Algebraic Structures and Their Applications Vol. 11 No. 3 (2024) pp 243-253.

Research Paper

SEMI STRONG OUTER MOD SUM CAYLEY GRAPHS

BADEKARA SOORYANARAYANA* AND M. JAYALAKSHMI

Abstract. Let A be an abelian group generated by a 2-element set $S = \{a, b : a^m = b^n = e, m, n \geq 2\}$, where e is the identity element of A. Let $\Gamma_{m,n} = Cay_g(A, S)$ be the undirected Cayley graph of A associated with S. In this paper, it is shown that $\Gamma_{2k+1,2l+1}$, $\Gamma_{2,2+l}$ and $\Gamma_{2k+1,6}$ are Semi Strong Outer Mod Sum Graphs, and $\Gamma_{k,l}$ is Anti-Outer Mod Sum Graph, for every $k, l \in \mathbb{Z}^+$.

1. Introduction

Let $G(V, E)$ be a graph and $f : V \to Z^+$ be an injective mapping. Let $N_f(v) = \sum_{u \in N(v)} f(u)$ and $N_f(V) = \{N_f(v) : v \in V\}$, where $N(v)$ be the open neighborhood of v in G. Then f is called an outer sum labeling if $N_f(V) \subseteq f(V)$. A graph G which admits an outer sum labeling is called an *outer sum graph* and was proposed by Sooryanarayana et al. in [10]. Further, in [8], Jayalakshmi et al. were studied *Outer Mod Sum Labeling* (OMSL) by taking the sum $N_f(v)$ under addition modulo m for an injective mapping $f : V(G) \to Z_m/_{\{0\}}$. A graph G which

DOI: 10.22034/as.2023.19840.1626

MSC(2010): Primary: 05E15, 05C75, 05C78, 05C25.

Keywords: Cayley graphs, Mod sum labeling, Outer mod sum labeling, Outer sum labeling, Sum labeling.

Received: 08 March 2023, Accepted: 27 December 2023.

*Corresponding author

admits an outer mod sum labeling is called an *Outer Mod Sum Graph* (OMSG). Further, for a given $m \in Z^+$, an injection $f : V \to Z_m/_{\{0\}}$ is called *Anti-Outer Mod Sum Labeling* (AOMSL), if $f(V) \cap \{N_f(v) \pmod m : v \in V\} = \emptyset$.

An OMSL is called *Semi Strong Outer Mod Sum Labeling* (SSOMSL) if $N_f(V) = f(V)$. Finally, an SSOMSL is called *Stong Outer Mod Sum Labeling* (SOMSL) if $N_f(v) \equiv f(v) \pmod m$ for all $v \in V$. A graph G which admits an SSOMSL is called *Semi Strong Outer Mod Sum Graph* (SSOMSG). These labelings were introduced and studied in [7, 9]. The terms not defined here may be found in [2, 5] and for the similar work and entire survey we refer to [3, 4, 6].

Communication delay in a random access machine due to a single large memory is avoided by using a hypercube structured multiprocessor, in which its memory is divided into pieces of constant size and distributed over the network. Cayley graphs [1] are the graphs used in the construction of interconnection networks in which vertices correspond to the processing elements, memory modules, etc. and edges correspond to communication lines.

Secure communication is crucial for many applications to protect data transmission between two network nodes. Many communications take place over long distances and are mediated by technology, increasing awareness of the importance of interception issues. In communications and information processing, encoding is the process by which information from a source is converted into symbols to be communicated. Decoding is the reverse process of converting these code symbols back into information understandable by a receiver. One reason for coding is to enable communication in places where ordinary spoken or written language is difficult or impossible. If a communication is not readily identifiable, then it is unlikely to attract attention for identification of parties, and the mere fact that a communication has taken place is often enough by itself to establish an evidential link in legal prosecutions. It is also important with computers, to be sure where the security is applied, and what is covered. Encryption methods are created to be extremely hard to break; many communication methods either use deliberately weaker encryption than possible, or have backdoors inserted to permit rapid decryption. In some cases, government authorities have required that backdoors be installed in secret.

In this paper, we establish the semi-strong outer mod sum property of certain Cayley graphs by assigning an SSOMSL to them. The SSOMSL assignments are useful to hide or secure the original address tags in communication. More precisely, we show the existence of the Cayley structure, which can be embedded with the property that the sum of the addresses of other nodes is the address of the processing node with some secret key integer m.

