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Research Paper

EDGE GEODETIC SEQUENCE IN GRAPHS

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Abstract. In this paper, we introduced the concept of edge geodetic sequences in graph and its generating function. Some general properties satisfied by this concept are studied. It is shown that for every generating function

$$G(x)=\sum_{i=1}^{\infty}a^{i-1}x^{i-1}\quad a\in N-\{1\},$$

there exists a recurrence graph G with edge geodetic decomposition $\pi=\{G_1,G_2,\ldots,G_n\ldots\}$.

1. Introduction

By a graph $G=(V,E)$, we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [3]. $N(v)=\{u\in V(G):uv\in E(G)\}$ is called the open neighborhood of the vertex v in G. The degree of a vertex $v\in V(G)$ is $|N(v)|$ and is denoted

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by $deg(v)$. The maximum and minimum degree of a graph G is denoted by Δ and δ respectively. A vertex of degree $p-1$ is called a universal vertex. If $e = \{u, v\}$ is an edge of a graph G with $deg(u) = 1$ and $deg(v) > 1$, then we call e a pendent edge, u a leaf and v a support vertex. For a non empty vertex subset $S \subset V(G)$ of a graph G, an induced subgraph of S in G, denoted by $\langle S\rangle_G$, is the subgraph of G, with vertex set $V(\langle S\rangle_G) = S$ and edge set $E(\langle S\rangle_G) = \{uv \in E(G) : u, v \in S\}$. A vertex v in a connected graph G is said to be a semi simplicial vertex of G if $\Delta(\langle N(v)\rangle) = |N(v)|-1$. A vertex v is a *simplicial* vertex of a graph G if $\langle N(v)\rangle$ is complete. Every simplicial vertex of a graph G is semi simplicial vertex. A graph G is said to be a semi complete graph if every vertex of G is semi simplicial. It is observed that a semi simplicial graph has no cut vertices and no end vertices. The graph with at least two universal vertices is a semi complete graph. However there are semi complete graphs having no universal vertices.A graph having unique universal vertex is not semi complete. The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. An u-v path of length $d(u, v)$ is called an u-v geodesic. A vertex x is said to lie on a u-v geodesic P if x is a vertex of P including the vertices u and v. The interval $I[u, v]$ consist of all vertices lies in u-v geodesic of G [1, 2]. The interval $I_e[u, v]$ consist of all edges lies in u-v geodesic of G. For $S \subseteq V$, $I_e[S] = \cup_{u,v\in S} I_e[u, v]$. A set $S \subseteq V$ is called an edge geodetic set of G if $I_e[S] = E$. The edge geodetic number $g_e(G)$ of G is the minimum order of its edge geodetic sets and any edge geodetic set of order $g_e(G)$ is an edge geodetic basis of G or g_e-set of G. The edge geodetic number of a graph was studied in [8] and further studied in [9]. For any connected graph G, $2 \leq g_e(G) \leq p$. Any connected graph having exactly one universal vertex has edge geodetic number $p-1$. A decomposition π of a graph G is a collection of edge- disjoint subgraphs $G_1, G_2, \ldots, G_n$ of G such that every edge of G belongs to exactly one $G_i, (1 \leq i \leq n)$. The decomposition $\pi = \{G_1, G_2, \ldots, G_n\}$ of a connected graph G is said to be a distinct edge geodetic decomposition if $g_e(G_i) \neq g_e(G_j), (1 \leq i \neq j \leq n)$. The maximum cardinality of π is called the distinct edge geodetic decomposition number of G and is denoted by $\pi_{dg_e}(G)$, where $g_e(G)$ is the edge geodetic number of G. A graph G is said to be distinct edge geodetic decomposable graph if $\pi_{dg_e}(G) \geq 2$. The decomposition $\pi = \{G_1, G_2, \ldots, G_n\}$ of a connected graph G is said to be an edge geodetic self decomposition if $g_e(G_i) = g_e(G), (1 \leq i \leq n)$. The maximum cardinality of π is called the edge geodetic self decomposition number of G and is denoted by $\pi_{sg_e}(G)$, where $g_e(G)$ is the edge geodetic number of G. A graph G is said to be an edge geodetic self decomposable graph if $\pi_{sg_e}(G) \geq 2$. The concepts of decomposition were recently studied in [4, 5, 6, 7]. A sequence is a list of objects (or events) which have been ordered in a sequential fashion; such that each member either comes before, or after, every other member. Generating functions provide a natural and elegant way to deal with sequences of numbers by associating a function of a continuous variable with a sequence. In this way generating functions provide a bridge between discrete and continuous mathematics. Every edge geodetic decompossable graph has a sequence of edge geodetic number [6]. This concept motivate us to form the edge geodetic generating function. In this paper we assume the sequence is strictly increasing. The following theorems are used in sequel.

