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Research Paper

NON-COMMUTATIVE HYPERGROUPOID OBTAINED FROM SIMPLE GRAPHS

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Abstract. The purpose of this paper is the study of non-weak commutative hypergroups associated with hypergraphs. In this regards, we construct a hyperoperation on the set of vertices of hypergraph and obtain some results and characterizations of them. Moreover, according to this hyperoperation, we investigate conditions under which the hypergroupoid is a join space hypergroup. Finally, we present an application to marketing social network.

1. Introduction

The concept of graph theory had been introduced by Euler in 1736. The graph theory is an absolute useful tool to describe the connections between members of a discrete set and it is applied in many different fields such as computer science, optimization, economics, number theory, geometry, topology.

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The notion of graph should have been generalized to solve new and complex problems and hence Berg had introduced the notion of hypergraph as generalization of graph around 1960. Hypergraphs can connect a set of members through a hyperedge, while each edge in simple graph can connect only two. This implies that simple graphs are a subset of hypergraphs. Hence some properties of hypergraphs are generalization of simple graphs. The concept of hypergraph has considered as a very useful piece of equipment to analyze and model complex structures in different fields of computer science, optimization problems and discrete mathematics. The presentation and properties regarding graphs and hypergraphs are available in [1, 2].

Marty had taken first step in hyperstructures through presentation of hypergroup in 1934 [14]. Algebraic hyperstructures represent a proper generalization of classical algebraic structures. While in classical structure the composition of two items is one item, in hyperstructures the composition of two items is a set. Hyperstructures have been taken into consideration by many researchers. Some results in this area is available in [4, 6, 7, 17]. In 1990 Vougiouklis introduced the notion of H_v-structures as a generalization of well-known algebraic hyperstructures such as semihypergroups, hypergroups, hyperrings and so on [16]. H_v-structures satisfy the weak axioms where the non-empty intersection replaces the equality.

Analyzing the connections among graph theory, hypergraphs, binary relations and algebraic structures has been considered in last centuries. Arthur Cayley had presented the definition of Cayley graph in 1878 in order to encode the abstract structure of a group by using a set of generators. It was first notable graph associated with a group as an algebraic structure and meanwhile Cayley graphs have been developed in algebraic graph theory.

In last decades, connections between graph theory and hyperstructures have been noticed by many researchers (see for instance [8, 9, 3, 11, 15, 12]). By considering Γ as a hypergraph, Corsini constructed and presented a hypergroupoid H_Γ associated with Γ which was commutative and called it hypergraph hypergroupoid [3]. Sufficient conditions were found to make H_Γ a hypergroup and it was proven that the hypergroup H_Γ can be a join space under specific conditions. Iranmanesh and Iradmusa constructed a hypergroup associated with a hypergraph by defining a hyperoperation which was called *PHO* [12]. Farshi, Davvaz and Mirvakili defined special ρ-relationship on the set of vertices in hypergraph and constructed a ρ-hypergroup [9]. Further they constructed a degree hypergroupoid via defining an operation on the set of degrees of vertices of a hypergraph which was a H_v-group. Sufficient conditions were found to make degree hypergroupoid a degree hypergroup [8]. Nikkhah, Davvaz and Mirvakili constructed another degree hypergroupoid which was a H_v-group and they investigated some properties to have a hypergroup [15]. Hamidi and Broumand considered the connections between hypergraphs and hypergroupoids and redefined the concept of hypergraph via the concept of hypergroupoid [11].

Researches in this area is not limited to these cases, and other enthusiasts have conducted some other researches. The point is, all hyperstructures were associated to graphs and hypergraphs were commutative or weak commutative, while non-commutative structures are a huge part of hyperstructures. Regarding all these, in this article we aim to present a hyperstructure associated to a hypergraph which is not commutative. In this regard, we consider a hypergraph and construct a hyperoperation on the set of its vertices. This hyperoperation make a system on the set of vertices and we analyze its properties such as associativity, commutativity, transposition axiom etc. Finally, we present a non-weak commutative H_v-group associated with a specific hypergraph.

2. Preliminaries

In this section we recall all basic definitions and results we require of hyperstructures and hypergraphs. Let H be a non-empty set and $\mathcal{P}^*(H)$ be the set of all non-empty subsets of H and $H \times H$ be the Cartesian product of H. In general, a *hyperoperation* $\circ$ on H is a map from $H \times H$ to $\mathcal{P}^*(H)$. More exactly, for all x, y of H, we have $x \circ y \subseteq H$. $x \circ y$ is called the *hyperproduct* of x and y. If $x \in H$ and A is a subset of H, then by $x \circ A$ we mean $x \circ A = \bigcup_{y \in A} x \circ y$. The hyperproduct of elements $x_1, \dots, x_n$ of H is denoted by $\prod_{i=1}^{n} x_i$ and is equal to $x_1 \circ \prod_{i=2}^{n} x_i$. An algebraic system $(H, \circ)$ endowed with a hyperoperation is called a *hypergroupoid*. A *semihypergroup* is a hypergroupoid $(H, \circ)$ where $\circ$ is associative i.e. for all x, y, z of H we have $x \circ (y \circ z) = (x \circ y) \circ z$. A semihypergroup $(H, \circ)$ is called a *hypergroup* whenever reproductive axiom is valid i.e., $x \circ H = H \circ x = H$, for all $x \in H$. A hypergroup is called *commutative* if $x \circ y = y \circ x$ for all $x, y \in H$. A hypergroupoid is called an H_v*-group* whenever reproductive axiom is valid and $x \circ (y \circ z) \cap (x \circ y) \circ z \neq \emptyset$. A *join space* is a commutative hypergroup $(H, \circ)$ such that the following condition holds for all $a, b, c, d \in H$:

$$a/b \cap c/d \neq \emptyset \Longrightarrow a \circ d \cap b \circ c \neq \emptyset,$$

where $a/b = \{x \in H \mid a \in x \circ b\}$. But in non commutative hypergroups, hyperoperation has given another inverse in addition to a/b: $b \backslash a = \{x \in H \mid a \in b \circ x\}$. So a non-commutative join space or transposition hypergroup is a hypergroup where for all $a, b, c, d \in H$ following condition holds[13]:

$$b \backslash a \cap c/d \neq \emptyset \Longrightarrow a \circ d \cap b \circ c \neq \emptyset.$$

