Journal of Algebraic Structures and Their Applications
ISSN: 2382-9761
www.as.yazd.ac.ir

Algebraic Structures and Their Applications Vol. 7 No. 2 (2020) pp 51-62.

Research Paper

AN UPPER BOUND ON THE DISTINGUISHING INDEX OF GRAPHS WITH MINIMUM DEGREE AT LEAST TWO

SAEID ALIKHANI* AND SAMANEH SOLTANI

Abstract. The distinguishing index of a simple graph G, denoted by $D'(G)$, is the least number of labels in an edge labeling of G not preserved by any non-trivial automorphism. We prove that for a connected graph G with maximum degree Δ, if the minimum degree is at least two, then $D'(G) \leq \lceil \sqrt{\Delta} \rceil + 1$. We also present graphs G for which $D'(G) \leq \lceil \sqrt{\Delta(G)} \rceil$.

1. Introduction

Let $G = (V, E)$ be a simple connected graph. We use the standard graph notation ([6]). In particular, $\mathrm{Aut}(G)$ denotes the automorphism group of G and is a form of symmetry in which the graph is mapped onto itself while preserving the edge-vertex connectivity. Formally, an automorphism of a graph G is a permutation σ of the vertex set V, such that the pair of vertices (u, v) form an edge if and only if the pair $(\sigma(u), \sigma(v))$ also form an edge. For simple connected graph G, and $v \in V$, the *neighborhood* of a vertex v is the set $N_G(v) = \{u \in V(G) :$

DOI:10.29252/as.2020.1793

MSC(2010):05C25, 05C15

Keywords: distinguishing index; edge-colourings; upper bound.

Received: 03 Dec 2019, Accepted: 01 May 2020.

∗Corresponding author

This first version of this paper has submitted (and is available) to ArXiv on 12 Feb 2017.

$uv \in E(G)\}$. The *degree of a vertex* v in a graph G, denoted by $\deg_G(v)$, is the number of edges of G incident with v. The number of neighbours of v in G is denoted by $\deg_G(v)$. We denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees of the vertices of G. A graph G is k*-regular* if $\deg_G(v) = k$ for all $v \in V$. The distance between two vertices u and v is denoted by $d(u, v)$ and is the number of edges in a shortest path connecting u and v. The *diameter of a graph* G is the largest distance between two vertices of G, and denoted by $\mathrm{diam}(G)$. A connected graph G is called 2-connected, if for every vertex $x \in V(G)$, $G - x$ is connected.

An *edge-colouring* of G with colours in C is a map $c : E \to C$. We say that $f \in \mathrm{Aut}(G)$ *preserves* the edge-colouring c if $c \circ f = c$. Call a colouring of G *distinguishing*, if the identity is the only automorphism which preserves it. In other words, such a colouring c of a graph G *breaks* an automorphism $f \in \mathrm{Aut}(G)$ if f does not preserve colours of c. The *distinguishing index* $D'(G)$ of a graph G is the least number d such that G has an edge labeling with d labels that is preserved only by the identity automorphism of G. The distinguishing edge labeling was first defined by Kalinowski and Pilśniak [8] for graphs (inspired by the well-known distinguishing number $D(G)$ which has been defined for general vertex labelings by Albertson and Collins [1]). The distinguishing index of some examples of graphs was exhibited in [8]. For instance, $D'(P_n) = 2$ for every $n \geq 3$, and $D'(C_n) = 3$ for $n = 3, 4, 5$, $D'(C_n) = 2$ for $n \geq 6$. Also, for complete graphs K_n, we have $D'(K_n) = 3$ for $n = 3, 4, 5$, $D'(K_n) = 2$ for $n \geq 6$. It is known that for every graph G of order at least three, we have $D'(G) \leq D(G) + 1$ ([8]). Recently Alikhani et.al in [2] characterized finite trees for which this inequality is sharp. They also showed that if G is a connected unicyclic graph, then $D'(G) = D(G)$. Authors in [8] showed that if G is a connected graph of order $n \geq 3$ and maximum degree Δ, then $D'(G) \leq \Delta$, unless G is C_3, C_4 or C_5. It follows for connected graphs that $D'(G) > \Delta(G)$ if and only if $D'(G) = \Delta(G) + 1$ and G is a cycle of length at most five. The equality $D'(G) = \Delta(G)$ holds for all paths, for cycles of length at least 6, for K_4, $K_{3,3}$ and for symmetric or bisymmetric trees. Also, Pilśniak showed that $D'(G) < \Delta(G)$ for all other connected graphs.

Theorem 1.1. [9] *Let G be a connected graph that is neither a symmetric nor an asymmetric tree. If the maximum degree of G is at least 3, then $D'(G) \leq \Delta(G) - 1$ unless G is K_4 or $K_{3,3}$.*

Pilśniak put forward the following conjecture.

Conjecture 1.2. [9] If G is a 2-connected graph, then $D'(G) \leq 1 + \lceil \sqrt{\Delta(G)} \rceil$.

In this paper, we prove the following theorem which proves the conjecture.