2. Preliminaries

Theorem 2.1. [6] *For any partition $n_1 < n_2 < n_3 < \cdots < n_k$ $(2 \leq n_i \leq p-2)$ of q there exists a graph G of order p and size q such that G has a distinct edge geodetic decomposition $\pi = \{G_1, G_2, \cdots, G_k\}$, where $g_e(G_i) = n_i (1 \leq i \leq k)$ and $p - q = 1$.*

Theorem 2.2. [6] *For any connected graph G, $\pi_{dg_e}(G) = p-2$ $(p \geq 4)$ if G has at least $p-2$ universal vertices.*

Theorem 2.3. [6] *For any connected graph G with $p \geq 4$, $1 \leq \pi_{dg_e}(G) \leq p-2$.*

Theorem 2.4. [8] *For the connected graph $G = K_{1,n}$, $g_e(G) = n$.*

3. Edge geodetic sequence in graphs

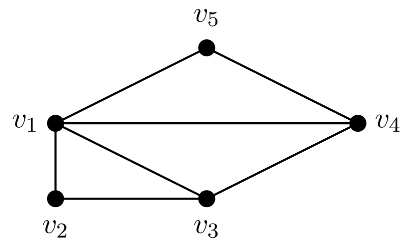
Definition 3.1. The decomposition $\pi = \{G_1, G_2, \ldots, G_n\}$ of a connected graph G is said to be a distinct edge geodetic decomposition if $g_e(G_i) \neq g_e(G_j), (1 \leq i \neq j \leq n)$. A sequence $\{g_i | i = 1, 2, \ldots\}$ is said to be an edge geodetic sequence of G if $g_i = g_e(G_i)$ for all $G_i \in \pi$, where $g_e(G_i)$ is the edge geodetic number of $G_i (1 \leq i \leq n)$.

Definition 3.2. The generating function

$$G(x) = \sum_{i=1}^{n} g_i x^{i-1},$$

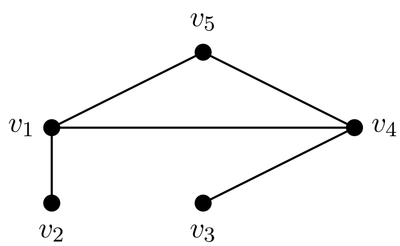
is called an edge geodetic generating function of G if $\{g_i\}$ is an edge geodetic sequence of G.

Example 3.3. For the graph G given in Figure 3.1, G_1 and G_2 [given in Figure 2.1(a) and Figure 2.1(b)] is a decomposition of G. Since $g_1(G_1) = 3$ and $g_1(G_2) = 2$, $\pi = \{G_1, G_2\}$ is a distinct edge geodetic decomposition of G. It can be easily verified that there is no other distinct edge geodetic decomposition of cardinality greater than 2. Therefore $\pi_{dg_e}(G) = 2$. Hence the edge geodetic sequence of G is $\{2, 3\}$. Hence the generating function $G(x)$ of G is $2 + 3x$.



