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Research Paper

LOCALLY ARTINIAN SUPPLEMENTED MODULES

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Abstract. In this paper, we introduce notions of RLA-local modules and locally artinian supplemented modules which are proper generalizations as notions of strongly local modules and ss-supplemented modules, respectively and we study some properties of these modules. In particular, we give a characterization of semiperfect rings and left perfect rings.

1. Introduction

A submodule N of an R-module M will show that $N \subseteq M$. $Rad(M)$ and $Soc(M)$ will indicate radical and socle of M, respectively. A non-zero module M is called *hollow* if every proper submodule of M is small in M, and is called *local* if the sum of all proper submodules of M is also a proper submodule of M. Note that local modules are hollow. M is called *locally artinian* if every finitely generated submodule of M is artinian [5, 31]. A submodule K of M is called a *supplement* of N in M if $M = N + K$ and $N \cap K \ll K$. The module M is called

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supplemented if every submodule of M has a supplement in M. A submodule K of M has *ample supplements* in M if every submodule T of M such that $M = K + T$ contains a supplement of K in M. The module M is called *amply supplemented* if every submodule of M has ample supplements in M [5]. In [6], Zhou and Zhang generalized the concept of socle of a module M to that of $Soc_s(M)$ by considering the class of all simple submodules of M that are small in M in place of the class of all simple submodules of M, that is, $Soc_s(M) = \sum\{ N \ll M \mid N$ is simple $\}$. It is clear that $Soc_s(M) \subseteq Rad(M)$ and $Soc_s(M) \subseteq Soc(M)$.

In this paper, we study notions of RLA-local and locally artinian supplemented modules thank to following notions:

In [3], a module M is called *strongly local* if it is local and $Rad(M)$ is semisimple. A submodule K of M is called an *ss-supplement* of N in M if $M = N + K$ and $N \cap K \subseteq Soc_s(K)$. The module M is called an *ss-supplemented* if every submodule of M has an ss-supplement in M. A submodule K of M has *ample ss-supplements* in M if every submodule T of M such that $M = K + T$ contains an ss-supplement of K in M. The module M is called *amply ss-supplemented* if every submodule of M has ample ss-supplements in M. This class of modules was first studied by Kaynar et al. in [3].

By examining the ss-supplemented modules previously defined in this study we defined and exemplified the concept of RLA-local supplemented modules, which is a more general concept than ss-supplemented modules, and gave its basic properties.

The goal of this paper is to show that, examples were given by defining RLA-local and locally artinian supplemented modules, and locally artinian supplemented modules were characterized on left artinian rings by giving some basic properties of locally artinian supplemeneted modules.

Throughout this paper, R will always denote an associative ring with identity element and modules will be left unital. $Rad(R)$ will denote the Jacobson radical of the ring R. We will use the notation $N \ll M$ to stress that N is small submodule of M. We refer to [1], [3] and [5] for any undefined notion arising in the text.

2. Results

In this section, we investigate some properties of RLA-local modules and locally artinian supplemented modules. We mainly study the relation between the notion of these modules and some other notions. In particular, we give a characterization of semiperfect rings and left perfect rings

Definition 2.1. We call a local module M *RLA-local module* if $Rad(M)$ is a locally artinian submodule of M. If a ring R is the RLA-local module as the left R-module, then we call R an *RLA-local ring.*

Since semisimple modules are locally artinian, we have the following implications hold on modules:

$$\text{strongly local} \Longrightarrow \text{RLA-local} \Longrightarrow \text{local}$$

The following example shows that the above inclusions are proper. Note that every local artinian module is an RLA-module.

Example 2.2. (1) Consider finitely generated $\mathbb{Z}$-module $\mathbb{Z}_8$. Since $\mathbb{Z}_8$ is local artinian, it is an RLA-local module. On the other hand, by [3, Example 18], $\mathbb{Z}_8$ is not a strongly local module.
(2) Given the Dedekind domain $\mathbb{Z}_{(p)} = \{\frac{a}{b} | \, a, b \in \mathbb{Z} \textit{ and } p \nmid b\}$, where $p \in \mathbb{Z}$ is a prime integer. Therefore, the ring $\mathbb{Z}_{(p)}$ is local which is not RLA-local.

