-1}y)$ by Nakayama's lemma. Hence $x^{k-1}y = ax^k$ for some $a = a_0 + \sum_{i=1}^{n-1}(a_i x^i + b_i x^{i-1}y) \in R$ where $a_i, b_i \in F$. Since the set $\{1, x^i, x^{i-1}y : 1 \leq i < n\}$ is a basis for ${}_FR$, we get a contradiction. So m^k is not cyclic.
(4) Assume $x = \sum_{i=k}^{n-1}(c_i x^i + d_i x^{i-1}y)$. So $c_k \neq 0$ or $d_k \neq 0$. Without loss of generality assume $c_k \neq 0$. If $a = \sum_{i=1}^{n-1}(a_i x^i + b_i x^{i-1}y) \in ann(x)$ then the coefficient of x^{1+k} in ax equals $a_1 c_k$ which implies $a_1 = 0$. Also the coefficient of $x^k y$ in ax equals $b_1 c_k$ which implies $b_1 = 0$. By a similar argument it is proved that if $i < n-k$ then $a_i = b_i = 0$. Hence $a \in m^{n-k}$. So $ann(x) = m^{n-k}$.

□

Theorem 3.2. *The ring R in Theorem 3.1 is an almost uniserial ring.*

Proof. Since m^i is generated by two elements, so $l(m^i/m^{i+1}) = 2$. Hence $l(m^i) = 2n - 2i$ and $l(R) = 2n - 1$. Let I be an ideal of R other than m^i's. Let k be the greatest integer such that $I \subseteq m^k$ and $x \in I \backslash m^{k+1}$. Hence by theorem 3.1, $ann(x) = m^{n-k}$.Also $Rx \cong R/ann(x)$ implies $l(Rx) = 2n - 2k - 1$. Since $2n - 2k - 1 = l(Rx) \leq l(I) < l(m^k) = 2n - 2k$ we conclude that $l(I) = l(Rx)$ which implies $I = Rx$. Hence $\{m^i : 1 \leq i < n\}$ is the set of all two generated non-cyclic ideals. So R is an almost uniserial ring by Theorem 2.2. □

Lemma 3.3. *Let R be a commutative ring and M be an artinian module. Then Rm is a Noetherian module for each $m \in M$.*

Proof. The isomorphism $Rm \cong R/ann(m)$ implies that $R/ann(m)$ is an artinian ring. So $R/ann(m)$ is a Noetherian ring. Hence Rm is a Noetherian module. □

Theorem 3.4. *Let R be a commutative ring and M be an artinian almost uniserial module. Then every proper submodule of M is a Noetherian module. In particular, every proper submodule of M has finite length.*

Proof. Let $m \in M \backslash N$. Then $N \subseteq Rm$ or $N \cong Rm$. Since Rm is noetherian by 3.3, N is a noetherian module. Since N is artinian too, N has finite length. □

Theorem 3.5. *Let M be an almost uniserial module and N be a maximal submodule of M. Then M or N is a cyclic module.*

Proof. Let $m \in M \backslash N$. Then $N \subseteq Rm$ or $N \cong Rm$. If $N \cong Rm$ then N is a cyclic module. Otherwise $Rm = M$ and M is a cyclic module. □

Corollary 3.6. *Let M be an almost uniserial module. Then M is finitely generated if and only if M has a maximal submodule.*

Proof. Assume M has a maximal submodule N. Then M or N is cyclic. Also M/N is cyclic. Hence M is finitely generated. The converse is clear. □

Remark 3.7. Let M be an almost uniserial $R-$module. If N and K are two non-comparable submodules of M then $N \cong K$. If $n \in N \backslash K$, then $K \nsubseteq Rn$ and $Rn \nsubseteq K$. So $N \cong K \cong Rn$. Hence every non-cyclic submodule of M is comparable to any other submodule of M. In particular, the set of non-cyclic submodules of M is a chain.

Theorem 3.8. *Let M be an almost uniserial $R-$module and $LT_n(M) = \{N \leq M : l(M/N) = n\}$. Then $|LT_n(M)| \leq 1$ or each element in $LT_n(M)$ is a cyclic module.*

Proof. Assume $|LT_n(M)| > 1$. Since each two elements of $LT_n(M)$ are non comparable, so each element in $LT_n(M)$ is a cyclic module by Remark 3.7. □

Theorem 3.9. *Let R be an almost uniserial ring and $J(R) = 0$. Then R is a Left principal ideal ring.*

Proof. If R is a local ring, then R is a division ring and the proof is complete Assume R is not a local ring. So $|LT_1(R)| \geq 2$. Hence every maximal left ideal is cyclic by Lemma 3.8. Let I be a non trivial left ideal. So $I \subsetneq J(R)$. Hence there is a maximal left ideal such that $I \subsetneq m$. So $I \cong m$ is a principal ideal. □

Theorem 3.10. *Let R be a PID and M be a finitely generated almost uniserial $R-$module. Then $M \cong R$ or there is a prime element in R such that $M \cong R/p^nR$ or $M \cong R/pR \bigoplus R/pR$.*