Theorem 1.3. *Let G be a connected graph of maximum degree Δ. If the minimum degree is at least two, then $D'(G) \leq \lceil \sqrt{\Delta} \rceil + 1$.*

For our purposes, we consider graphs with specific construction that are from dutch-windmill graphs. Because of this, in Section 2, we compute the distinguishing index of the dutch

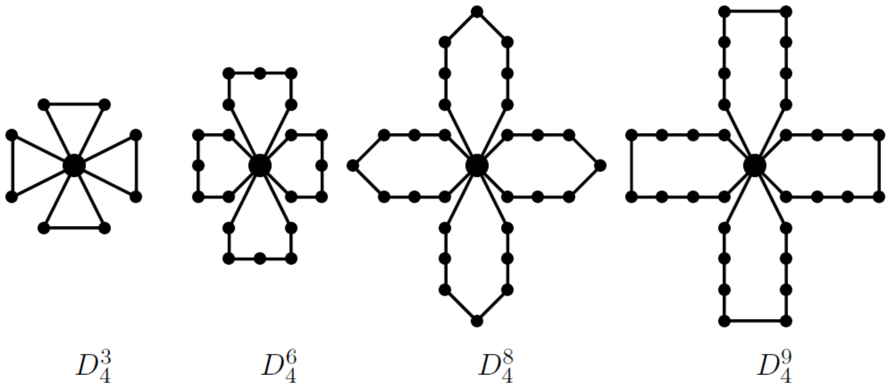

Figure 1. Examples of the dutch windmill graphs.

windmill graphs. In Section 3, we use those results to prove the main result. In the last section we present graphs G for which $D'(G) \leq \lceil \sqrt{\Delta(G)} \rceil$.

2. Distinguishing index of dutch windmill graphs

To obtain the upper bound for the distinguishing index of connected graphs with minimum degree at least two, we characterize such graphs with minimum number of edges. For this characterization we need the concept of dutch windmill graphs. The *dutch windmill graph* D_n^k is the graph obtained by taking n, $(n \geq 2)$ copies of a cycle C_k, $(k \geq 3)$ with a vertex in common (see Figure 1). If $k = 3$, then we call D_n^3, a friendship graph. In the following theorem we compute the distinguishing number of the dutch windmill graphs.

Theorem 2.1. *For every $n \geq 2$ and $k \geq 3$, $D(D_n^k) = \min\{r : \dfrac{r^{k-1} - r^{\lceil \frac{k-1}{2} \rceil}}{2} \geq n\}$.*

Proof. We consider two cases:

Case 1) k is odd. There is a natural number m such that $k = 2m + 1$. We can consider a blade of D_n^k as in Figure 2. Let $(x_1^{(i)}, x_1'^{(i)}, \ldots, x_m^{(i)}, x_m'^{(i)})$ be the label of vertices $(v_1, v_1', \ldots, v_m, v_m')$ of the ith blade where $1 \leq i \leq n$. Suppose that $L = \{(x_1^{(i)}, x_1'^{(i)}, \ldots, x_m^{(i)}, x_m'^{(i)}) |\ 1 \leq i \leq n; x_j^{(i)}, x_j'^{(i)} \in \mathbb{N}, 1 \leq j \leq m\}$ is a labeling of the vertices of D_n^k except its central vertex. In an r-distinguishing labeling we must have:

- (i) There exists $j \in \{1, \ldots, m\}$ such that $x_j^{(i)} \neq x_j'^{(i)}$ for all $i \in \{1, \ldots, n\}$.
- (ii) For $i_1 \neq i_2$ we must have $(x_1^{(i_1)}, x_1'^{(i_1)}, \ldots, x_m^{(i_1)}, x_m'^{(i_1)}) \neq (x_1^{(i_2)}, x_1'^{(i_2)}, \ldots, x_m^{(i_2)}, x_m'^{(i_2)})$ and $(x_1^{(i_1)}, x_1'^{(i_1)}, \ldots, x_m^{(i_1)}, x_m'^{(i_1)}) \neq (x_1'^{(i_2)}, x_1^{(i_2)}, \ldots, x_m'^{(i_2)}, x_m^{(i_2)})$.

The number of $(2m)$-arrays of labels which using r labels that satisfying the condition (i) is $\dfrac{r^{2m} - r^m}{2}$. According to the condition (ii) and since there are n blades, we conclude that $D(D_n^k) = \min\{r : \dfrac{r^{2m} - r^m}{2} \geq n\}$.

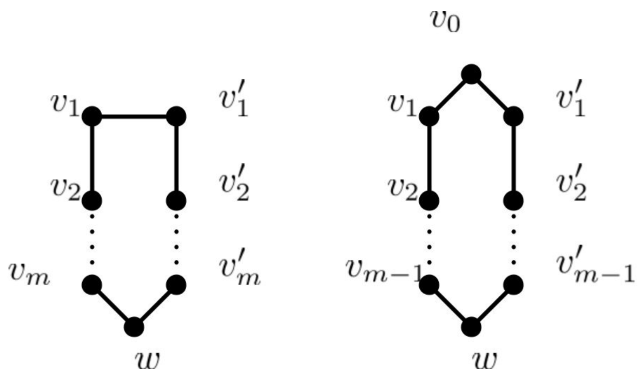

FIGURE 2. The considered polygon (or a cycle of size k) in the proof of Theorem 2.