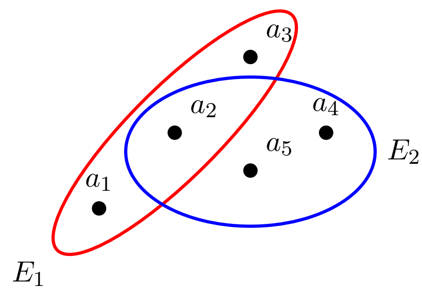


FIGURE 1. An example of hypergraph with 2 hyperedges.

A *hypergraph* is a pair $\Gamma = (H, E = \{E_i\}_{i=1}^{n})$, where H is a finite set of vertices and $E = \{E_1, \ldots, E_m\}$ is a set of hyperedges which are non-empty subsets of H such that $\bigcup_{i=1}^{m} E_i = H$. If for all $1 \leq i \leq m$, $|E_i| = 2$ and if the hypergraph is simple, the hypergraph becomes an ordinary simple graph and if there exist i such that $E_i = 1$, it is considered as a loop in a graph. Figure 1 is an example of a hypergraph with 2 hyperedges $E_1 = \{a_1, a_2, a_3\}$ and $E_2 = \{a_2, a_4, a_5\}$.

Let $\Gamma = (H, E)$ be a hypergraph and $x, y \in H$. A hyperedge sequence $(E_1, \ldots, E_k)$ is called a path of length k from x to y if the following conditions are satisfied:

(1) $x \in E_1$ and $y \in E_k$,
(2) $E_i \neq E_j$ for $i \neq j$,
(3) $E_i \cap E_{i+1} \neq \emptyset$ for $1 \leq i \leq k-1$.

In a hypergraph, two vertices x and y are called connected if contains a path from x to y. The vertices are called adjacent by a single hyperedge. We use x—y to denote the adjacency of vertices x and y. $x \not\!\!-\!\!- y$ is also used whenever x and y are not adjacent to each other. A hypergraph is said to be connected if every pair of vertices in the hypergraph is connected. The degree of a vertex is the number of hyperedges which contains the vertex and is shown by $deg(x)$ $(deg(x) = \{E_i / x \in E_i\})$. For example in Figure 1, $deg(a_2) = 2$ and $deg(a_1) = 1$. The length of shortest path between vertices x and y is denoted by $d(x, y)$ and the diameter of Γ is defined as follows:

$$diam(\Gamma) = \begin{cases} \max\{d(x,y)\}, & if\ \Gamma\ is\ connected, \\ \infty, & otherwise. \end{cases}$$

In [3], Corsini considered a hypergraph $\Gamma = (H, \{E_i\}_i)$ and defined a hyperoperation $\circ$ on H as follows:

$$\forall x, y \in H^2,\ x \circ y = E(x) \cup E(y),$$

where $E(x) = \bigcup_{x \in E_i} E_i$. The hypergroupoid $H_\Gamma = (H, \circ)$ is called a hypergraph hypergroupoid or an h.g. hypergroupoid. An associative h.g hypergroup is called an h.g. hypergroup. Corsini proved that:

Theorem 2.1. [3] *The hypergroupoid H_Γ has the following properties for each $(x, y) \in H^2$:*

(1) $x \circ y = x \circ x \cup y \circ y$,
(2) $x \in x \circ x$,
(3) $y \in x \circ x \Longleftrightarrow x \in y \circ y$.

Also, he proved that:

Theorem 2.2. [3]

(1) *A hypergroupoid $(H, \circ)$ satisfying (1), (2) and (3) of Theorem 2.1 is a hypergroup if and only if the following condition is valid:*

$$\forall (a, c) \in H^2, c \circ c \circ c - c \circ c \subseteq a \circ a \circ a.$$

(2) *A hypergroup $(H, \circ)$ satisfying (1), (2) and (3) of Theorem 2.1 is a join space.*

3. Special hyperoperation

By using adjacency and paths in a graph, we can connect hypergraphs to hypergroups.

Definition 3.1. Let $\Gamma = (H, \{E_i\}_i)$ be a hypergraph. Define the hyperoperation $\circ$ on H as follows:

$$\forall (x, y) \in H^2, \quad x \circ y = E(x) \cup E^c(y),$$

where $E(x) = \bigcup_{x \in E_i} E_i$ and $E^c(y) = H \setminus E(y)$.

Example 3.2. Consider the hypergraph in Figure 1 which is mentioned that $E_1 = \{a_1, a_2, a_3\}$ and $E_2 = \{a_2, a_4, a_5\}$, then Table 1 is obtained.