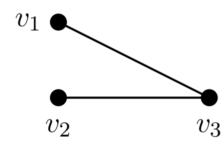
G

Figure 3.1



G_1

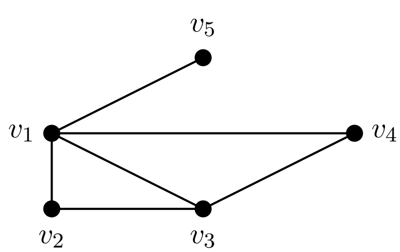
Figure 3.1(a)



G_2

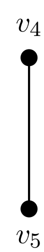
Figure 3.1(b)

Remark 3.4. *A graph G can have more than one edge geodetic sequence and consequently it may have more than one edge geodetic generating function. For the graph G given in Figure 3.1, G_3 and G_4 [given in Figures 3.1(c) and 3.1(d)] is a decomposition of G. Since $g_e(G_3) = 4$ and $g_e(G_4) = 2$, $\pi_2 = \{G_3, G_4\}$ is also a distinct edge geodetic decomposition of G. Hence the edge geodetic sequence G is $g_n = 2, 4$ and therefore the edge geodetic generating function of G is $G(x) = 2 + 4x$.*



G_3

Figure 3.1(c)



G_4

Figure 3.1(d)

Definition 3.5. The edge geodetic decomposition $\pi = \{G_1, G_2, ..., G_n\}$ of $G = (V, E)$ is said to be an edge geodetic partition if $\sum_{i=1}^{n} g_e(G_i) = q$

Remark 3.6. An edge geodetic decomposition need not be an edge geodetic partition of G. For the graph G given in Figure 3.1, $g_e(G_1) + g_e(G_2) = 5 \neq q$ so that $\pi = \{G_1, G_2\}$ is not an edge geodetic partition of G.

Theorem 3.7. *Let $G = (V, E)$ be a distinct edge geodetic decomposable graph. Then an edge geodetic generating function $G(x)$ is unique if edge geodetic decomposition is a partition of q, where $q = |E|$.*

Proof. Let $\pi_1 = \{G_1, G_2, \ldots, G_n\}$ be a partition of G. Then we have to prove G has unique edge geodetic generating function. It is enough to prove that G has unique edge geodetic sequence. Suppose that G has another edge geodetic sequence for an edge geodetic decomposition $\pi_2 = \{H_1, H_2, \ldots, H_m\}$. Then $\sum_{i=1}^{n} g_e(G_i) = q$ and $\sum_{j=1}^{m} g_e(H_j) = q$. Hence $\sum_{i=1}^{n} g_e(G_i) = \sum_{j=1}^{m} g_e(H_j)$, which implies that $m = n($ if $m < n$ then π_2 cannot be a maximum decomposition). If $G_i \cong H_j$ for all i, j $(1 \le i \le n), (1 \le j \le m)$, then we are done. Suppose that there exists at least one $G_i \in \pi_1$ and $H_j \in \pi_2$ such that $G_i \ncong H_j$ (for some i and j) and $g_e(G_i) = g_e(H_j) (i \neq j)$. Hence $\mid E(G_i) - E(H_j) \mid \le 1$ (otherwise sum will be greater than q). Without loss of generality, assume that $\mid E(G_i) \mid \ge \mid E(H_j) \mid$. Suppose that $\mid E(G_i) - E(H_j) \mid = 0$ then $\mid E(G_i)| = |E(H_j) \mid$ which implies $G_i = K_3$ and $H_j = K_{1,3}$ and vice versa. If $\mid E(G_i) - E(H_j) \mid = 1$(for some i and j), then one of the elements of π_1 can be K_2 and exactly one element of π_1 (say), $G_r (1 \le r \le n)$ such that $\mid E(G_r) \mid = 1 + g_e(G_r)$. Thus the edge geodetic sequence of G from π_1 and π_2 are equal so that $G(x)$ is unique. □

Theorem 3.8. *Let $G = (V, E)$ be a distinct edge geodetic decomposable graph with edge geodetic generating function $G(x)$. Then $G(1) = q$ if and only if an edge geodetic decomposition is the partition of $q = |E|$.*