2. The Cayley graph $\Gamma_{p,q}$

Let A be a group generated by a 2-element set $S = \{a, b : a^p = b^q = e \text{ and } p, q \geq 2\}$, where e is the identity of A. The un-directed Cayley graph of A associated with a non-empty set $T \subseteq A - \{e\}$, denoted by $Cay_g(A, T)$, is defined on the elements of A with the property that two vertices x, y are adjacent in $Cay_g(A, T)$ if and only if $xy^{-1} \in T \cup T^{-1}$. Let $\Gamma_{p,q} = Cay_g(A, S)$. Then $V(\Gamma_{p,q}) = \{a^i b^j : 0 \leq i \leq p-1, 0 \leq j \leq q-1\}$ and $E(\Gamma_{p,q}) = \{\{a^i b^j, a^{i\oplus_p 1} b^j\}, \{a^i b^j, a^i b^{j\oplus_q 1}\} : 0 \leq i \leq p-1, 0 \leq j \leq q-1\}$. Further, $N(a^i b^j) = \{a^i b^{j\ominus_q 1}, a^i b^{j\oplus_q 1}, a^{i\ominus_p 1} b^j, a^{i\oplus_p 1} b^j\}$ and $\Gamma_{p,q} \cong \Gamma_{q,p}$.

3. Semi strong and anti outer mod sum Cayley graphs

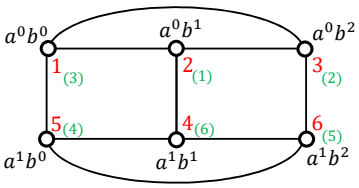

Figure 1. An SSOMSL of the graph $\Gamma_{2,3}$ under modulo 7.

Theorem 3.1. *For any integer $k \geq 3$, the graph $\Gamma_{2,k}$ is an SSOMSG.*

Proof. Let $G = \Gamma_{2,k}$. Define a function $f : V(G) \to Z^+$ in two cases as follows.

Case 1: k is odd.

For $k = 3$, it is easy to see that the function f defined for the graph $G = \Gamma_{2,3}$ in Figure 1, is an SSOML under modulo 7.

Let $k \geq 5$. For each j, $0 \leq j \leq k-1$, define

$$f(a^i b^j) = \begin{cases} 2j+1, & \text{if } i = 0, \\ 4k-2j-1, & \text{if } i = 1. \end{cases}$$

Then,

$$\begin{aligned} N_f(a^0b^0) &= f(a^0b^{k-1}) + f(a^0b^1) + f(a^1b^0) \\ &= [2(k-1)+1] + [2(1)+1] + [4k-2(0)-1] \\ &= 4k - 2(k-1) - 1 = f(a^1b^{k-1}), \\ N_f(a^0b^j) &= f(a^0b^{j-1}) + f(a^0b^{j+1}) + f(a^1b^j) \\ &= [2(j-1)+1] + [2(j+1)+1] + [4k-2j-1] \\ &= 4k + 2j + 1 \\ &\equiv 2j+1, \qquad (\text{mod } 4k) = f(a^0b^j) \text{ for all} i, 1 \leq j \leq k-2, \end{aligned}$$

and

$$\begin{aligned} N_f(a^0b^{k-1}) &= f(a^0b^0) + f(a^0b^{k-2}) + f(a^1b^{k-1}) \\ &= [1] + [2(k-2)+1] + [4k-2(k-1)-1] = 4k-1 = f(a^1b^0). \end{aligned}$$

Also,

$$\begin{aligned} N_f(a^1b^0) &= f(a^1b^{k-1}) + f(a^1b^1) + f(a^0b^0) \\ &= [4k-2(k-1)-1] + [4k-2(1)-1] + [2(0)+1] \\ &\equiv 2(k-1)+1, \pmod{4k} = f(a^0b^{k-1}), \\ N_f(a^1b^j) &= f(a^1b^{j-1}) + f(a^1b^{j+1}) + f(a^0b^j) \\ &= [4k-2(j-1)-1] + [4k-2(j+1)-1] + 2j + 1 \\ &\equiv 4k-2j-1, \pmod{4k} = f(a^1b^j) \text{ for all} i, 1 \le j \le k-2, \\ N_f(a^1b^{k-1}) &= f(a^1b^0) + f(a^1b^{k-2}) + f(a^0b^{k-1}) \\ &= [4k-2(0)-1] + [4k-2(k-2)-1] + [2(k-1)+1] \\ &\equiv 1, \pmod{4k} = f(a^0b^0). \end{aligned}$$

Thus, $N_f(a^ib^j) \in f(V)$. Further, if $N_f(u) = N_f(v)$ for any $u, v \in V(G)$, then $f(a^ib^j) = f(a^lb^m)$ for some $0 \le i, l \le 1$ and $0 \le j, m \le k-1$. But then, $i = l$ (else, without loss of generality taking $i = 0$ and $l = 1$, it follows that $2j+1 = 4k-2m-1 \Rightarrow 2(j+m+1) = 4k \Rightarrow 2k = j+m+1 \le 2k-1$, a contradiction) and hence $i = m$ (so $u = v$). Thus, f is an SSOMSL of G under addition modulo $4k$ in this case.