$\circ$	a_1	a_2	a_3	a_4	a_5
a_1	H	E_1	H	E_1	E_1
a_2	H	H	H	H	H
a_3	H	E_1	H	E_1	E_1
a_4	E_2	E_2	E_2	H	H
a_5	E_2	E_2	E_2	H	H

Table 1. Cayley table.

Notice that $x \in E(x)$ for every $x \in H$. This point is an important issue specially about simple, undirected and loopless graphs which we aim to study as an essential group of hypergraphs. Some of this graphs are provided in the next example.

Example 3.3. (1) Let $\Gamma = K_n$ be a complete graph with n vertices, then $(H, \circ)$ is a total hypergroup, i.e., for every $x, y \in H$ we have $x \circ y = H$ since $E(x) = H$. For instance you can see K_4 in Figure 2.

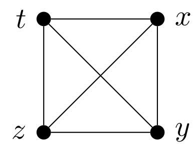


FIGURE 2. K_4.

(2) Let $\Gamma = C_4$ be a cycle graph with 4 vertices, then for every $x, y \in H$, if $x \neq y$ then $x \circ y = E(x)$ and $x \circ x = H$.

(3) Let $\Gamma = C_5$ be a cycle graph with 4 vertices, then for every $x, y \in H$,

$$x \circ y = \begin{cases} H, & x = y, \\ H \setminus \{z\}, & x \text{—} y \text{—} z \textit{ and } x \not\text{—} z, \\ E(x), & \textit{otherwise}. \end{cases}$$

(4) Let $\Gamma = P_3$ and $P_3 : x\text{—}y\text{—}z$, then for every $x, y \in H$, $x \circ y = E(x)$ if $x \neq y$ and $x \circ x = H$.

(5) Let $\Gamma = P_4$ and $P_4 : x\text{—}y\text{—}z\text{—}t$, then Table 2 is obtained.

$\circ$	x	y	z	t
x	H	$H \setminus \{z\}$	$E(x)$	$E(x)$
y	H	H	$E(y)$	$E(y)$
z	$E(z)$	$E(z)$	H	H
t	$E(t)$	$E(t)$	$H \setminus \{y\}$	H

TABLE 2. Cayley table for P_4.

Theorem 3.4. *Let H be the vertex set of a hypergraph Γ, then the hypergroupoid $\mathfrak{H} = (H, \circ)$ has the following properties for each $(x, y) \in H^2$:*

(1) $x \circ x = H$,

(2) $x \in x \circ y$,

(3) $y \in x \circ y$ *if and only if* $x\text{—}y$,

(4) *If* $x \not\!\!-\, z$ *and* $z \in x \circ y$ *then* $z \not\!\!-\, y$,

(5) $x \circ (x \circ x) = (x \circ x) \circ x$.

Proof. Omitted by obvious. □

Theorem 3.5. $\mathfrak{H} = (H, \circ)$ *is a quasihypergroup.*

Proof. It is easy to see that $x \circ H \subseteq H$ and $H \circ x \subseteq H$, for every $x \in H$. By part (1) of Theorem 3.4, we have $H \subseteq x \circ x \subseteq x \circ H \cup H \circ x$, where $x \in H$. □

Theorem 3.6. $\mathfrak{H} = (H, \circ)$ *is an* H_v*-group.*

Proof. Since $E(x) \subseteq x \circ (y \circ z) \cap (x \circ y) \circ z$. □

Lemma 3.7. *If* $diam(\Gamma) \leq 2$ *then* $x \circ y \cap y \circ x \neq \emptyset$*. Therefore* $\mathfrak{H} = (H, \circ)$ *is a weak commutative* H_v*-group.*

Proof. By Theorem 3.6, $\mathfrak{H} = (H, \circ)$ is an H_v-group. Now let $x, y \in H$, if $x — y$ then $\{x, y\} \subseteq E(x) \cap E(y)$ and so $x \circ y \cap y \circ x \neq \emptyset$. If $x \not\!\!-\, y$, then there exists $z \in H$ such that $x — z — y$, hence $z \in E(x) \cap E(y)$ and therefore $x \circ y \cap y \circ x \neq \emptyset$. □

Example 3.8. (1) Let $\Gamma = S_n$ be a star graph, $diam(\Gamma) = 2$ and $H = \{a, v_1, \ldots, v_n\}$ where a is universal vertex and $n \geq 2$, then for every $x, y \in H$,

$$x \circ y = \begin{cases} H, & x = y, \\ H, & x = a,\ y \neq a, \\ \{x, y\}, & x \neq a,\ y = a, \\ H \setminus \{y\}, & x \neq a \neq y,\ x \neq y. \end{cases}$$

Therefore $\mathfrak{H} = (H, \circ)$ is a weak commutative H_v-group.

(2) Let $\Gamma = W_n$ be a wheel graph, $diam(\Gamma) = 2$ and $H = \{a, v_1, \ldots, v_n\}$ where a is universal vertex and $n \geq 3$, then for every $x, y \in H$,

$$x \circ y = \begin{cases} H, & x = y, \\ H, & x = a,\ y \neq a, \\ E(x), & x \neq a,\ y = a, \\ H \setminus \{v_i \mid v_i \in E(y),\ v_i \not\!\!-\, x\}, & x \neq a \neq y,\ x \neq y. \end{cases}$$

So $\mathfrak{H} = (H, \circ)$ is a weak commutative H_v-group.