Proof. Let $\pi(G) = \{G_1, G_2, \ldots, G_n\}$ be an edge geodetic decomposition of G. Suppose that $G(1) = q$. We have

$$\begin{aligned} G(x) &= \sum_{i=1}^{n} g_i x^{i-1} \\ &= \sum_{i=1}^{n} g_e(G_i) x^{i-1}, \\ \Rightarrow G(1) &= \sum_{i=1}^{n} g_e(G_i) \\ \Rightarrow \sum_{i=1}^{n} g_e(G_i) &= q. \end{aligned}$$

Therefore π is an edge geodetic partition of G. The converse is clear. □

Theorem 3.9. *Let T be a star of size q. Then the distinct edge geodetic decomposition sequence of T is a Fibonacci sequence if and only if q is either*

$$\sum_{i=1}^{n}\left[\left(1+\frac{2}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{i-1}+\left(1-\frac{2}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{i-1}\right],$$

or

$$\left\{\sum_{i=1}^{n}\left(1+\frac{2}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{i-1}+\left(1-\frac{2}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{i-1}\right\}-1.$$

Proof. Suppose that the edge geodetic sequence of T is a Fibonacci sequence. Since $2 \leq g_e(G) \leq p$, $\{g_n\} = 2, 3, 5, \ldots$ is the Fibonacci sequence of T. Then the sequence satisfies the recurrence relation

$$\begin{gathered} q_n + q_{n+1} = q_{n+2} \quad (n \geq 1), \\ q_{n+2} - q_{n+1} - q_n = 0, \end{gathered} \tag{1}$$

with the initial conditions $q_1 = 2$ and $q_2 = 3$. Let $q_n = A{r_1}^{n-1} + B{r_2}^{n-1}$ be a solution of the given equation(1).
$\therefore$ The characteristic equation of (1) is

$$r^2 - r - 1 = 0. \tag{2}$$

Solving the quadratic equation(2) and by using initial values we can get

$$\begin{aligned} q_n &= \left(1+\frac{2}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}+\left(1-\frac{2}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}, \\ \therefore G(x) &= \sum_{i=1}^{n} q_i x^{i-1} \quad (\because g_e(G_i) = q_i \quad if \quad G_i \neq K_2 \quad for \quad all \quad i) \\ &= \sum_{i=1}^{n}\left[\left(1+\frac{2}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{i-1}+\left(1-\frac{2}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{i-1}\right]x^{i-1}. \end{aligned} \tag{3}$$

where n is the number of terms in the sequence. For $G_i \neq K_2$

$$q = \sum_{i=1}^{n}\left[\left(1+\frac{2}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{i-1}+\left(1-\frac{2}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{i-1}\right].$$

If $G_i = K_2(1 \leq i \leq n)$, then $g_e(G_i) - 1 = q_i$.
Since $G_j \neq K_2(1 \leq i \neq j \leq n)$,

$$q = \left\{\sum_{i=1}^{n}\left[\left(1+\frac{2}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{i-1}+\left(1-\frac{2}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{i-1}\right]\right\}-1.$$

Conversely suppose that q is either

$$\sum_{i=1}^{n}\left[\left(1+\frac{2}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{i-1}+\left(1-\frac{2}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{i-1}\right],$$

or

$$\left\{\sum_{i=1}^{n}\left(1+\frac{2}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{i-1}+\left(1-\frac{2}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{i-1}\right\}-1.$$

Case(i): Let

$$q=\sum_{i=1}^{n}\left[\left(1+\frac{2}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{i-1}+\left(1-\frac{2}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{i-1}\right].$$

Then q can be partitioned as $2,3,5,8,\ldots,q_n$ where

$$q_n=\left(1+\frac{2}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{i-1}+\left(1-\frac{2}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{i-1}.$$

It is clear that $\{q_n\}=2,3,5,8,\ldots,q_i$ is a Fibonacci sequence. since $2<3<5<\cdots<q_{p-2}$, by Theorem 2.1, there exists a star T such that T has a decomposition $\pi=\{G_1,G_2,\ldots,G_n\}$(by Theorem 2.3, $n\le p-2$), where

$$(4)\qquad g_e(G_i)=q_i\left(1+\frac{2}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{i-1}+\left(1-\frac{2}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{i-1},\qquad (1\le i\le n),$$

and $p-q=1$. Assume that p is sufficiently large and since $g_e(G_i)\neq g_e(G_i)$, $\pi=\{G_1,G_2,...,G_n\}$ is a distinct edge geodetic decomposition sequence of T.