Case 2) k is even. There is a natural number m such that $k = 2m$. We can consider a blade of D_n^k as in Figure 2. Let $(x_0^{(i)} x_1^{(i)}, x_1'^{(i)}, \ldots, x_{m-1}^{(i)}, x_{m-1}'^{(i)})$ be the label of vertices $(v_0, v_1, v_1', \ldots, v_{m-1}, v_{m-1}')$ of ith blade where $1 \leq i \leq n$. Suppose that $L = \{(x_0^{(i)}, x_1^{(i)}, x_1'^{(i)}, \ldots, x_{m-1}^{(i)}, x_{m-1}'^{(i)})|\ 1 \leq i \leq n; x_0^{(i)}, x_j^{(i)}, x_j'^{(i)} \in \mathbb{N}, 1 \leq j \leq m-1\}$ is a labeling of the vertices of D_n^k except its central vertex. In an r-distinguishing labeling we must have:

(i) There exists $j \in \{1, \ldots, m-1\}$ such that $x_j^{(i)} \neq x_j'^{(i)}$ for all $i \in \{1, \ldots, n\}$.

(ii) For $i_1 \neq i_2$ we must have

$$(x_0^{(i_1)}, x_1^{(i_1)}, x_1'^{(i_1)}, \ldots, x_{m-1}^{(i_1)}, x_{m-1}'^{(i_1)}) \neq (x_0^{(i_2)}, x_1^{(i_2)}, x_1'^{(i_2)}, \ldots, x_{m-1}^{(i_2)}, x_{m-1}'^{(i_2)}),$$
$$(x_0^{(i_1)}, x_1^{(i_1)}, x_1'^{(i_1)}, \ldots, x_{m-1}^{(i_1)}, x_{m-1}'^{(i_1)}) \neq (x_0^{(i_2)}, x_1'^{(i_2)}, x_1^{(i_2)}, \ldots, x_{m-1}'^{(i_2)}, x_{m-1}^{(i_2)}).$$

There are $\dfrac{r^{2m-1} - r^m}{2}$ possible $(2m-1)$-arrays of labels using r labels satisfying condition (i) (r choices for x_0 and $\dfrac{r^{2(m-1)} - r^{m-1}}{2}$ choices for $x_1^{(i_1)}, x_1'^{(i_1)}, \ldots, x_{m-1}^{(i_1)}, x_{m-1}'^{(i_1)}$). According to the condition (ii) and since there are n blades, we conclude that $D(D_n^k) = \min\{r : \dfrac{r^{2m-1} - r^m}{2} \geq n\}$.□

The following theorem gives the distinguishing index of D_n^k.

Theorem 2.2. *For any $n \geq 2$ and $k \geq 3$, $D'(D_n^k) = \min\{r : \dfrac{r^k - r^{\lceil \frac{k}{2} \rceil}}{2} \geq n\}$.*

Proof. There is a natural bijection between the edges of D_n^k and the non-central vertices of D_n^{k+1}, so if we consider the non-central vertices of D_n^{k+1} as the edges of D_n^k, then we have $D'(D_n^k) = D(D_n^{k+1})$. Therefore the result follows from Theorem 2.□

3. Proof of conjecture

In this section, we shall prove Conjecture 1. To do this, first we state some preliminaries. By the result obtained by Fisher and Isaak [5] and independently by Imrich, Jerebic and Klavžar [7] the distinguishing index of complete bipartite graphs is as follows:

Theorem 3.1. [5, 7] *Let p, q, d be integers such that $d \geq 2$ and $(d-1)^p < q \leq d^p$. Then*

$$D'(K_{p,q}) = \begin{cases} d & \text{if } q \leq d^p - \lceil \log_d p \rceil - 1, \\ d+1 & \text{if } q \geq d^p - \lceil \log_d p \rceil + 1. \end{cases}$$

If $q = d^p - \lceil \log_d p \rceil$ then the distinguishing index $D'(K_{p,q})$ is either d or $d+1$ and can be computed recursively in $O(\log(q))$ time.

Corollary 3.2. [9] If $p \leq q$, then $D'(K_{p,q}) \leq \lceil \sqrt[p]{q} \rceil + 1$.

Let G be a connected graph of order $n > 3$ and let $c : E(G) \to \{1, 2, \ldots, k\}$ be a colouring of the edges of G for some positive integer k (where adjacent edges may be coloured the same). The *colour code* of a vertex v of G (with respect to c) is the ordered k-tuple $c(v) = (a_1, a_2, \ldots, a_k)$ (or simply, $c(v) = a_1 a_2 \ldots a_k$), where a_i is the number of edges incident with v that are coloured i for $1 \leq i \leq k$, see [4].