Lemma 3.9. $diam(\Gamma) = 1$ *if and only if* $\mathfrak{H} = (H, \circ)$ *is a commutative* H_v*-group.*

Proof. If $diam(\Gamma) = 1$, then for all $x, y \in H$ we have $x \circ y = H$ and so $\mathfrak{H} = (H, \circ)$ is a commutative H_v-group. Conversely, let $\mathfrak{H} = (H, \circ)$ be a commutative H_v-group and $diam(\Gamma) > 1$. Hence, there exist $x, y, z \in H$ such that $x — y — z$ and $x \not— z$. Therefore $z \in y \circ x$ and $z \notin x \circ y$. This means that $(H, \circ)$ is not commutative which is contradiction. □

Lemma 3.10. *If $diam(\Gamma) = 2$ then $(x \circ y) \circ z = H$ but in general $x \circ (y \circ z) \neq H$.*

Proof. Since $diam(\Gamma) = 2$ then $E(E(x)) = \cup_{t \in E(x)} E(t) = H$ and $(x \circ y) \circ z = H$ for every $x, y, z \in H$. But for example if $\Gamma = C_5$ as is shown in Figure 3, then $x \circ (y \circ z) = \{z, t, x, r\} \neq H$.

□

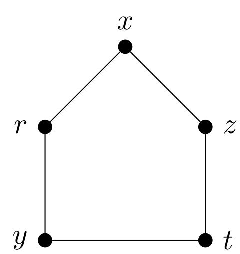


FIGURE 3. C_5.

Theorem 3.11. *$diam(\Gamma) = 1$ if and only if $\mathfrak{H} = (H, \circ)$ is a commutative hypergroup.*

Proof. If $diam(\Gamma) = 1$, then $(x \circ y) \circ z = H$ and $x \circ (y \circ z) = H$ for all $x, y, z \in H$ and commutativity is obvious by Lemma 3.9. Conversely when $\mathfrak{H} = (H, \circ)$ is a commutative hypergroup, it means $\circ$ is associative and commutative on H and it occurs when $diam(\Gamma) = 1$.

□

Theorem 3.12. *Let H be the vertex set of a hypergraph Γ. Then the H_v-group $\mathfrak{H} = (H, \circ)$ has the following properties for all $x, y, z \in H$:*

(1) $x \circ (x \circ y) = H$,
(2) $(x \circ y) \circ x = H$,
(3) *If $x — y$ or $x \not— z$, then $x \circ (y \circ z) = H$.*

Proof. (1) Since $E(x) \in x \circ (x \circ y)$ and $E^c(x) \in x \circ (x \circ y)$.

(2) and (3) can be proved in the same way. □

Theorem 3.13. (1) *If* $diam(\Gamma) = 2$*, then there exist* $x, y \in H$ *such that* $x \not\perp y$ *and* $x \circ (y \circ x) \neq H$ *while* $(x \circ y) \circ x = H$.

(2) *If* $diam(\Gamma) \geq 3$*, then there exist* $x, y \in H$ *such that* $(x \circ y) \circ y \neq H$ *while* $x \circ (y \circ y) = H$.

Proof. (1) $y \notin E(x)$ and $y \in y \circ x$, therefore $E^c(y) \in x \circ (y \circ x)$. These mean $y \notin x \circ (y \circ x)$, while $(x \circ y) \circ x = H$ by Lemma 3.10.

(2) Since $diam(\Gamma) \geq 3$, there exist $x, y \in H$ such that $d(x, y) \geq 3$. Since $x \circ y = H \setminus E(y)$, $z \not\perp y$ for every $z \in x \circ y$. Therefore $y \notin (x \circ y) \circ y$. □

As we know $x \circ x = H$ for every $x \in H$. Therefore we can define a relation on H:

Definition 3.14. [5] Let $(H, \circ)$ be a hypergroup. Then define a relation β on H as follows:

$$x \beta_n y \Longleftrightarrow \exists a_1, a_2, \ldots, a_n \in H, \text{ such that } \{x, y\} \in \prod_{i=1}^{n} a_i. \tag{1}$$

Let $\beta = \bigcup_n \beta_n$. Let β^* transitive closer of β.

In hypergroup β is an equivalence relation and so

Proposition 3.15. *Let* $\Gamma = (H, E)$ *and* $diam(\Gamma) = 1$*, then* β_Γ *and* β^* *coincide where* $\beta = \beta^*$.

Proof. Since $\mathfrak{H} = (H, \circ)$ is a hypergroup. □

Example 3.16. Let $\Gamma = K_n$, $diam(\Gamma) = 1$ and $\mathfrak{H} = (H, \circ)$ be a hypergroup for all $n \in \mathbb{N}$. If we define β_Γ on H as it is defined in relation 1, then β_Γ is an equivalence relation on H and it coincides with β^*.

The equivalence of the β relation in Hv-groups is an open problem. Here we do not have a hypergroup for $diam(\Gamma) \geq 2$, but the relation β_Γ is an equivalence relation.

Example 3.17. Let $\Gamma = C_5$, $diam(\Gamma) = 2$ and $\mathfrak{H} = (H, \circ)$ be an H_v-group. Then $\beta_\Gamma = H \times H$ because $x \circ x = H$ and so $\beta_\Gamma = \beta^*_\Gamma$.

Proposition 3.18. *Let* $\Gamma = (H, E)$ *and* $diam(\Gamma) \geq 2$*, then* $\beta_\Gamma = \beta^*_\Gamma$.