Case(ii): Let us consider the sequence $\{1,3,5,8,\ldots\}$=$\{2,3,5,8,\ldots\}-1$, then from the equation (3)

$$G_1(x)=G(x)-1=\left[\sum_{i=1}^{n}q_ix^{i-1}\right]-1,$$

where $G_1(x)$ is the corresponding generating function of the sequence. Then by Theorem 3.8, $G_1(1)=q=$

$$\left[\sum_{i=1}^{n}\left(1+\frac{2}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{i-1}+\left(1-\frac{2}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{i-1}\right]-1.$$

As in the case(i), $\pi=\{G_1,G_2,\ldots,G_{p-2}\}$, where $G_1=K_2,G_2=K_{1,3},G_3=K_{1,5},\ldots,G_{p-2}=K_{1,q_n}$ be a decomposition of G. Since $g_e(G_1)=2,g_e(G_2)=3,g_e(G_3)=5,g_e(G_3)=8,\ldots,$ π is a distinct edge geodetic decomposition of T and the sequence $\{2,3,5,8,\ldots\}$ is a Fibonacci sequence. Hence the result holds. □

Theorem 3.10. *Let $G = (V, E)$ be an edge geodetic decomposable graph of size q and the decomposition is the partition of q. Then the edge geodetic sequence is an arithmetic sequence if and only if*

$$q = \frac{n}{2}[2a + (n-1)d] \quad where \quad a \geq 2, \quad n \leq p-2. \tag{5}$$

Proof. Let $\pi = \{G_1, G_2, \ldots, G_n\}$ be an edge geodetic decomposition of G and the edge geodetic sequence$\{g_e(G_1), g_e(G_2), \ldots, g_e(G_n)\}$ be an arithmetic sequence so that sequence is of the form $\{a, a+d, a+2d, \ldots, a+(n-1)d\}$. Since the decomposition is the partition of G,

$$\sum_{i=1}^{n} g_e(G_i) = q.$$

Then it is clear that

$$q = \frac{n}{2}[2a + (n-1)d] \quad where \quad a \geq 2, \quad n \leq p-2.$$

Conversely suppose that

$$q = \frac{n}{2}[2a + (n-1)d] \quad where \quad a \geq 2, \quad n \leq p-2.$$

Then q can be partitioned as $\{n_1 = a, n_2 = a+d, n_3 = a+2d, ..., n_{p-2} = a+(n-1)d\}$. Moreover $n_1 < n_2 < n_3 < \cdots < n_{p-2}$. Then by Theorem 2.1, there exists a graph G which has distinct edge geodetic decomposition $\pi = \{G_1, G_2, \ldots, G_n\}(n \leq p-2)$ such that $g_e(G_i) = n_i$ and

$$\sum_{i=1}^{n} g_e(G_i) = q.$$

Thus the distinct edge geodetic decomposition is the partition of q and the sequence $\{n_i\}$ is arithmetic. □

Theorem 3.11. *Let G be a distinct edge geodetic decomposable graph with edge geodetic generating function $G(x)$. Then the sum of the edge geodetic number is the coefficient of x^{n-1} in the expansion of*

$$\frac{G(x)}{1-x}. \tag{6}$$

Proof. Let $\pi(G) = \{G_1, G_2, \ldots, G_n\}$ be an edge geodetic decomposition of G and $g_i = g_e(G_i)(1 \leq i \leq n)$. Then

$$\begin{aligned} G(x) &= \sum_{i=1}^{n} g_e(G_i)x^{i-1} \\ &= \sum_{i=1}^{n} g_i x^{i-1} \\ &= g_1 + g_2x + \cdots + g_nx^{n-1} \\ \therefore \frac{G(x)}{1-x} &= [g_1 + g_2x + \cdots + g_nx^{n-1}][1 + x + x^2 + \cdots + x^{n-1} + \cdots]. \end{aligned}$$