Lemma 3.3. [4] Let c be a k-colouring of the edges of a graph G. The maximum number of different color codes of the vertices of degree r in G is $\binom{r+k-1}{r}$

Now, we are ready to prove the main theorem of this paper.

Proof of Theorem 1. If $\Delta \leq 5$, then the result follows from Theorem 1. So, we suppose that $\Delta \geq 6$. Let v be a vertex of G with the maximum degree Δ. If G does not contain a dutch windmill graph or a triangle attached to G at the vertex v (a proper subgraph H of G is called *attached* if it has only one vertex adjacent to vertices outside H), then we define G' to be G. Otherwise, we delete the attached dutch windmill graph or attached triangle attached to G at the vertex v, until we obtain a subgraph G'. We first label the attached dutch windmill graph or attached triangle, and then we construct an edge labeling with $1 + \lceil \sqrt{\Delta} \rceil$ labels stabilizing all vertices of G' by every automorphism preserving the labeling. By Theorem 2, we can label the edges of a dutch windmill graph attached to G at the vertex v for which v is the central point of the dutch windmill graph, with at most $1 + \lceil \sqrt{\Delta} \rceil$ labels from label set $\{0, 1, \ldots, \lceil \sqrt{\Delta} \rceil\}$, distinguishingly, such that the label 0 is used for at least one edge. If there exists a triangle attached to G at v, then we label the two incident edges to v with 0 and 1, and the other edge of the triangle with label 2.

Let $N^{(1)}(v) = \{v_1, \ldots, v_{|N^{(1)}(v)|}\}$ be the vertices of G' at distance one from v, and continue the labeling by the following steps:

Step 1) In this step, we want to find a labeling of the edges of $G'[N^{(1)}(v)]$ such that each vertex of $N^{(1)}(v)$ is fixed by every automorphism preserving this labeling. Since $|N^{(1)}(v)| \leq \Delta$, for $0 \leq i \leq \lceil\sqrt{\Delta}\rceil - 1$ and $1 \leq j \leq \lceil\sqrt{\Delta}\rceil$, we label the edges $vv_{i\lceil\sqrt{\Delta}\rceil+j}$ with label i, and we do not use the label 0 any more. With respect to the number of incident edges to v with label 0, we conclude that the vertex v is fixed under each automorphism preserving the labeling. Also, since the dutch windmill or the triangle graph attached to G at v has been labeled distinguishingly, so the vertices of attached graph are fixed under each automorphism preserving the labeling. Hence, every automorphism preserving the labeling must map the set of vertices of G' at distance i from v to itself setwise, for any $1 \leq i \leq \mathrm{diam}(G')$. We denote the set of vertices of G' at distance i from v by $N^{(i)}(v)$ for $2 \leq i \leq \mathrm{diam}(G')$.

If $N^{(2)}(v) = \emptyset$, then let $E_{k,j}$ be the set of unlabeled edges of G' incident to the vertex $v_{j\lceil\sqrt{\Delta}\rceil+k}$. For every $0 \leq j \leq \lceil\sqrt{\Delta}\rceil - 1$, we can label the elements of each $E_{k,j}$ with labels $\{1, \ldots, \lceil\sqrt{\Delta}\rceil\}$ such that for every pair of $(E_{k,j}, E_{k',j})$, where $k \neq k'$, there exists a label l, $1 \leq l \leq \lceil\sqrt{\Delta}\rceil$, such that the number of label l used for labeling of elements of $E_{k,j}$ and $E_{k',j}$ are distinct (this is possible, by Lemma 3). Therefore, all elements of $N^{(1)}(v)$ is fixed under each automorphism preserving the labeling.

Thus we suppose that $N^{(i)}(v) \neq \emptyset$, for some $i \geq 2$. Now, we partition the vertices of $N^{(1)}(v)$ into two sets $M_1^{(1)}$ and $M_2^{(1)}$ as follows:

$$M_1^{(1)} = \{x \in N^{(1)}(v) : \ N(x) \setminus \{v\} \subseteq N(v)\},$$
$$M_2^{(1)} = \{x \in N^{(1)}(v) : \ N(x) \setminus \{v\} \nsubseteq N(v)\}.$$

The sets $M_1^{(1)}$ and $M_2^{(1)}$ are mapped to $M_1^{(1)}$ and $M_2^{(1)}$, respectively, setwise, under each automorphism preserving the labeling. For $0 \leq i \leq \lceil\sqrt{\Delta}\rceil - 1$, we set $L_i = \{v_{i\lceil\sqrt{\Delta}\rceil+j} \ : \ 1 \leq j \leq \lceil\sqrt{\Delta}\rceil\}$. By this notation, we get that for $0 \leq i \leq \lceil\sqrt{\Delta}\rceil - 1$, the set L_i is mapped to L_i setwise, under each automorphism preserving the labeling. Let the sets $M_{1i}^{(1)}$ and $M_{2i}^{(1)}$ for $0 \leq i \leq \lceil\sqrt{\Delta}\rceil - 1$ be as follows:

$$M_{1i}^{(1)} = M_1^{(1)} \cap L_i, \ M_{2i}^{(1)} = M_2^{(1)} \cap L_i.$$

It is clear that the sets $M_{1i}^{(1)}$ and $M_{2i}^{(1)}$ are mapped to $M_{1i}^{(1)}$ and $M_{2i}^{(1)}$, respectively, setwise, under each automorphism preserving the labeling. Since for any $0 \leq i \leq \lceil\sqrt{\Delta}\rceil - 1$, we have $|M_{1i}^{(1)}| \leq \lceil\sqrt{\Delta}\rceil$, we can label all incident edges to each element of $M_{1i}^{(1)}$ with labels $\{1, 2, \ldots, \lceil\sqrt{\Delta}\rceil\}$, such that for any two vertices of $M_{1i}^{(1)}$, say x and y, there exists a label k, $1 \leq k \leq \lceil\sqrt{\Delta}\rceil$, such that the number of label k for the incident edges to vertex x is different from the number of label k for the incident edges to vertex y (this is possible by Lemma 3). Hence, it can be deduced that each vertex of $M_{1i}^{(1)}$ is fixed under each automorphism preserving the labeling, where $0 \leq i \leq \lceil\sqrt{\Delta}\rceil - 1$. Thus every vertex of $M_1^{(1)}$ is fixed under each automorphism preserving the labeling. In sequel, we want to label the edges incident to vertices of $M_2^{(1)}$ such that $M_2^{(1)}$ is fixed under each automorphism preserving the labeling, pointwise. For this purpose, we partition the vertices of $M_{2i}^{(1)}$ into the sets $M_{2i_p}^{(1)}$, $(1 \leq p \leq \Delta-1)$ as follows:

$$M_{2i_p}^{(1)} = \{x \in M_{2i}^{(1)} \ : \ |N(x) \cap N^{(2)}(v)| = p\}.$$

Since the set $N^{(i)}(v)$, for any i, is mapped to itself, it can be concluded that $M_{2i_p}^{(1)}$ is mapped to itself, setwise, under each automorphism preserving the labeling, for any i and p. Let $M_{2i_p}^{(1)} = \{x_{i_{p1}}, \ldots, x_{i_{ps_p}}\}$. It is clear that $|M_{2i_p}^{(1)}| \leq |M_{2i}^{(1)}| \leq \lceil\sqrt{\Delta}\rceil$. Let $x_{i_{pk}} \in M_{2i_p}^{(1)}$, and $N(x_{i_{pk}}) \cap N^{(2)}(v) = \{x'_{i_{pk1}}, x'_{i_{pk2}}, \ldots, x'_{i_{pkp}}\}$. We assign to the p-tuple $(x_{i_{pk}}x'_{i_{pk1}}, \ldots, x_{i_{pk}}x'_{i_{pkp}})$ of edges, a p-tuple of labels such that for every $x_{i_{pk}}$ and $x_{i_{pk'}}$, $1 \leq k, k' \leq s_p$, there exists a label l in their corresponding p-tuples of labels for which the number of label l in the corresponding p-tuples of $x_{i_{pk}}$ and $x_{i_{pk'}}$ is distinct. For constructing $|M_{2i_p}^{(1)}|$ numbers of such p-tuples we need, $\min\{r : \binom{p+r-1}{r-1} \geq |M_{2i_p}^{(1)}|\}$ distinct labels. Since for any $1 \leq p \leq \Delta - 1$, we have

$$\min\left\{r : \binom{p+r-1}{r-1} \geq |M_{2i_p}^{(1)}|\right\} \leq \min\left\{r : \binom{p+r-1}{r-1} \geq \lceil\sqrt{\Delta}\rceil\right\} \leq \lceil\sqrt{\Delta}\rceil,$$

so we need at most $\lceil\sqrt{\Delta}\rceil$ distinct labels from label set $\{1, 2, \ldots, \lceil\sqrt{\Delta}\rceil\}$ for constructing such j-tuples. For instance, let $p = 1$, and $M_{2i_1}^{(1)} = \{x_{i_{11}}, \ldots, x_{i_{1s_1}}\}$. By our method, we label the edge $x_{i_{11}}x'_{i_{1k1}}$ with label k for $1 \leq k \leq s_1$ where $s_1 \leq \lceil\sqrt{\Delta}\rceil$. Hence, the vertices of $M_{2i_p}^{(1)}$, for any $1 \leq p \leq \Delta - 1$, are fixed under each automorphism preserving the labeling. Therefore, the vertices of $M_{2i}^{(1)}$ for any $0 \leq i \leq \lceil\sqrt{\Delta}\rceil - 1$, and so the vertices of $M_2^{(1)}$ are fixed under each automorphism preserving the labeling. Now, we can get that all vertices of $N^{(1)}(v)$ are fixed. If there exist unlabeled edges of G' with the two endpoints in $N^{(1)}(v)$, then we assign them an arbitrary label, say 1.