Proof. Since $x \circ x = H$ so for every $a, b \in H$ we have $a \beta_\Gamma b$ and so β_Γ is an equivalence relation.
□

Example 3.19. (1) Let $\Gamma = C_6$ then for every $x, y \in H$,

$$x \circ y = \begin{cases} H, & x = y, \\ H \setminus \{z\}, & x—y—z \text{ and } x \not— z, \\ H \setminus \{y, t\}, & x—z—y—t \text{ and } x \not— t, \\ E(x), & d(x, y) = 3. \end{cases}$$

Then $\mathfrak{H} = (H, \circ)$ is not weak commutative. When $d(x, y) = 3$ we have $E(x) \cap E(y) = \emptyset$. It means $x \circ y \cap y \circ x = \emptyset$.

(2) Let $\Gamma = C_7$ then $(H, \circ)$ is a weak commutative H_v-group. $diam(\Gamma) = 3$ and $|H| = 7$, so for every $x, y \in H$ there exist z such that $z \notin E(x)$ and $z \notin E(y)$. Hence $z \in x \circ y$ and $z \in y \circ x$.

(3) Let $\Gamma = C_n$ and $diam(\Gamma) \geq 4$ then for every $x, y \in H$,

$$x \circ y = \begin{cases} H, & x = y, \\ H \setminus \{z\}, & x—y—z \text{ and } x \not— z, \\ H \setminus \{y, t\}, & x—z—y—t \text{ and } x \not— t, \\ H \setminus E(y), & d(x, y) \geq 3. \end{cases}$$

$\mathfrak{H} = (H, \circ)$ is a weak commutative H_v-group.

Theorem 3.20. *If $\Gamma = C_n$, $n \geq 3$ and $n \neq 6$, then $\mathfrak{H} = (H, \circ)$ is a weak commutative H_v-group.*

Proof. Let $\Gamma = C_n$ and $n \leq 5$, so $diam(\Gamma) \leq 2$ and $\mathfrak{H} = (H, \circ)$ is a weak commutative H_v-group by Lemma 3.7. Now consider $n \geq 7$, hence $diam(\Gamma) \geq 3$, $|H| \geq 7$ and there exist $z \in H$ for every $x, y \in H$ such that $z \in x \circ y \cap y \circ x$. □

Example 3.21. (1) Let $\Gamma = P_n$ be a linear graph and $H = \{v_1, \ldots, v_n\}$. We know $1 \leq deg(v_i) \leq 2$ and $diam(\Gamma) = n-1$. For every $v_i, v_j \in H$,

$$v_i \circ v_j = \begin{cases} H, & v_i = v_j, \\ H \setminus \{v_{i+2}\}, & j = i+1 \ (d(v_i, v_j) = 1), \\ H \setminus \{v_{i+2}, v_{i+3}\}, & j = i+2 \ (d(v_i, v_j) = 2), \\ H \setminus E(v_j), & d(v_i, v_j) \geq 3 \ (j \geq i+3 \ or \ j \leq i-3), \\ H \setminus \{v_{i-2}\}, & j = i-1 \ (d(v_i, v_j) = 1), \\ H \setminus \{v_{i-2}, v_{i-3}\}, & j = i-2 \ (d(v_i, v_j) = 2), \\ H, & j = 1, \ i = 2, \ (d(v_i, v_j) = 1), \\ H \setminus \{v_1\}, & j = 1, \ i = 3, \ (d(v_i, v_j) = 2), \\ H, & j = n, \ i = n-1, \ (d(v_i, v_j) = 1), \\ H \setminus \{v_n\}, & j = n, \ i = n-2, \ (d(v_i, v_j) = 2). \end{cases}$$

(2) Let $\Gamma = P_n$ and $n \leq 3$, then $(H, \circ)$ is a weak commutative H_v-group.

(3) Let $\Gamma = P_7$ and $diam(\Gamma) = 6$, then $(H, \circ)$ is a weak commutative H_v-group.

Theorem 3.22. *If $\Gamma = P_n$ and $4 \leq n \leq 6$, then $\mathfrak{H} = (H, \circ)$ is not a weak commutative H_v-group.*

Proof. since $3 \leq diam(\Gamma) \leq 5$, there exist x, y such that $d(x, y) = 3$, $E(x) \cap E(y) = \emptyset$ and $x \circ y = E(x)$, $y \circ x = E(y)$. Therefore $x \circ y \cap y \circ x = \emptyset$. □

Theorem 3.23. *If $\Gamma = P_n$ and $n \geq 7$, then $\mathfrak{H} = (H, \circ)$ is a weak commutative H_v-group.*

Proof. It is easy to see when $d(x, y) \leq 2$ for every $x, y \in H$, then $x \circ y \cap y \circ x \neq \emptyset$.
Now let $d(x, y) \geq 3$, then $v_i \circ v_j = H \setminus E(v_j)$, $v_j \circ v_i = H \setminus E(v_i)$ and $E(v_i) \cap E(v_j) = \emptyset$. Since $diam(\Gamma) \geq 6$, $|H| \geq 7$, therefore there exist $v_k \in H$ such that $v_k \notin E(v_i)$ and $v_k \notin E(v_j)$. That means $v_k \in v_i \circ v_j$ and $v_k \in v_j \circ v_i$. □

Definition 3.24. Let $F = (V, E)$, $|V| = n$ and $n \geq 2$. Now consider $a \in V$ as center of F such that $deg(a) = n-1$. F is called $a-$friendship graph.

Example 3.25. (1) Let S_n be a star graph with one central vertex and $n-1$ vertices which are adjacent to the central vertex. S_n is an $a-$friendship graph.

(2) Let W_n be a wheel graph which is formed by joining a vertex as center to all vertices of a cycle. W_n ia an $a-$friendship graph.

(3) Let F_n be a friendship graph which is formed by connecting n copies of cycle graph C_3 with a common vertex as center. F_n is an $a-$friendship graph.