Case 2: k is even and $k \neq 2$.

For each $0 \le i \le 1$ and $0 \le j \le k-1$, define

$$f(a^ib^j) = \begin{cases} ik + \frac{j}{2} + 1, & \text{if } j \text{ is even,} \\ ik + k - \frac{j-1}{2}, & \text{if } j \text{ is odd.} \end{cases}$$

Then, for each even j, $2 \le j \le k-2$,

$$\begin{aligned} N_f(a^0b^j) &= f(a^0b^{j-1}) + f(a^0b^{j+1}) + f(a^1b^j) \\ &= [0 + k - \tfrac{(j-1)-1}{2}] + [0 + k - \tfrac{(j+1)-1}{2}] + [k + \tfrac{j}{2} + 1] \\ &= (2k+1) + k - \tfrac{j}{2} + 1 \\ &\equiv k - \tfrac{(j-1)-1}{2}, \pmod{2k+1} = f(a^0b^{j-1}), \\ N_f(a^1b^j) &= f(a^1b^{j-1}) + f(a^1b^{j+1}) + f(a^0b^j) \\ &= [k + k - \tfrac{(j-1)-1}{2}] + [k + k - \tfrac{(j+1)-1}{2}] + [0 + \tfrac{j}{2} + 1] \\ &= (2k+1) + k - \tfrac{j}{2} + 1 \\ &\equiv k + k - \tfrac{(j-1)-1}{2}, \pmod{2k+1} = f(a^1b^{j-1}), \end{aligned}$$

$$\begin{aligned}
N_f(a^0b^0) &= f(a^0b^{k-1}) + f(a^0b^1) + f(a^1b^0)\\
&= [0+k-\tfrac{(k-1)-1}{2}] + [0+k-\tfrac{(1)-1}{2}] + [k+0+1]\\
&\equiv k-\tfrac{k}{2}+1, \qquad (\text{mod } 2k+1) = f(a^0b^{k-1}),\\
N_f(a^1b^0) &= f(a^1b^{k-1}) + f(a^1b^1) + f(a^0b^0)\\
&= [k+k-\tfrac{(k-1)-1}{2}] + [k+k-\tfrac{(1)-1}{2}] + [0+0+1]\\
&\equiv k+k-\tfrac{(k-1)-1}{2}, \qquad (\text{mod } 2k+1) = f(a^1b^{k-1}).
\end{aligned}$$

For each odd j, $1 \le j \le k-1$,

$$\begin{aligned}
N_f(a^0b^j) &= f(a^0b^{j-1}) + f(a^0b^{j+1}) + f(a^1b^j)\\
&= [0+\tfrac{(j-1)}{2}+1] + [0+\tfrac{(j+1)}{2}+1] + [k+k-\tfrac{j-1}{2}]\\
&= (2k+1) + \tfrac{j+1}{2} + 1\\
&\equiv 0+\tfrac{j+1}{2}+1, \qquad (\text{mod } 2k+1) = f(a^0b^{j+1}),\\
N_f(a^0b^{k-1}) &= f(a^0b^{k-2}) + f(a^0b^0) + f(a^1b^{k-1})\\
&= [0+\tfrac{k-2}{2}+1] + [0+0+1] + [k+k-\tfrac{(k-1)-1}{2}]\\
&= 2k+2 \equiv 1, \qquad (\text{mod } 2k+1) = f(a^0b^0),\\
N_f(a^1b^j) &= f(a^1b^{j-1}) + f(a^1b^{j+1}) + f(a^0b^j)\\
&= [k+\tfrac{j-1}{2}+1] + [k+\tfrac{j+1}{2}+1] + [0+k-\tfrac{j-1}{2}]\\
&\equiv k+\tfrac{j+1}{2}+1, \qquad (\text{mod } 2k+1) = f(a^1b^{j+1}),\\
N_f(a^1b^{k-1}) &= f(a^1b^{k-2}) + f(a^1b^0) + f(a^0b^{k-1})\\
&= [k+\tfrac{k-2}{2}+1] + [k+0+1] + [0+k-\tfrac{(k-1)-1}{2}]\\
&= 2k+2 \equiv k+1, \qquad (\text{mod } 2k+1) = f(a^1b^0).
\end{aligned}$$

Finally, as in the Case 1 above, if $N_f(u) = N_f(v)$ for any $u, v \in V(G)$, then $f(a^ib^j) = f(a^lb^m)$ for some $0 \le i, l \le 1$ and $0 \le j, m \le k-1$. But then, both i and l are even, or, both are odd (else, taking i as even and l as odd, it follows that $\frac{j}{2}+1 = k-\frac{m-1}{2} \Rightarrow 2(j+m+1) = 4k \Rightarrow 2k-2 = j+m-1 \le 2k-3$, a contradiction) and hence $i = m$ implies that $u = v$. Thus, f is an SSOMSL of G under addition modulo $2k+1$ in this case. Hence the theorem. □

Remark 3.2. The graph $G = \Gamma_{2,2}$ is isomorphic to the cycle C_4, so $N_f(u) = N_f(v)$ for diagonal vertices u, v under any function $f : V(G) \to Z^+$. Hence G is not an SSOMSG.