Now the coefficient of x^{n-1} is

$$\begin{aligned} &= g_1 + g_2 + \cdots + g_n \\ &= \sum_{i=1}^{n} g_e(G_i). \end{aligned}$$

Hence the proof is completed. □

Theorem 3.12. *Let G be a distinct edge geodetic decomposable graph of order p with at least $p-2$ universal vertices and $g_e(G_i) \neq p$ for all $G_i \in \pi$. Then*

$$G(x) = \frac{2-x}{(1-x)^2}, \tag{7}$$

for infinitely many p and $|x| < 1$.

Proof. Let G be a distinct edge geodetic decomposable graph with at least $p-2$ universal vertices. Then by Theorem 2.2, $\pi_{dg_e}(G) = p-2$. Since $2 \leq g_e \leq p$ and $g_e(G_i) \neq p$ for all $G_i \in \pi$, $g_e(G_i) = g_i = i + 2(0 \leq i \leq p-4)$ for all $G_i \in \pi$. Then

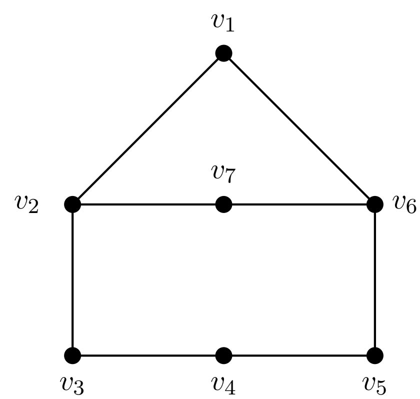
$$\begin{aligned} G(x) &= 2 + 3x + 4x^2 + \cdots + (p-2)x^{p-4} \\ &= \frac{1}{x}\left\{2x + 3x^2 + 4x^3 + \cdots + (p-2)x^{p-3}\right\} \\ &= \frac{1}{x}\left\{1 + 2x + 3x^2 + 4x^3 + \cdots + (p-2)x^{p-3} - 1\right\} \\ &= \frac{1}{x}\left\{\frac{1}{(1-x)^2} - 1\right\}, as \quad p \to \infty \\ &= \frac{1}{x}\left\{\frac{2x - x^2}{(1-x)^2}\right\} - 1 \\ &= \frac{2-x}{(1-x)^2}, |x| < 1. \end{aligned}$$

□

4. Edge geodetic constant sequence of graphs

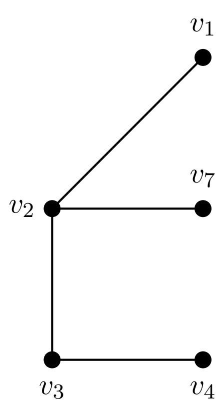
Definition 4.1. A sequence $\{g_i | i = 1, 2, \ldots\}$ is said to be an edge geodetic constant sequence of G if $g_i = g_e(G_i)$ for all $G_i \in \pi$, where $g_e(G_i)$ is the edge geodetic number of $G_i (1 \leq i \leq n)$ and π is an edge geodetic self decomposition of G.

Example 4.2. *For the graph G given in Figure 4.1, G_1 and G_2 [given in Figure 4.1(a) and Figure 4.1(b)] is a decomposition of G. Since $g_e(G_1) = g_e(G_2) = g_e(G) = 3$, $\pi = \{G_1, G_2\}$ is an edge geodetic self decomposition of G. It is easily verified that there is no edge geodetic self decomposition of cardinality more than 3. Therefore $\pi_{sg_e}(G) = 2$.* Hence the edge geodetic sequence of G is $\{3, 3\}$. Hence the generating function $G(x)$ of G is $3 + 3x$.