Step 2) Here we consider $N^{(2)}(v)$. We partition this set such that the vertices of $N^{(2)}(v)$ with the same neighbours in $M_2^{(1)}$, lie in a set. In other words, we can write $N^{(2)}(v) = \bigcup_i A_i$, such that A_i contains that elements of $N^{(2)}(v)$ having the same neighbours in $M_2^{(1)}$, for any i. Since all vertices in $M_2^{(1)}$ are fixed, so the set A_i is mapped to A_i setwise, under each automorphism of G' preserving the labeling. Let $A_i = \{w_{i1}, \ldots, w_{it_i}\}$, and we have

$$N(w_{i1}) \cap M_2^{(1)} = \cdots = N(w_{it_i}) \cap M_2^{(1)} = \{v_{i1}, \ldots, v_{ip_i}\}.$$

We consider the following two cases:

Case 1) If for every w_{ij} and $w_{ij'}$ in A_i, where $1 \leq j, j' \leq t_i$, there exists a k, $1 \leq k \leq p_i$, for which the label of edge $w_{ij}v_{ik}$ is different from label of edge $w_{ij'}v_{ik}$, then all vertices of G' in A_i are fixed under each automorphism preserving the labeling. Now, we can label the unlabeled edges of G' which are incident to the vertices in A_i and another their endpoint is $N^{(3)}(v)$, arbitrarily.

Case 2) If there exist w_{ij} and $w_{ij'}$ in A_i, where $1 \leq j, j' \leq t_i$, such that for every k, $1 \leq k \leq p_i$, the label of edge $w_{ij}v_{ik}$ and $w_{ij'}v_{ik}$ are the same, then we can make a labeling such that the vertices in A_i have the same property as Case 1, and so are fixed under each automorphism preserving the labeling, by using at least one of the following actions:

- By permuting the coordinates of j-tuple of labels assigned to the incident edges to v_{ik} with an end point in $N^{(2)}(v)$.
- By using a new j-tuple of labels, with labels $\{1, 2, \ldots, \lceil\sqrt{\Delta}\rceil\}$, for incident edges to v_{ik} with an end point in $N^{(2)}(v)$, such that (by notations in Step 1) for every $x_{i'_{jk'}}$ and $x_{i'_{jk''}}$, $1 \leq k', k'' \leq s_j$, there exists a label l in their corresponding j-tuples of labels with different number of label l in their coordinates, where $1 \leq i' \leq \lceil\sqrt{\Delta}\rceil$.
- By labeling the unlabeled edges of G' with the two end points in $N^{(2)}(v)$ which are incident to the vertices in A_i.
- By labeling the unlabeled edges of G' which are incident to the vertices in A_i, and another their endpoint is $N^{(3)}(v)$.
- By labeling the unlabeled edges of G' with the two end points in $N^{(3)}(v)$ for which the end points in $N^{(3)}(v)$ are adjacent to some of vertices in A_i.

Using at least one of above actions, it can be seen that every two vertices w_{ij} and $w_{ij'}$ in A_i have the property as Case 1. Thus we conclude that all vertices in A_i, for any i, and so all vertices in $N^{(2)}(v)$, are fixed under each automorphism preserving the labeling. Now, we can label the unlabeled edges of G' which are incident to the vertices in A_i and another their endpoint is $N^{(3)}(v)$, arbitrarily. If there exist unlabeled edges of G with the two endpoints in $N^{(2)}(v)$, then we assign them an arbitrary label, say 1.

By continuing this method, in the next step we partition $N^{(3)}(v)$ exactly by the same method as partition of $N^{(2)}(v)$ to the sets A_i's in Step 2, and so we can make a labeling such that the elements of $N^{(i)}(v)$ are fixed pointwise, under each automorphism preserving the labeling, for any $3 \leq i \leq \mathrm{diam}(G')$.□

For a 2-connected planar graph G, the distinguishing index may attain $1 + \lceil\sqrt{\Delta(G)}\rceil$. For example, consider the complete bipartite graph $K_{2,q}$ with $q = r^2$, where r is a positive integer r. By Theorem 3, $D'(K_{2,q}) = r + 1$.

4. Graphs with $D'(G) \leq \lceil\sqrt{\Delta}\rceil$

In this section, we present graphs G with specific construction such that $D'(G) \leq \lceil\sqrt{\Delta}\rceil$. To do this we state the following definition.

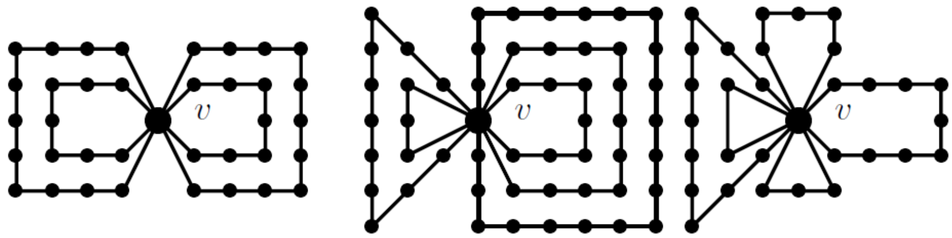

FIGURE 3. Examples of Wind(v).