Theorem 3.3. *For any $k \in Z^+$, the graph $\Gamma_{2k+1,6}$ is SSOMSG.*

Proof. Let $G = \Gamma_{2k+1,6}$. Let $f : V(G) \to Z_{14k+7}/_{\{0\}}$ be defined by $f(a^ib^j) = 7i+j+1$, for each $0 \le i \le 2k$ and $0 \le j \le 5$. Then $N_f(a^0b^0) = 14k+17 \equiv 10 \pmod{14k+7}$; $N_f(a^0b^j) = 14k+4j+11 \equiv 4(j+1) \pmod{14k+7}$ for $1 \le j \le 4$; $N_f(a^0b^5) = 14k+25 \equiv 18 \pmod{14k+7}$; and for $1 \le i \le 2k$, $0 \le j \le 5$, $N_f(a^ib^j) = f(a^ib^{j-1}) + f(a^ib^{j+1}) + f(a^{i+1}b^j) +$

$f(a^{i-1}b^j) = [7i + f(a^0b^{j-1})] + [7i + f(a^0b^{j+1})] + [7(i-1) + f(a^0b^j)] + [7i + f(a^1b^j)] = 28i + [f(a^0b^{j-1}) + f(a^0b^{j+1}) + f(a^1b^j)] + [f(a^0b^j) - 7] = 28i + [N_f(a^0b^j) - f(a^{2k}b^j)] + [j + 1 - 7] = 28i + [N_f(a^0b^j) - (7(2k) + j + 1)] + [j + 1 - 7] = 28i + N_f(a^0b^j) - (14k + 7) \equiv N_f(a^0b^j) + 28i \pmod{14k+7}$.

Now, by the definition of f, to prove f is an SSOMSL of $\Gamma_{2k+1,6}$ it is suffices to execute a bijection between $\{N_f(v) : v \in V(\Gamma_{2k+1,6})\}$ and $Z_{14k+7} - \{7i : 0 \leq i \leq 2k\} = f(V)$. For which, we show that $N_f(a^ib^j) = N_f(a^rb^s)$ only if $i = r$ and $j = s$, and $N_f(a^ib^j) \notin P = \{7i : 0 \leq i \leq 2k\}$. We first see that $N_f(a^0b^j) \in \{10, 8, 12, 16, 20, 18\}$ and hence $N_f(a^0b^j) \notin P$, under modulo $14k + 7$ for any $0 \leq j \leq 5$.

If possible, let $N_f(a^ib^j) = N_f(a^rb^s)$ for some $0 \leq j, s \leq 5$ and $1 \leq r \leq 2k$ (Note that if $r = 0$ then certainly $j = s$). Then $N_f(a^0b^j) + 28i \equiv N_f(a^0b^s) + 28r \pmod{14k+7}$. But then, $|N_f(a^0b^j) - N_f(a^0b^s)| \equiv 28|i - r| \pmod{14k+7}$. But as $|N_f(a^0b^j) - N_f(a^0b^s)| \in \{0, 2, 4, 6, 8, 10, 12\}$, we get $28|i - r| = 0$ and which is possible only if $j = s$ and $r = i$ (since $1 \leq i, r \leq 2k$).

Finally, if $N_f(a^ib^j) \in P$, then $N_f(a^0b^j) + 28i \equiv 7t$ for some $t \in \mathbb{Z}^+$ under modulo $4k + 7$. Hence $N_f(a^0b^j) \in P$ under modulo $14k + 7$, a contradiction. Hence the theorem. □

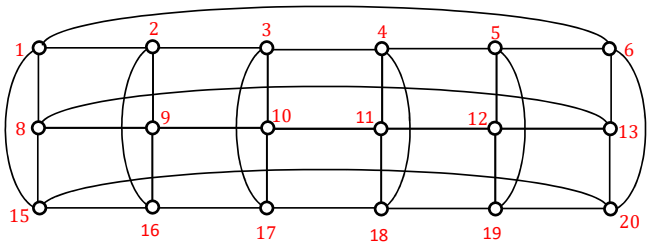

FIGURE 2. An SSOMSL of the Cayley graph $\Gamma_{3,6}$.

The above theorems are the exceptional cases of the following theorem.