Proposition 2.3. *If M is an RLA-local module, then every factor module of M is RLA-local.*

Proof. Assume $N \subset M$. It is clear that $\frac{M}{N}$ is local as a homomorphic image of the local module M. Since local modules are good hollow, it follows from [5, 23.3 (a)] that $Rad(\frac{M}{N}) = \pi(Rad(M))$, where $\pi : M \longrightarrow \frac{M}{N}$ is the canonical homomorphism. Therefore, $Rad(\frac{M}{N}) = \frac{Rad(M)}{N}$ is locally artinian by [5, 31.2 (1) (i)]. Hence $\frac{M}{N}$ is an RLA-local module. □

Definition 2.4. Let M be a module. M is called *locally artinian supplemented* if every submodule U of M has a locally artinian supplement V in M, that is, V is a supplement of U in M such that $U \cap V$ is locally artinian. M is called *amply locally artinian supplemented* if every submodule U of M has ample locally artinian supplements in M. Here a submodule U of M has ample locally artinian supplements in M if every submodule L of M such that $M = U + L$ contains a locally artinian supplement $L^{'}$ of U in M.

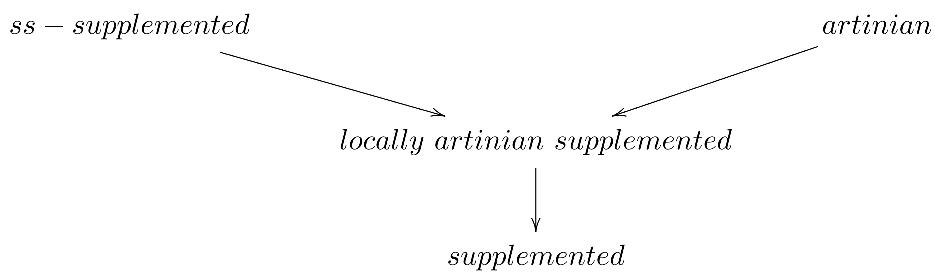
We begin by giving some counterexamples seperating locally artinian supplemented modules, ss-supplemented modules, artinian modules and supplemented modules. Note that artinian modules are supplemented, and over a left artinian ring every left module is locally artinian supplemented.

Example 2.5. For a prime integer $p \in \mathbb{Z}$, take the left $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty}$. Then M is artinian and so it is locally artinian supplemented. However, M is not ss-supplemented according to [3, Example 17].

Example 2.6. Let R be a left artinian ring and M be the left R-module $R^{(\mathbb{N})}$. Then M is a locally artinian supplemented module which is not artinian.

Example 2.7. Let R be a local Dedekind domain with quotient field K. Therefore $_RK$ is a hollow module and so it is supplemented. Since $Soc(_RK) = 0$, $_RK$ has no semi-artinian submodules. It means that $_RK$ is not locally artinian supplemented.

Under given Examples, we clearly have the following implication on modules:



Lemma 2.8. *Let M be a supplemented module and $Rad(M)$ be a locally artinian submodule of M. Then M is locally artinian supplemented.*

Proof. Let K be an arbitrary submodule of M. Since M is supplemented, we can write $M = K + L$ and $K \cap L \ll L$ for some submodule $K \subseteq M$. Then $K \cap L \subseteq Rad(M)$ because $K \cap L \ll M$. By the hypothesis and [5, 31.2 (1)(i)], we obtain that $K \cap L$ is a locally artinian submodule of M. Therefore M is locally artinian supplemented. □

Theorem 2.9. *Let M be a module with $Rad(M) \ll M$. Then the following statements are equivalent:*

(1) *M is locally artinian supplemented;*
(2) *M is supplemented and $Rad(M)$ has a locally artinian supplement in M;*
(3) *M is supplemented and $Rad(M)$ is locally artinian.*

Proof. (1) $\Rightarrow$ (2) Since M is locally artinian supplemented, M is supplemented and it is obvious that $Rad(M)$ has a locally artinian supplement in M.
(2) $\Rightarrow$ (3) Since $Rad(M) \ll M$, M is a locally artinian supplement of $Rad(M)$ in M. So, $M = Rad(M) + M$, $Rad(M) = Rad(M) \cap M \ll M$ and $Rad(M)$ is a locally artinian module.
(3) $\Rightarrow$ (1) Clear from Lemma 2.8. □

Let $f : P \longrightarrow M$ be an epimorphism. f is called a *cover* if $Ker(f) \ll P$, and a cover f is called a *projective cover* if P is a projective module. A ring R is called *(semi)perfect* if every (finitely generated) left R-module has a projective cover ([5]). It is known in [5, 42.6] that R is semiperfect if and only if ${}_RR$ is supplemented. Using this fact along with the above Theorem we obtain the following:

Corollary 2.10. *Let R be a ring. Then ${}_RR$ is locally artinian supplemented if and only if it is a semiperfect ring and $Rad(R)$ is locally artinian.*

Theorem 2.11. *Every RLA-local module is amply locally artinian supplemented.*

Proof. Let $M = U + V$. Since M is local, it is amply supplemented and so there exists a submodule $V^{'}$ of V such that $M = U + V^{'}$ and $U \cap V^{'} \ll V^{'}$. Therefore $U \cap V^{'} \subseteq Rad(V^{'}) \subseteq Rad(M)$. It follows from the hypothesis that $U \cap V^{'}$ is locally artinian. Hence M is amply locally artinian supplemented □

Recall from [5, 31.2 (ii)] that every submodule of a locally artinian module is locally artinian.