Proof. According to structure theorem for finitely generated modules over a principal ideal domain, $M \cong \bigoplus_{i=1}^{k} Rm_i$ where each Rm_i is an indecomposable module. If $ann(m_i) = 0$ then $Rm_i \cong R$. If $ann(m_i) = \langle a \rangle$ then indecomposability and chinese remainder theorem implies $ann(m) = p^nR$ for a prime element p in R. Hence if $k = 1$ then $M \cong R$ or $M \cong R/p^nR$. . If $k \geq 2$ then $k = 2$ by Proposition 2.1and $Rm_1 \cong Rm_2$ is a simple module. Hence $ann(m_1) = ann(m_2)$ is a maximal ideal. So there is a prime element $p \in R$ such that $ann(m_1) = pR$. Hence $M \cong R/pR \bigoplus R/pR$. □

Theorem 3.11. *Let R be a PID and K be the field of fractions of R and $E = K/R$. If M is a torsion almost uniserial $R-$module then there exists a prime element $p \in R$ such that $M \cong R/pR \bigoplus R/pR$ or M is isomorphic to a submodule of $E(p)$.*

Proof. Assume $M \ncong R/pR \bigoplus R/pR$ for any prime element $p \in R$. We prove M is a localy cylic module. Assume N is a finitely generated submodule of M which is not cyclic. So $N \cong R/pR \bigoplus R/pR$ by Theorem 3.10. So $N \neq M$. Let $a \in M \backslash N$. Then $Ra \nsubseteq N$. Since R is a PID, every submodule of a cyclic module is cyclic. If $N \subseteq Ra$ or $N \cong Ra$ then N is a cyclic module which is a contradiction. So N is cyclic. Hence M is a locally cyclic module. Thus M is isomorphic to a submodule of $E = K/R$ by Theorem 2.4. Also M is an indecomposable $R-$module by Theorem 2.1. Hence there is a prime element $p \in R$ such that $M = M(p)$ which is isomorphic to a submodule of $E(p)$. □

Theorem 3.12. *Let R be a discrete valuation ring(DVR) and K be the field of fractions of R. Then K is a uniserial module. Also If M is a torsion free almost uniserial $R-$module then M is isomorphic to a submodule of K. In particular M is a uniserial module.*

Proof. Let $m = pR$ be the unique maximal ideal of R. Then every ideal of R equals p^nR for some $n \in \mathbb{N}$. Also every $0 \subset \cdots \subset p^2R \subset pR \subset R \subset p^{-1}R \subset p^{-2}R \subset \cdots \subset K$. Also every element of K equals up^n for some $u \in U(R)$ and $n \in \mathbb{Z}$. Let N be a proper non zero submodule of K. Note that if $up^n \in N$ then for any $a \in R$ we have $p^na = (u^{-1}a)p^nu \in N$. Hence $p^nR \subseteq N$. If $up^n \in K\backslash N$ then $Rp^n \nsubseteq N$. So $\{n \in \mathbb{Z} : p^nR \subseteq N\}$ is bounded from below. So it has a minimum say t. If $p^nu \in N$ then $p^nR \subseteq N$. Hence $t \leq n$. So $p^nu \in p^tR$. Thus $N \subseteq p^tR$. Also $p^tR \subseteq N$ which implies $N = p^tR$. Hence K is uniserial. Let M be a torsion free almost uniserial $R-$module. So M is a locally cyclic module by Theorem 3.10. Hence M is isomorphic to a submodule of K by Theorem 2.4. □

Lemma 3.13. *Let M be an artinian $R-$module. Then M is not isomorphic to any of its proper submodules.*

Proof. Let N be a proper submodule of M which is isomorphic to M and is minimal with this property. Assume $f : M \to N$ is an isomorphism. Then $f_{|N} : N \to f(N)$ is an isomorphism. Hence $M \cong N \cong f(N) < N$ which is a contradiction by minimality of N. □

Theorem 3.14. *Let M be an almost uniserial artinian $R-$module. Also N and K are two non comparable submodules of M. Then $\{T : T < N\} = \{T : T < K\}$ and $\{T : N < T\} = \{T : K < T\}$. Also $N \cap K$ is the unique maximal submodule of N and K and $N + K$ is the unique module which contains N and is minimal with this property.*

Proof. Let $T < N$ be a proper submodule of N. If $T \nless K$, then $T \cong K \cong N$. This contrdicts Lemma 3.13. So $T < K$. If $N < T$ and $K \nless T$, then $T \cong K \cong N$ which is a contradiction by Lemma 3.13. If $T < N$, then $T < K$. Hence $T \leq N \cap K$. So $N \cap K$ is the unique maximal submodule of N. If $N < T$, then $K < T$. Hence $N + K \leq T$. So $N + K$ is the unique module which contains N and is minimal with this property. □

Corollary 3.15. *Let M be an almost uniserial artinian $R-$module. For any submodule N of M set $L(N) = \{T : T < N\}$. Then $\{L(N) : N \leq M\}$ is a chain.*

Proof. If $N \leq K$ then $L(N) \leq L(K)$. If $N \cong K$, then $L(N) = L(K)$ by Theorem 3.14. □

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