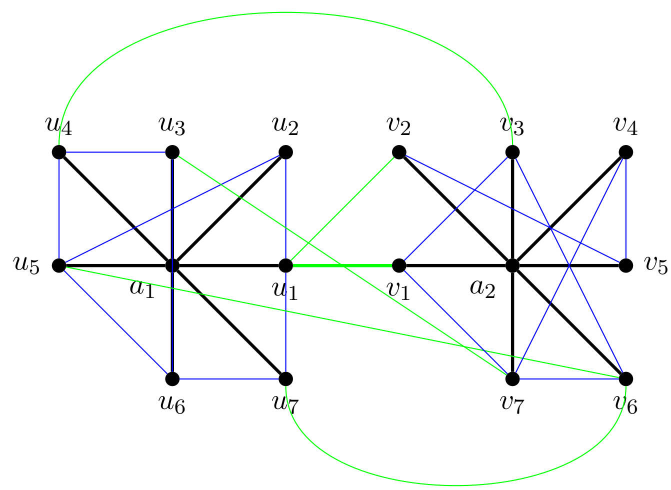


FIGURE 4. An example of $(a_1, a_2)-$friendship graph.

(4) Let $Wd(k, n)$ be a windmill graph which is constructed for $k \geq 2$ and $n \geq 2$ by joining n copies of the complete graph K_k at a shared universal vertex. $Wd(k, n)$ is an $a-$friendship graph.

Definition 3.26. Let $G_1 = (V_1, E_1)$ be a_1-friendship graph and $G_2 = (V_2, E_2)$ be a_2-friendship graph. Now define the connected graph $G = (V_1 \cup V_2, E)$ such that for every $e = x_1x_2$ of E which is an edge of G and $x_1, x_2 \in V_1 \cup V_2$, one of the following occurs:

(1) $e \in E_1$,

(2) $e \in E_2$,

(3) $x_1 \in V_1 \setminus \{a_1\}$ and $x_2 \in V_2 \setminus \{a_2\}$.

G is called $(a_1, a_2)-$friendship graph.

Example 3.27. (1) Figure 4 shows an $(a_1, a_2)-$friendship graph. Black edges should be existed, green edge between u_1 and v_1 should be existed too, but it can be replaced with one of other green edges. This means at least one edge between u_i and v_i should be existed. Blue and other green edges are arbitrary and their existence depend on the connections in the graph. The vertices and same edges can be more in same graphs.

(2) C_6 which is a cycle graph with 6 vertices, is a $(x, t)-$friendship graph. It is shown in Figure 5.

(3) P_4: $x — y — z — t$ is an $(x, t)-$friendship graph.

(4) P_5: $x — y — z — t — r$ is an $(x, t)-$friendship graph or $(y, r)-$friendship graph.

(5) P_6: $x — y — z — t — r — s$ is a $(y, r)-$friendship graph.

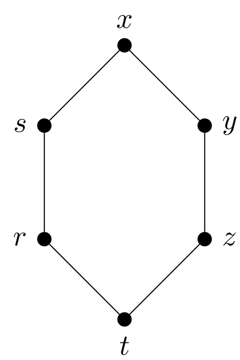


FIGURE 5. C_6 as an $(x,t)-$friendship graph.

Proposition 3.28. *If G be an $(a_1,a_2)-$friendship graph, then $3 \leq diam(G) \leq 5$.*

Proof. $d(a_1,a_2)=3$ according to the Definition 3.26 and moreover, G includes a_1-friendship graph and a_2-friendship graph. □

Lemma 3.29. *Let G be an $(a_1,a_2)-$friendship graph and $\Gamma = G$, then for every $x,y \in V$,*

$$x \circ y = \begin{cases} V, & x=y, \\ V_1, & x=a_1,\ y=a_2, \\ V_2, & x=a_2,\ y=a_1, \\ V \setminus \{v_i \mid v_i \in V_2,\ v_i — y\}, & x=a_1,\ a_1 \neq y \neq a_2, \\ V \setminus \{v_i \mid v_i \in V_1,\ v_i — y\}, & x=a_2,\ a_1 \neq y \neq a_2, \\ E(x) \cup V_2, & a_1 \neq x \neq a_2,\ y=a_1, \\ E(x) \cup V_1, & a_1 \neq x \neq a_2,\ y=a_2, \\ V \setminus E(y), & a_1 \neq x \neq a_2,\ a_1 \neq y \neq a_2. \end{cases}$$

Theorem 3.30. *Let $\Gamma = G$ be a connected hypergraph, $G=(V,E)$ is an $(a_1,a_2)-$friendship graph if and only if $\mathfrak{V}=(V,\circ)$ is a non-weak commutative H_v-group.*

Proof. Since $d(a_1,a_2)=3$ and $V_1 \cap V_2 = \emptyset$, while $a_1 \circ a_2 = V_1$ and $a_2 \circ a_1 = V_2$. Therefore $a_1 \circ a_2 \cap a_2 \circ a_1 = \emptyset$. Conversely let $\mathfrak{V}=(V,\circ)$ be a non-weak commutative H_v-group and suppose that G is not an $(a_1,a_2)-$friendship graph. This implies that for every $x,y \in V$ such that $E(x) \cap E(y) = \emptyset$, there is $t \in V$ such that $t \notin E(x)$ but $t \in E^c(y)$ and likewise $t \notin E(y)$ but $t \in E^c(x)$. Hence $t \in x \circ y \cap y \circ x$ and it is contradiction. □