Theorem 3.4. *For any $k, l \in \mathbb{Z}^+$, the graph $\Gamma_{2k+1,2l+1}$ is an SSOMSG.*

Proof. Let $G = \Gamma_{2k+1,2l+1}$. Let $f : V(G) \to \{1, 4, 7, \ldots, 12kl + 6(k + l) + 1\} \subseteq Z^+$ be defined by $f(a^ib^j) = 1 + 3[(2l + 1)i + j]$, for each $0 \leq i \leq 2k$ and $0 \leq j \leq 2l$. Take $m = 3(2k + 1)(2l + 1)$. Then $N_f(a^0b^0) = m + 6l + 7 \equiv 6l + 7 = 1 + 3[2l + 2] \pmod{m}$, $N_f(a^0b^j) = N_f(a^0b^0) + 3(4j - 2l - 1) \equiv 12j + 4 \pmod{m} = 1 + 3[4j + 1]$ for $1 \leq j \leq 2l - 1$, and $N_f(a^0b^{2l}) = N_f(a^0b^0) + 3(8l - 4l - 2) \equiv 1 + 18l \pmod{m} = 1 + 3[6l]$.

i \ j	0	1	...	j	...	$2l-1$	$2l$
0	$6l+7$			$12j+4$			$18l+1$
1	↓			↓			↓
...							
i				$+12(2l+1)i$			
...	↓			↓			↓
$2k$							

TABLE 1. Neighborhood sum of each vertex a^ib^j in an SSOMSL of the graph $\Gamma_{2k+1,2l+1}$ under modulo $3(2k+1)(2l+1)$.

Further,

$$\begin{aligned} N_f(a^ib^j) &= N_f(a^0b^j) + 3(2l+1)(4i-\eta) \\ &\equiv N_f(a^0b^j) + 3(2l+1)(4i) - 3(2l+1)\eta \pmod m \\ &\equiv N_f(a^0b^j) + 12(2l+1)i, \pmod m, \end{aligned} \tag{1}$$

where $\eta = \begin{cases} 2k+1, & \text{if} \quad 1 \le i \le 2k-1, \\ 2(2k+1), & \text{if} \quad i = 2k. \end{cases}$

Now, substituting $N_f(a^0b^j)$ in Equation (1), gives

$$N_f(a^ib^j) \equiv \begin{cases} 1+3[2l+2+4(2l+1)i], \pmod m & \text{if} \quad j=0, \\ 1+3[1+4[(2l+1)i+j]], \pmod m & \text{if} \quad 1 \le j \le 2l-1, \\ 1+3[6l+4(2l+1)i], \pmod m & \text{if} \quad j=2l. \end{cases}$$

Also, for every $1 \le j \le s \le 2l$,

$$N_f(a^0b^s) - N_f(a^0b^j) \equiv \begin{cases} 0, \pmod m & \text{if} \quad j=s=0, \\ 3(4s-2l-1), \pmod m & \text{if} \quad j=0<s\le 2l-1, \\ 6(2l-1), \pmod m & \text{if} \quad j=0 \text{ and } s=2l, \\ 12(s-j), \pmod m & \text{if} \quad 1\le j<s\le 2l-1, \\ 3(6l-4j-1), \pmod m & \text{if} \quad 1\le j<s=2l, \\ 0, \pmod m & \text{if} \quad j=s=2l. \end{cases}$$

By the definition of f and the above computation, $N_f(v) = 1+3(n) \in f(V(\Gamma_{2k+1,2l+1}))$, for some integer n under addition modulo m. Hence, to prove f is an SSOMSL of $\Gamma_{2k+1,2l+1}$ it suffices to show that $N_f(a^ib^j) \pmod m = N_f(a^rb^s) \pmod m$ only if $i=r$ and $j=s$.

If possible, let $N_f(a^ib^j) \pmod m = N_f(a^rb^s) \pmod m$. Without loss of generality, we take $j \leq s$ and $k \leq l$. Then

Case 1: $i = 0$ and $r = 0$.

When $s = 0$ or $j = 2l$, the result is obvious (since $j = s$ in these cases).

Sub case 1a: $j = 0$.

If $1 < s < 2l$, then $N_f(a^0b^s) - N_f(a^0b^0) \equiv 0 \pmod m \Rightarrow 3(4s - 2l - 1) \equiv 0 \pmod m \Rightarrow 3(4s-2l-1) = 0$ (because $-m = -3(2l+1)(2k+1) < -(2l+1) < -2l+3 < -2l+4-1 < -2l+4s-1 < 3(4s-2l-1) < 3(4(2l)-2l-1) = 3(6l-1) < 3(6l) \leq 3(3)(2l) < 3(2k+1)(2l+1) = m$). Thus, $4s - 2l - 1 = 0 \Rightarrow s = \frac{2l+1}{4} \notin \mathbb{Z}^+$, a contradiction.