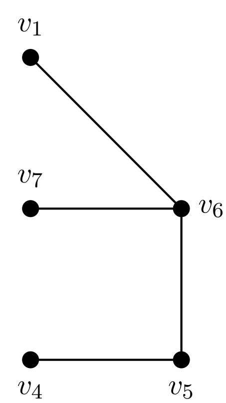
G

Figure 4.1



G_1

Figure 4.1(a)



G_2

Figure 4.1(b)

Theorem 4.3. *Let G be an edge geodetic self decomposable graph with edge geodetic number $g_e(G)$. Then $G(x)$ is converges to*

$$\frac{g_e(G)}{(1-x)},$$

if $|x| < 1$ *and diverges if* $|x| \geq 1$.

Proof. Let G be an edge geodetic self decomposable graph with edge geodetic number $g_e(G)$ and $\pi = \{G_1, G_2, \ldots, G_n\}$ be an edge geodetic self decomposition of G. Then $g_e(G) = g_e(G_i)(1 \leq i \leq n)$. Thus $g_i = g_e(G_i)$. Moreover

$$\begin{aligned} G(x) &= \sum_{i=0}^{n} g_e(G_i)x^i \\ &= g_e(G)\sum_{i=0}^{n} x^i \qquad \because g_e(G_i) = g_e(G) \\ &= \frac{g_e(G)}{(1-x)}, for \quad infinitely \quad many \quad p \quad and \quad |x| < 1 \end{aligned}$$

$(\because n \leq p - 2)$.

□

Theorem 4.4. *Let* G *be an edge geodetic self decomposable graph of size* q. *Then*

$$G(x) = 2\frac{(1-x^{q+1})}{(1-x)}, |x| \neq 1,$$

if and only if $g_e(G) = 2$.

Proof. Suppose that

$$G(x) = 2\frac{(1-x^{q+1})}{(1-x)}, |x| \neq 1.$$

Let $g_e(G) = k(k \geq 3)$ and $\pi = \{G_1, G_2, \ldots, G_n\}$ be the edge geodetic self decomposition of G. Then $g_e(G_i) = k(1 \leq i \leq n)$ and $\pi_{sg_e}(G) \leq \frac{q}{k}$.

$$G(x) = 2\frac{(1-x^{\frac{q}{k}+1})}{(1-x)}, |x| \neq 1,$$

which is a contradiction. Therefore $g_e(G) = 2$. Conversely let $g_e(G) = 2$. Then $\pi = \{G_1, G_2, \ldots, G_q\}$ where $G_i = K_2(1 \leq i \leq n)$ is the unique edge geodetic self decomposition of G. Hence the edge geodetic sequence is $\{g_n\} = 2, 2, \ldots, 2$ (q terms). Thus

$$\begin{aligned} G(x) &= \sum_{i=0}^{q} g_e(G_i)x^i \\ &= 2\frac{(1-x^{q+1})}{(1-x)}, |x| \neq 1. \end{aligned}$$

□

Theorem 4.5. *The edge geodetic self decomposition* π *is an edge geodetic partition if* π *is a star decomposition.*

Proof. Let $\pi = \{G_1, G_2, \ldots, G_n\}$ be an edge geodetic self decomposition of G and π is a star decomposition of G. Assume that $g_e(G) = m$. Then $g_e(G_i) = m$ and hence $G_i = K_{1,m}(1 \leq i \leq n)$. Thus $g_e(G_i) = |E(G_i)|$. Moreover

$$q = \sum_{i=1}^{n} g_e(G_i),$$

so that π is an edge geodetic partition of G. □

Corollary 4.6. *The edge geodetic self decomposition of complete bipartite graph $G = K_{m,n}(2 < m \leq n)$ is an edge geodetic partition of G.*

Proof. Let $V = \{v_1, v_2, \ldots, v_m\}$ and $W = \{w_1, w_2, \ldots, w_n\}(2 < m \leq n)$ be the partition of $V(G)$. Then $g_e(G) = m$ and G can be decomposed as $\pi = \{G_1, G_2, \ldots, G_n\}$, where $G_i = K_{1,m}(1 \leq j \leq n)$ with rooted vertex $w_i(1 \leq j \leq n)$. Then by Theorem 4.5, π is an edge geodetic partition of G. □

Definition 4.7. Let $\pi = \{G_1, G_2, \ldots, G_n, \ldots\}$ be an edge geodetic decomposition of G and $\{g_i = g_e(G_i)|G_i \in \pi\}$ be an edge geodetic sequence of G. Then G is said to be recurrence graph if each G_i is obtained from G_{i-1} recursively.