Corollary 3.31. *Let $G=(V,E)$ be an $(a_1,a_2)-$friendship graph and $G'=(V',E')$ be a subgraph of $G=(V,E)$ such that $a_1,a_2 \in V'$ and $d(a_1,a_2)=3$. If $\Gamma = G'$ then $\mathfrak{V}'=(V',\circ)$ is a non-weak commutative H_v-group.*

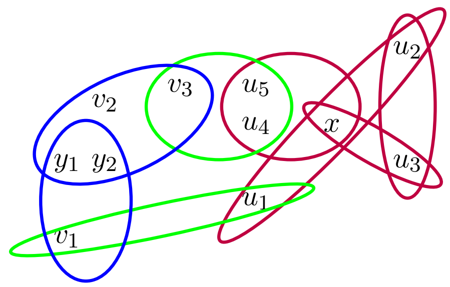


FIGURE 6. An $(X, Y)-$friendship hypergraph.

Theorem 3.32. *Let $\Gamma = (H, E)$ be a connected hypergraph and $diam(\Gamma) \geq 6$, then $(H, \circ)$ is a weak commutative H_v-group.*

Proof. Since $diam(\Gamma) \geq 6$, there is at least one path which its length is 6. This implies that for every $x, y \in H$ there exist $t \in H$ such that $t \in x \circ y \cap y \circ x$. □

We can generalize the concept of friendship graphs to hypergraphs as follows:

Definition 3.33. Let $\Omega = (H, E)$ be a hypergraph with n hyperedges. Now suppose that $X \subseteq H$ such that for every $x \in X$ and $y \in H \setminus \{x\}$ there exist i such that $x, y \in E_i$. In other words x is adjacent of y and x—y. Ω is called an $X-$friendship hypergraph.
Now let $\Omega_1 = (H_1, \{E_i\}_i)$ where $1 \leq i \leq n$ be an $X-$friendship hypergraph and $\Omega_2 = (H_2, \{E_j\}_j)$ where $1 \leq j \leq m$ be a $Y-$friendship hypergraph, now define connected hypergraph $\Gamma = (H_1 \cup H_2, \{E_k\}_k)$ such that for every E_k where $t > m + n$ and $1 \leq k \leq t$, one of the following occurs:

(1) $E_k \subseteq \{E_i\}_i$,
(2) $E_k \subseteq \{E_j\}_j$,
(3) for every $z \in E_k$, $z \in (H_1 \setminus X) \cup (H_2 \setminus Y)$.

Γ is called an $(X, Y)-$friendship hypergraph.

Example 3.34. Figure 6 is an $(X, Y)-$friendship hypergraph such that $X = \{x\}$ and $Y = \{y_1, y_2\}$. Further x—u_i for every u_i where $1 \leq i \leq 5$ and y_1—v_i, y_2—v_i for every v_j where $1 \leq j \leq 3$, also $d(x, y_1) = 3$ and $d(x, y_2) = 3$.

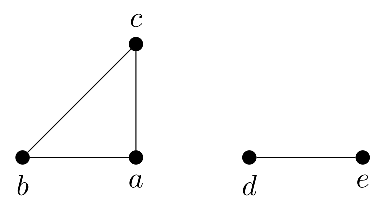
Theorem 3.35. *Let $\Gamma = (H, E)$ be a connected hypergraph, Γ is an $(X, Y)-$friendship hypergraph if and only if $\mathfrak{H} = (H, \circ)$ is a non-weak commutative H_v-group.*

Proof. Omitted by obvious. □

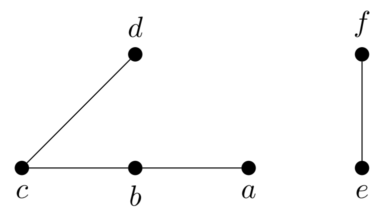
Being connected is an essential condition in 3.30, but the question is that what is commutativity condition in disconnected graphs and hypergraphs. We are going to provide some examples and answer this question.

Example 3.36. (1) Let Γ be an edgeless graph with 2 vertices x and y. Then $x \circ y \cap y \circ x = \emptyset$. Therefore $\mathfrak{H} = (H, \circ)$ is a non-weak commutative H_v-group where $H = \{x, y\}$.

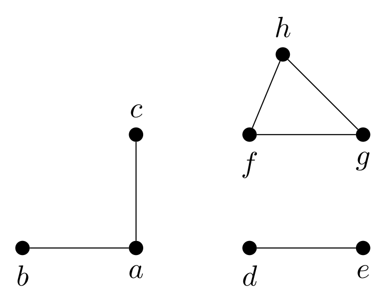
(2) Let $\Gamma = (H, E)$ be a disconnected graph with two connected components which is shown below. It is clear that $\mathfrak{H} = (H, \circ)$ is a non-weak commutative H_v-group.



(3) Let $\Gamma = (H, E)$ be a disconnected graph with two connected components which is shown below. Clearly $\mathfrak{H} = (H, \circ)$ is a weak commutative H_v-group.



(4) Let $\Gamma = (H, E)$ be a disconnected graph with three connected components which is shown below. $\mathfrak{H} = (H, \circ)$ is a weak commutative H_v-group.