Proposition 2.12. *Let M be a module and U be a maximal submodule of M. A submodule V of M is a locally artinian supplement of U if and only if $M = U + V$ and V is an RLA-local module.*

Proof. (⇒) Let V be a locally artinian supplement of U in M. So we can write $M = U + V$, $U \cap V \ll V$ and $U \cap V$ is locally artinian. Since U is a maximal submodule of M and V is supplement of U, V is local module by [5, 41.1]. It follows that $Rad(V) = U \cap V$. So V is an RLA-local module.

(⇐) Since V is an RLA-local module, V is local and $Rad(V)$ is a locally artinian module. Since V is local and U is maximal submodule of M, $U \cap V \subseteq Rad(V)$. It means that $U \cap V$ is locally artinian and $U \cap V \ll V$. Therefore $M = U + V$, $U \cap V \ll V$ and $Rad(V) = U \cap V$ is a locally artinian module, as required. □

To prove that the finite sum of locally artinian supplemented modules is locally artinian supplemented, we use the following standard lemma (see, [5, 41.2]).

Lemma 2.13. *Let M be a module and U, V be submodules of M with U locally artinian supplemented. If $U + V$ has a locally artinian supplement in M, U also has a locally artinian supplement in M.*

Proof. Let M be locally artinian supplement of $U + V$ in M and L be locally artinian supplement of $(K+V)\cap U \subseteq U$. Then $M = U+V+K$, $(U+V)\cap K \ll K$ and $(U+V)\cap K$ is locally artinian. $U = [(K + V) \cap U] + L$ $(K + V) \cap L = [(K + V) \cap U] \cap L \ll L$ and $(K + V) \cap L$ is a locally artinian module. So $M = U+V+K = [(K+V)\cap U]+L+(V+K) = V+(K+L)$. Since $K\cap(U+V)$ and $L\cap(K+V) \ll L$, then we have $V\cap(K+L) \subseteq [K\cap(V+L)]+[L\cap(K+V)] \subseteq [K \cap (U + V)] + [L \cap (K + V)] \ll K + L$, as required. □

Proposition 2.14. *Let U, V be any submodules of a module M such that $M = U + V$. If U and V are locally artinian supplemented, then M is locally artinian supplemented.*

Proof. Let K be any submodule of M. The trivial submodule 0 is a locally artinian supplement of $M = U + V + K$ in M. Since U is locally artinian supplemented, $V + K$ has a locally artinian supplement in M by Lemma 2.13. Again applying Lemma 2.13, we also have that K has a locally artinian supplement in M. This shows that M is locally-artinian supplemented. □

Using this fact we obtain the following corollary.

Corollary 2.15. *Every finite sum of locally artinian supplemented modules is locally artinian supplemented.*

Proposition 2.16. *If a module M is (amply) locally artinian supplemented, then every factor module of M is (amply) locally artinian supplemented.*

Proof. Let M be a locally artinian supplemented module and $\frac{M}{N}$ be a factor module of M. By the assumption, for any submodule U of M which contains N, there exists a submodule V of M such that $M = U + V$, $U \cap V \ll V$ and $U \cap V$ is locally artinian. Let $\pi : M \longrightarrow \frac{M}{N}$ the canonical projection. Then we have that $\frac{M}{N} = \frac{U}{N} + \frac{V+N}{N}$ and $\frac{U}{N} \cap \frac{V+N}{N} = \frac{(U \cap V)+N}{N} = \pi(U \cap V) \ll \pi(V) = \frac{V+N}{N}$ by [5, 19.3 (4)]. Since $U \cap V$ is locally artinian, $\pi(U \cap V) = \frac{U}{N} \cap \frac{V+N}{N}$ is locally artinian. That is $\frac{V+N}{N}$ is a locally artinian supplement of $\frac{U}{N}$ in $\frac{M}{N}$, as required. □

Proposition 2.17. *Let M be a module. If every submodule of M is locally artinian supplemented, then M is amply locally artinian supplemented.*

Proof. Let K and L be two submodules of M such that $M = K + L$. Since L is locally-artinian supplemented, there exists a submodule L' of L such that $L = (K \cap L) + L'$ and $K \cap L' \ll L'$ is locally artinian. Note that $M = K + L = K + (K \cap L) + L' = K + L'$. It means that K has ample locally artinian supplements in M. Hence M is amply locally artinian supplemented. □