If $s = 2l$, then $N_f(a^0b^{2l}) - N_f(a^0b^0) \equiv 0 \pmod m \Rightarrow 6(2l - 1) = 0$(because $0 \leq 6(2l - 1) < 3(2)(2l + 1) < 3(2k + 1)(2l + 1) = m) \Rightarrow l = 1/2$, a contradiction.

Sub case 1b: $1 \leq j \leq 2l - 1$.

If $j \leq s \leq 2l - 1$, then $N_f(a^0b^s) - N_f(a^0b^j) \equiv 0 \pmod m \Rightarrow 12(s - j) = 3(2k + 1)(2l + 1)t$ for some even integer t. Thus, $12(s - j) = 0$ (because $t < 2$ as $12(s - j) < 12(2l - 1) < 12(2l + 1) < 2(3)(3)(2l + 1) < 2(3(2k + 1)(2l + 1)) = 2m) \Rightarrow (s - j) = 0 \Rightarrow j = s$, as desired.

If $j < s = 2l$, then $N_f(a^0b^s) - N_f(a^0b^j) \equiv 0 \pmod m \Rightarrow 3(6l - 4j - 1) \equiv 0 \pmod m \Rightarrow 3(6l - 4j - 1) = 0$ (because $-m = -3(2l + 1)(2k + 1) < -3(2l + 1) = 6l - 12l - 3 < 6l - 8l - 3 < 6l - 4j - 3 < 3(6l - 4j - 1) < 3(6l) < 3(3)(2l) < 3(2k + 1)(2l + 1) = m$). So, $6l - 4j - 1 = 0 \Rightarrow j = \frac{6l-1}{4} \notin Z^+$, a contradiction.

Case 2: $1 \leq i \leq 2k$

From Equation (1), $N_f(a^rb^s) - N_f(a^ib^j) = N_f(a^0b^s) - N_f(a^0b^j) + 12(2l + 1)(r - i) \equiv 0 \pmod m$.

If $r = i$, then $N_f(a^ib^s) - N_f(a^ib^j) \equiv 0 \pmod m \Rightarrow N_f(a^0b^s) - N_f(a^0b^j) \equiv 0 \pmod m$. Hence the result follows by the above Case 1. Let $r \neq i$. Then

Sub case 2a: $j = 0$.

If $s = 0$, then $N_f(a^rb^s) - N_f(a^ib^j) \equiv 0 \pmod m \Rightarrow N_f(a^0b^0) - N_f(a^0b^0) + 12(2l+1)|r-i| \equiv 0 \pmod m \Rightarrow 12(2l + 1)|r - i| = mt \Rightarrow 4|r - i| = (2k + 1)t$. So, t is an even integer and $t \geq 4$. But $t = 4\frac{|r-i|}{2k+1} \leq 4\frac{2k-1}{2k+1} < 4$, a contradiction.

If $1 \leq s \leq 2l - 1$, then $N_f(a^rb^s) - N_f(a^ib^j) \equiv 0 \pmod m \Rightarrow N_f(a^0b^s) - N_f(a^0b^0) + 12(2l + 1)|r - i| \equiv 0 \pmod m \Rightarrow 3(4s - 2l - 1) + 12(2l + 1)|r - i| = mt \Rightarrow (4s - 2l - 1) + 4(2l + 1)|r - i| = (2l + 1)(2k + 1)t$. This is valid only if t is odd. Now, for the case $4s > 2l + 1$, $4s - (2l + 1) + 4(2l + 1)|r - i| = (2l + 1)(2k + 1)t \Rightarrow 4s = (2l + 1)(2kt + t + 1 - 4\xi)$, where

$\xi = |r-i|$. So, $2s = (2l+1)(kt + \frac{t+1}{2} - 2\xi) \in Z^+$ (since t is odd) and $kt + \frac{t+1}{2} - 2\xi$ is an even integer ≥ 1 (as $s > 0$). But then, $0 < 2s < 2(2l) \Rightarrow 0 < (2l+1)(kt + \frac{t+1}{2} - 2\xi) < 2l \Rightarrow 0 < (kt + \frac{t+1}{2} - 2\xi) < \frac{2l}{2l+1} < 1$, a contradiction. Similarly, if $4s \leq 2l+1$, then $(2l+1) - 4s + 4(2l+1)|r-i| = (2l+1)(2k+1)t \Rightarrow 4s = (2l+1)(1 + 4\xi - 2kt - t)$, where $\xi = |r-i|$. So, $2s = (2l+1)(-kt - \frac{t-1}{2} + 2\xi) \in Z^+$ (since t is odd) and $-kt - \frac{t-1}{2} + 2\xi$ is an even integer ≥ 1 (as $s > 0$). But then, $0 < 2s < 2(2l) \Rightarrow 0 < (2l+1)(-kt - \frac{t-1}{2} + 2\xi) < 2l \Rightarrow 0 < (-kt - \frac{t-1}{2} + 2\xi) < \frac{2l}{2l+1} < 1$, again a contradiction.