Definition 4.1. Let G be a connected graph with $\delta(G) \geq 2$. The graph G is called a δ-minimally graph, if the minimum degree of each spanning subgraph of G, except G, is less than $\delta(G)$.

It can be concluded from Definition 4 that if e is an edge of a connected δ-minimally graph with end points u and w, then without loss of generality we can assume that $\deg_G u = \delta$ and $\deg_G w \geq \delta$. In fact the distance between the two vertices of degree greater than δ is at least two.

The simplest connected 2-minimally graphs are cycles C_n and complete bipartite graphs $K_{2,n}$. Now, we explain more on the structure of a 2-minimally graph. Let call a path of length at least two in the graph a *simple path*, if all its internal vertices have degree two. Let G be a connected 2-minimally graph.

- If the degree of all vertices of G is two, then G is a cycle graph.
- If there exists a vertex v of G with degree at least three. We consider the two following cases:

Case 1) If v is the only vertex of G with degree greater than two, then G is a graph which is made by identifying the central points of some dutch windmill graphs $D_{n_i}^{p_i}$ where $p_i \geq 3$, and hence $\Delta(G) = 2\sum_{i \in I} n_i$ where I is a set of indices. In this case we denote G by Wind(v) (for instance, see Figure 3).

Case 2) If G has other vertex w of degree greater than two, then either there exists at least a simple path between v and w, or there exists a vertex of degree at least three on each path between v and w. In the latter case, we can obtain a vertex u of G with degree greater than two such that there exists at least a simple path between v and u (see Figure 4).

Now we present graphs with $D'(G) \leq \lceil \sqrt{\Delta} \rceil$.

Theorem 4.2. *Let G be a connected 2-minimally graph with maximum degree Δ. If G is not a cycle C_3, C_4, C_5 or a complete bipartite graph K_{2,r^2} for some integer r, then $D'(G) \leq \lceil \sqrt{\Delta} \rceil$.*

Proof. If $\Delta = 2$, then G is a cycle. It is known that the distinguishing index of cycle graph of order at least 6 is two. Hence, we suppose that G is not a cycle, so $\Delta \geq 3$. Let v be a vertex

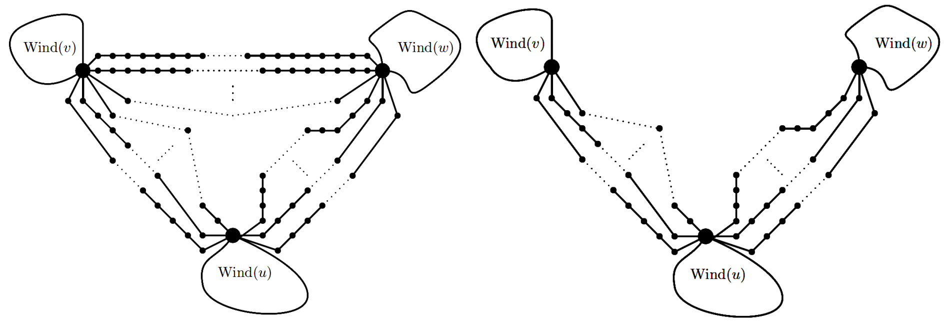

FIGURE 4. The state of vertices of degree greater than two in a connected 2-minimally graph.

of G of maximum degree Δ. Suppose that $V' = \{v_1, v_2, \ldots, v_k\}$ are all vertices of G which are of degree at least three such that there exists at least a simple path between v and v_i, for any $1 \leq i \leq k$ (it is possible that $V' = \emptyset$). Let there exist n_{ij} disjoint simple paths of length j between v and v_i, for any $1 \leq i \leq k$ and $2 \leq j \leq \mathrm{diam}(G)$ where n_{ij} is a non-negative integer and $\sum_{j=2}^{\mathrm{diam}(G)} n_{ij} > 0$. We can label these n_{ij} simple paths of length j with at most $\lceil\sqrt{\Delta}\rceil$ labels, by using n_{ij} numbers of j-tuples such that the coordinates of each j-tuple are in the set $\{1, 2, \ldots, \lceil\sqrt{\Delta}\rceil\}$, for any $1 \leq i \leq k$ and $2 \leq j \leq \mathrm{diam}(G)$, and for every two paths of length j, say P_1 and P_2 with labels $(a_1, \ldots, a_j)$ and $(b_1, \ldots, b_j)$ respectively, there exists l, $1 \leq l \leq j$, such that $a_l \neq b_l$. Let P be a simple path between v and v_i for some $1 \leq i \leq k$, such that the label of edge of P which is incident to v, is different from the label of edge of P which is incident to v_i. We do not use labeling of the simple path P, for any other simple path (with the same length) between any two vertices of degree greater than two. Since G is not a complete bipartite graph K_{2,r^2} for some integer r, we can label these paths distinguishingly with at most $\lceil\sqrt{\Delta}\rceil$ labels. Now, we label the induced subgraph $\mathrm{Wind}(x)$, for any vertex x of degree greater than two, if there exists, with at most $\lceil\sqrt{\Delta}\rceil$ labels distinguishingly by Theorem 2, such that the distinguishing labeling of $\mathrm{Wind}(v)$ is nonisomorphic to the remaining distinguishing labeling of $\mathrm{Wind}(x)$, where $x \in V(G) - \{v\}$. Thus any automorphism of G preserving this labeling should fix $v, v_1, \ldots, v_k$ and all vertices of degree two on the simple paths between v and v_i for any $1 \leq i \leq k$. Since for any $1 \leq i, j \leq k$ where $i \neq j$, the vertices v_i and v_j are fixed, so all the simple paths between v_i and v_j, if there exist, are mapped to each other under each automorphism of G preserving this labeling. Hence we can label all edges of these simple paths with at most $\sqrt{\Delta}$ labels, by assigning distinct ordered tuples of labels of length of the simple paths between v_i and v_j such that all vertices of these paths are fixed under each automorphism of G preserving this labeling.