Lemma 2.18. *Let M be amply locally artinian supplemented module and V be a supplement submodule in M. Then V is amply locally artinian supplemented.*

Proof. Let V be a supplement of a submodule U of M. Let X and Y be submodules of V such that $V = X + Y$. Then $M = (U + X) + Y$. Since M is amply locally artinian supplemented, $U + X$ has a locally artinian supplement $Y' \subseteq Y$ in M. It follows that $X + Y' \subseteq V$. By the minimality of V, we have $V = X + Y'$. In addition, $X \cap Y' \subseteq (U + X) \cap Y' \ll Y'$, that is, $X \cap Y' \ll Y'$. Since $(U + X) \cap Y'$ is locally artinian, $X \cap Y'$ is also locally artinian by [2, 8.1.5]. It means that Y' is a locally artinian supplement of X in V. Finally, V is amply locally artinian supplemented. □

Proposition 2.19. *Let M be a module. Then, M is amply locally artinian supplemented if and only if every submodule U of M is of the form $U = K + L$, where K is locally artinian supplemented and $L \ll M$ is a locally artinian module.*

Proof. Let U be a submodule of M. Since M is locally artinian supplemented, U has a locally artinian supplement V in M. Then $M = U + V$. By the assumption, there exists a submodule K of U such that K is a locally artinian supplement of V in M. Put $L = U \cap V$. Since V is a locally artinian supplement of U in M, $U = U \cap M = U \cap (K + V) = K + (U \cap V) = K + L$ by Modular Law. Note that K is locally artinian supplemented by Lemma 2.18. Since $L \ll V$, we obtain that $L \ll M$. So the proof is completed. □

Proposition 2.20. *Let M be a π-projective and locally artinian supplemented module. Then M is amply locally artinian supplemented.*

Proof. Let U and V be submodules of M such that $M = U + V$. Since M is π-projective, there exists an endomorphism φ of M such that $\varphi(M) \subseteq U$ and $(1 - \varphi)(M) \subseteq V$. Note that $(1 - \varphi)(U) \subseteq U$. Let K be a locally artinian supplement of U in M. Then $M = \varphi(M) + (1 - \varphi)(M) = \varphi(M) + (1 - \varphi)(U + K) \subseteq U + (1 - \varphi)(K)$, so that $M = U + (1 - \varphi)(K)$. Note that $(1 - \varphi)(K)$ is a submodule of v. Let $y \in U \cap (1 - \varphi)(K)$. Then, $y \in V$ and $y = (1 - \varphi)(x) = x - \varphi(x)$ for some $x \in K$. Then $x = y + \varphi(x) \in U$ so that $y = (1 - \varphi)(x) \in (1 - \varphi)(U \cap K)$. Since $U \cap (1 - \varphi)(K) \subseteq (1 - \varphi)(U \cap K)$, inverse inclusion can be shown by similar method as $U \cap (1 - \varphi)(K) = (1 - \varphi)(U \cap K)$ in [5, 19.3.(4)]. Since $U \cap K$ is locally artinian, $(1 - \varphi)(U \cap K)$ is locally artinian. Since $M = U + (1 - \varphi)(K)$, $U \cap (1 - \varphi)(K) \ll (1 - \varphi)(K)$ and $U \cap (1 - \varphi)(K) = (1 - \varphi)(K)$, M is amply locally artinian supplemented. □

Since every projective module is π-projective, we can obtain the following result.

Corollary 2.21. *Every projective locally artinian supplemented module is amply locally artinian supplemented.*

Now, we shall characterize the rings over which all modules are (amply) locally artinian supplemented.

Lemma 2.22. *Let M be a projective module. Then M is locally artinian supplemented if and only if it is supplemented and $Rad(M)$ is locally-artinian.*

Proof. Suppose that M is a projective supplemented module. Therefore we have $Rad(M) \ll M$ by [5, 42.5]. Then the proof is obvious from Theorem 2.9. □

Theorem 2.23. *The following statements are equivalent for a ring R.*

(1) *R is a left perfect ring and $Rad(R)$ is locally artinian;*

(2) *Every free left R-module is (amply) locally artinian supplemented;*

(3) *every left R-module is (amply) locally artinian supplemented;*

Proof. (1) $\Rightarrow$ (2) Let $F = R^{(I)}$ for some index set I. By [5, 43.9], F is supplemented. It follows from [5, 31.2 (2)] that $Rad(F) = Rad(R^{(I)}) = Rad(R)^{(I)}$ is locally artinian. Hence, by Theorem 2.9, F is locally artinian supplemented.
(2) $\Rightarrow$ (3) Since every module is a homomorphic image of a free left module, the proof follows from Proposition 2.14.
(3) $\Rightarrow$ (1) By Theorem 2.9 and [5, 43.9]. □

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