If $s = 2l$, then $N_f(a^r b^s) - N_f(a^i b^j) \equiv 0 \pmod m \Rightarrow N_f(a^0 b^{2l}) - N_f(a^0 b^j) + 12(2l+1)|r-i| \equiv 0 \pmod m \Rightarrow 6(2l-1) + 12(2l+1)|r-i| \equiv 0 \pmod m \Rightarrow 6(2l-1) + 12(2l+1)|r-i| = 3(2l+1)(2k+1)t \Rightarrow 2(2l-1) + 4(2l+1)|r-i| = (2l+1)(2k+1)t$ for some even integer $t \geq 4$ (otherwise $2l-1 < 0$ or $2l-1$ is an integer multiple of $2l+1$ if $t = 0$ or $t = 2$, respectively, which is not possible). Also, when $t = 4$, $(2l-1) + 2(2l+1)|r-i| = 2(2l+1)(2k+1) \Rightarrow (2l-1)$ is even , a contradiction. Thus, $t \geq 6$. But, as $t = \frac{2(2l-1)+4(2l+1)|r-i|}{(2l+1)(2k+1)} < \frac{2(2l+1)+4(2l+1)(2k+1)}{(2l+1)(2k+1)} = \frac{2}{(2k+1)} + 4 < 5$, again a contradiction.

Sub case 2b: $1 \leq j < s \leq 2l-1$.

If $r > i$, then $N_f(a^r b^s) - N_f(a^i b^j) \equiv 0 \pmod m \Rightarrow N_f(a^0 b^s) - N_f(a^0 b^j) + 12(2l+1)(r-i) \equiv 0 \pmod m \Rightarrow 12[(s-j) + (2l+1)(r-i)] = mt$. Let $c = s-j$ and $d = r-i$. Then $0 \leq c \leq 2l-1$, $1 \leq d \leq 2k-1$ and $12(c + d(2l+1)) = 3(2k+1)(2l+1)t$. Thus, $4(c + d(2l+1)) = (2k+1)(2l+1)t \Rightarrow t \geq 4$ (else left had side is even and right hand side is odd). Also, $t = \frac{4c}{(2k+1)(2l+1)} + \frac{4d}{2k+1} \leq \frac{4(2l-1)}{(2k+1)(2l+1)} + \frac{4c}{2k+1} < \frac{4(2l+1)}{(2k+1)(2l+1)} + \frac{4d}{2k+1} = \frac{4}{(2k+1)} + \frac{4d}{2k+1} = \frac{4}{2k+1}(1+d) \leq \frac{4}{2k+1}(1+2k-1) = 4\frac{2k}{2k+1} < 4$, a contradiction.

If $r < i$, then $N_f(a^i b^j) - N_f(a^r b^s) \equiv 0 \pmod m \Rightarrow N_f(a^0 b^j) - N_f(a^0 b^s) + 12(2l+1)(i-r) = 12[(j-s) - (2l+1)(i-r)] = mt$. Let $c = j-s$ and $d = i-r$. Then, as above, $0 \leq c \leq 2l-1$, $1 \leq d \leq 2k-1$ and $12(c - d(2l+1)) = mt$. Thus, $t = \frac{4(c-d(2l+1))}{(2k+1)(2l+1)} \leq \frac{4c-3}{(2k+1)(2l+1)} (\because d, l > 0) \leq \frac{4(2l-1)-3}{(2k+1)(2l+1)} = \frac{4(2l+1)-11}{(2k+1)(2l+1)} = \frac{4}{2k+1} - \frac{11}{(2k+1)(2l+1)} < 1$. Therefore, $t = 0$ and hence $12(c - d(2l+1)) = 0 \Rightarrow c = d(2l+1)$ and hence $d(2l+1) = c \leq 2l-1 \Rightarrow d(2l+1) \leq 2l+1-2 \Rightarrow 2 \leq (2l+1)(1-d) \Rightarrow d = 0$, a contradiction (since $d \geq 1$).

Sub case 2c: $1 \leq j < s = 2l$.