For any $1 \leq i \leq k$, we consider v_i, and suppose that $v_{i1}, \ldots, v_{ik_i}$ are all vertices of $V(G) \setminus \{v_1, \ldots, v_k\}$ with degree at least three such that there exists at least a simple path between v_i

and v_{ij} for any $1 \leq j \leq k_i$. Now we do the same method as labeling of simple paths between v and $\{v_1, \ldots, v_k\}$, for all simple paths between v_i and $\{v_{i1}, \ldots, v_{ik_i}\}$ with at most $\lceil \sqrt{\Delta} \rceil$ labels. Also, we do the same method as labeling of simple paths between v_i and v_j, for all simple paths between v_{ip} and v_{iq} with at most $\lceil \sqrt{\Delta} \rceil$ labels, where $1 \leq p, q \leq k_i$. Note that we do not use labeling of P for any simple path with the same length as P between v_i and $\{v_{i1}, \ldots, v_{ik_i}\}$. Thus the vertices $\{v_i, v_{i1}, \ldots, v_{ik_i}\}$ and all vertices of the simple paths between them are fixed under each automorphism of G preserving this labeling. After the finite number of steps we can obtain a distinguishing edge labeling of G with at most $\lceil \sqrt{\Delta} \rceil$ labels.$\square$

5. Conclusion

We gave an upper bound for the distinguishing index of graphs G with minimum degree at least two. This result proves a conjecture by Pilśniak (2017). We also studied graphs G with $D'(G) \leq \lceil \sqrt{\Delta} \rceil$. We think that the following conjecture is true, but until now all attempts to prove this failed. So, we end this paper by proposing the following conjecture.

Conjecture 5.1. Let G be a connected graph with maximum and minimum degree Δ and δ, respectively.

(i) If G is a connected δ-minimally graph with $\delta(G) \geq 3$ such that G is not a complete bipartite or δ-regular graph, then $D'(G) \leq \lceil \sqrt[\delta(G)]{\Delta(G)} \rceil$.

(ii) If G is a connected graph with $\delta(G) \geq 3$, then $D'(G) \leq 1 + \lceil \sqrt[\delta(G)]{\Delta(G)} \rceil$.

6. Acknowledgments

The authors would like to express their gratitude to the referees for their careful reading and helpful comments.

References

[1] M.O. Albertson, K.L. Collins, *Symmetry breaking in graphs*, Electron. J. Combin., **3** (1996) #R18.

[2] S. Alikhani, S. Klavžar, F. Lehner, S. Soltani, *Trees with distinguishing index equal distinguishing number plus one*, Discuss. Math. Graph Theory, **40** (2020) 875-884.

[3] S. Alikhani, S. Soltani, *Distinguishing number and distinguishing index of certain graphs*, Filomat, **31** No. 14 (2017) 4393-4404.

[4] G. Chartrand, H. Escuadro, F. Okamoto, P. Zhang, *Detectable colorings of graphs*, Utilitas Math., **69** (2006) 13-32.

[5] M. J. Fisher, G. Isaak, *Distinguishing colorings of Cartesian products of complete graphs*, Discrete Math., **308** No. 11 (2008) 2240-2246.

[6] R. Hammack, W. Imrich and S. Klavžar, *Handbook of product graphs (second edition)*, Taylor & Francis group, 2011.

[7] W. Imrich, J. Jerebic, S. Klavžar, *The distinguishing number of Cartesian products of complete graphs*, European J. Combin., **29** (2008) 922-929.

[8] R. Kalinowski, M. Pilśniak, *Distinguishing graphs by edge colourings*, European J. Combin., **45** (2015) 124-131.

[9] M. Pilśniak, *Improvig upper Bounds for the Distinguishing Index*, Ars Math. Contemp., **13** (2017) 259-274.

S. Alikhani

Department of Mathematics, Yazd University,

89195-741, Yazd, Iran

alikhani@yazd.ac.ir

S. Soltani

Department of Mathematics, Yazd University,

89195-741, Yazd, Iran

s.soltani1979@gmail.com