Theorem 4.8. *For every generating function*

$$G(x) = \sum_{i=1}^{\infty} a^{i-1}x^{i-1}, \qquad a \in N - \{1\},$$

there exists a connected recurrence graph G with edge geodetic decomposition $\pi = \{G_1, G_2, \ldots, G_n, \ldots\}$ such that $G_{i+1} = K_1 + \left[(g_e^{i-1} - 1)K_1 \cup G_i\right](i \geq 1), G_1 = K_1 + K_1$, if $g_e = 2$ and $G_{i+1} = K_1 + \left\{\left[g_e^i(g_e - 1) - 1\right] K_1 \cup G_i\right\}(i \geq 1), G_1 = K_1 + g_eK_1$, $g_e \neq 2.$, where $g_e(G_1) = a$.

Proof. Suppose that $g_e = a \neq 2$. Let $G_1 = K_1 + g_eK_1 = K_{1,a}$ and $\{u_1, v_1, v_2 ..., v_a\}$ be the set of $a + 1$ vertices of G_1. Then by Theorem 2.4, $g_e = a$. The graph G_2 is obtained from G_1 by adding new $(a^2 - a)$ vertices $\{u_2, v_{a+1}, v_{a+2}, \ldots, v_{a^2}\}$ and introduce new edges $u_2u_1, u_2v_i(1 \leq i \leq a^2)$. Thus $G_2 = K_1 + \left\{(a^2 - (a + 1))K_1 \cup G_1\right\} = K_1 + \{[g_e(g_e - 1) - 1] K_1 \cup G_1\}$. Moreover G_2 can be decomposed as $\pi = \{G_1 = K_{1,a}, K_{1,a^2}\}$. Hence by Theorem 2.4, we obtain the edge geodetic sequence $\left\{a, a^2\right\}$. Assume the result is true for $i = k$. That is $G_k = K_1 + \left\{\left[g_e^{k-1}(g_e - 1) - 1\right] K_1 \cup G_{k-1}\right\}$ and G_k can be decomposed with the edge geodetic sequence $\left\{a, a^2, a^3, \ldots, a^{k+1}\right\}$. Obtain the graph G_{k+1} from G_k by adding new $(a^{k+1} - a^k)$ vertices $\{u_{k+1}, v_{a^k+1}, v_{a^k+2}, \ldots, v_{a^{k+1}}\}$ and introduce the new edges $u_{k+1}u_1, u_{k+1}u_2, \ldots, u_{k+1}u_k, u_{k+1}v_1, u_{k+1}v_2, \ldots, u_{k+1}v_{a^{k+1}}$. Hence

$$G_{k+1} = K_1 + \left\{ \left[g_e^k(g_e - 1) - 1 \right] K_1 \cup G_k \right\}.$$

Moreover G_{k+1} can be decomposed as $\pi = \{K_{1,a}, K_{1,a^2}, \ldots, K_{1,a^k}, K_{1,a^{k+1}}\}$ with rooted vertices $u_1, u_2, \ldots, u_{k+1}$ respectively. This implies that the edge geodetic sequence of G_{k+1} is $\left\{a, a^2, a^3, \ldots, a^{k+1}\right\}$. Hence by mathematical induction the result is true for all i and $a \neq 2$. If $a = 2$, starting with $G_1 = K_1 + K_1$ and follow the above procedure we obtain $G_{i+1} = K_1 + \left[(g_e^{i-1} - 1)K_1 \cup G_i\right]$. □

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