In this case, $N_f(a^i b^j) - N_f(a^r b^s) \equiv 0 \pmod m \Rightarrow N_f(a^0 b^j) - N_f(a^0 b^{2l}) + 12(2l+1)|i-r| \equiv 0 \pmod m \Rightarrow 3(6l-4j-1) + 12(2l+1)|i-r| = mt$ for some $t \in Z^+$ (since $r \neq i$). But then, $(6l-4j-1) + 4(2l+1)|i-r| = (2l+1)(2k+1)t$ which implies that t is odd and $t < 4$ (since $|6l - (4j+1)| < \frac{m}{3}$ and $4(2l+1)|r-i| < \frac{4}{3}m$). Therefore,

$$(6l-4j-1) = (2l+1)((2k+1)t - 4|i-r|) \text{ for some } t \in \{1,3\}. \tag{2}$$

But, $-2l+3 = 6l-8l+3 = 6l-4(2l-1)-1 \leq 6l-4j-1 \leq 6l-4(1)-1 = 6l-5$ implies that $-2l+3 \leq (2l+1)((2k+1)t-4|i-r|) \leq 6l-5 \Rightarrow -1+\frac{4}{2l+1} \leq (2k+1)t-4|i-r| \leq 3-\frac{8}{2l+1} \Rightarrow 0 \leq (2k+1)t-4|i-r| \leq 2 \Rightarrow 0 \geq 4|i-r|-(2k+1)t \geq -2$.

When $t=1$, this shows that $2k+1 \geq 4|i-r| \geq 2k-1$ and hence $4|i-r| = 2k$ (being an even number). Substituting this in Equation (2), gives $\frac{6l-4j-1}{2l+1} = (2k+1)-2k = 1 \Rightarrow j = l-\frac{1}{2} \notin Z^+$, a contradiction. Similarly, when $t=3$, the above equation shows that $6k+3 \geq 4|i-r| \geq 6k+1$ and hence $4|i-r| = 6k+2$. Substituting this in Equation (2), gives $\frac{6l-4j-1}{2l+1} = 3(2k+1)-(6k+2) = 1 \Rightarrow j = l-\frac{1}{2} \notin Z^+$, a contradiction.

Case 3: $1 \leq i \leq 2k$ and $j = 2l$.

In this case $2l = j \leq s \Rightarrow j = s$. If $r \neq i$, then $N_f(a^ib^{2l}) - N_f(a^rb^{2l}) \equiv 0 \pmod m \Rightarrow N_f(a^0b^{2l}) - N_f(a^0b^{2l}) + 12(2l+1)|r-i| \equiv 0 \pmod m \Rightarrow 0+12(2l+1)|r-i| = 3(2k+1)(2l+1)t$, for some $t \in \mathbb{Z}^+$. This implies that $4|r-i| = (2k+1)t$ and hence t is an even integer and $t > 2$ (if $t=2$, then $2|r-i| = (2k+1)$, a contradiction, because the left hand side is even and the right hand side is odd). But, $t = 4\frac{|r-i|}{2k+1} \leq 4\frac{2k-1}{2k+1} < 4$, again a contradiction. Hence the theorem. □

Theorem 3.5. *For any $p, q \in Z^+$, the graph $\Gamma_{p,q}$ is AOMSG whenever $p, q \geq 3$.*

Proof. Define a function $f : (V(\Gamma_{p,q})) \to \{1, 3, 5, \ldots, 2pq-1\}$ as $f(a^ib^j) = 2(qi+j)+1$ for each $0 \leq i \leq p-1$ and $0 \leq j \leq q-1$. Then $N_f(a^0b^0) = 2(p+1)q+4 \equiv 2q+4 \pmod{2pq}$, $N_f(a^0b^j) = 8j+4$ for $1 \leq j \leq q-2$ and $N_f(a^0b^{q-1}) = 6q-4$. Finally, for each $1 \leq i \leq p-1$, $N_f(a^ib^j) \equiv N_f(a^0b^j) + 8qi \pmod{2pq}$. This shows that $N_f(a^ib^j) =$ even integer $\notin f(V(\Gamma_{p,q}))$

i \ j	0	1	…	j	…	$q-2$	$q-1$
0	$2(p+1)q+4$			$8j+4$			$6q-4$
1	↓			↓			↓
⋮							
i				$+8qi$			
⋮	↓			↓			↓
$p-1$							

TABLE 2. Neighborhood sum of each vertex a^ib^j in an AOMSL of $\Gamma_{p,q}$ under modulo $2pq$.

for every $a^ib^j \in f(V(\Gamma_{p,q}))$. Therefore, f is an AOMSL of $\Gamma_{p,q}$. Hence the theorem. □

4. Acknowledgments

The authors are very much thankful to the Management, and the Principal of Dr. Ambedkar Institute of Technology for their constant support during the preparation of this paper. Also special thanks to the referees for several useful comments.

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Badekara Sooryanarayana

Department of Mathematics, Dr. Ambedkar Institute of Technology,

B.D.A. Outer Ring Road, Malallahalli,

Bengaluru, India.

dr_bsnrao@yahoo.co.in

Jayalakshmi M

Department of Mathematics, Dr. Ambedkar Institute of Technology,

B.D.A. Outer Ring Road, Malallahalli,

Bengaluru, India.

jayachatra@yahoo.co.in