Theorem 3.37. *Let $\Gamma = (H, E)$ be a disconnected hypergraph with at least 3 connected components, then $\mathfrak{H} = (H, \circ)$ is a weak commutative H_v-group.*

Proof. It is clear that $(H, \circ)$ is not commutative, but about weak commutativity, consider 3 connected components for Γ. $E^c(x) \cap E^c(y) \neq \emptyset$ For every $x, y \in H$, so $x \circ y \cap y \circ x \neq \emptyset$. □

Theorem 3.38. *Let $G = (H, E)$ and $\Gamma = G$ be a disconnected hypergraph with 2 connected components $G_i = (H_i, E_i)$, $i \in \{1, 2\}$. $\mathfrak{H} = (H, \circ)$ is a weak commutative H_v-group if and only if there exist $i \in \{1, 2\}$ such that if $|G_i| = n$, then $deg(x) < n - 1$ for every $x \in H_i$.*

Proof. Let $\mathfrak{H} = (H, \circ)$ be a weak commutative H_v-group and suppose that $|G_1| = n$, $|G_2| = m$ and there exist $a_1 \in H_1$, $a_2 \in H_2$ such that $deg(a_1) = n - 1$ and $deg(a_2) = m - 1$. Then $a_1 \circ a_2 \cap a_2 \circ a_1 = \emptyset$ which is contradiction. Conversely suppose that $|G_1| = n$ and $deg(x) < n-1$ for every $x \in H_1$. This implies that there exist $z \in H_1$ such that $x \not\!\!-\, z$ and $z \in E^c(x)$. On the other hand $z \in E^c(y)$ for every $y \in H_2$. Therefore $x \circ y \cap y \circ x \neq \emptyset$. The proof is immediate in other cases. □

Corollary 3.39. *Let $G = (H, E)$ and $\Gamma = G$ be a disconnected hypergraph with 2 connected components $G_i = (H_i, E_i)$, $i \in \{1, 2\}$. If G_i is an a_i-friendship graph for every $i \in \{1, 2\}$, then $\mathfrak{H} = (H, \circ)$ is a non-weak commutative H_v-group.*

Proof. Since $a_1 \circ a_2 \cap a_2 \circ a_1 = \emptyset$. □

Here we are going to discuss join space hypergroup and transposition condition.

Theorem 3.40. *Let $\Gamma = (H, E)$; $diam(\Gamma) = 1$ if and only if $\mathfrak{H} = (H, \circ)$ is a join space H_v-group.*

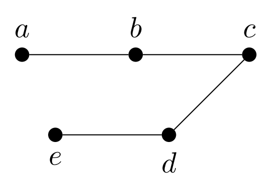
Proof. By the Theorem 3.11 and since $a \circ b = H$ for every $a, b \in H$. □

Theorem 3.41. *Let $\Gamma = (H, E)$ be a connected hypergraph.*

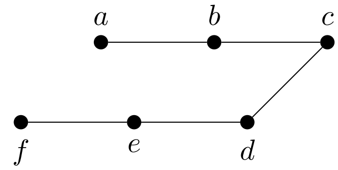
(1) *If $diam(\Gamma) = 2$ or $diam(\Gamma) \geq 6$, then $\mathfrak{H} = (H, \circ)$ is a transposition H_v-group.*
(2) *If $3 \leq diam(\Gamma) \leq 5$, then the result of part (1) does not hold.*

Proof. (1) Let $diam(\Gamma) = 2$, then for every $a, b \in H$, if a—b the proof is immediate. But if $a \not\!\!-\, b$, it implies that there exist e such that a—e—b, so for every $c, d \in H$; $e \in a \circ d \cap b \circ c$. Obviously transposition condition holds. Now let $diam(\Gamma) \geq 6$, then for every $a, b, c, d \in H$; $a \circ d \cap b \circ c \neq \emptyset$. Clearly transposition condition satisfies.

(2) Let $diam(\Gamma) = 3$ and there exist $a, b, c, d \in H$ such that a—c—d—b. Therefore $a \circ d \cap b \circ c = \emptyset$ while $b \backslash a \cap c/d \neq \emptyset$. Now let $diam(\Gamma) = 4$ and there exist a path like below in the graph:



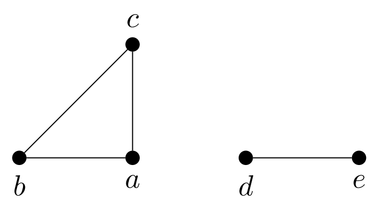
This is obvious that $a \circ e \cap e \circ b = \emptyset$ while $e \backslash a \cap b/e \neq \emptyset$. Finally let $diam(\Gamma) = 5$ and there exist a path like below in the graph:



Clearly $a \circ e \cap e \circ b = \emptyset$ while $e \backslash a \cap b/e \neq \emptyset$. □

Theorem 3.42. *Let $G = (H, E)$ and $\Gamma = G$ be a disconnected hypergraph with 2 connected components $G_i = (H_i, E_i)$, $i \in \{1, 2\}$. If G_i is an a_i-friendship graph for every $i \in \{1, 2\}$, then $\mathfrak{H} = (H, \circ)$ is not a transposition H_v-group.*

Proof. Consider Γ a hypergraph as is shown below:



It is easy to see that $a \circ e \cap e \circ b = \emptyset$ while $e \backslash a \cap b/e \neq \emptyset$. □

Corollary 3.43. *Let $G = (H, E)$ and $\Gamma = G$ be a disconnected hypergraph with 2 connected components $G_i = (H_i, E_i)$, $i \in \{1, 2\}$ and there exist i such that $|G_i| = n$ and $deg(x) < n - 1$ for every $x \in H_i$. Then $\mathfrak{H} = (H, \circ)$ is a transposition H_v-group.*

Proof. Since $a \circ d \cap b \circ c \neq \emptyset$ for every $a, b, c, d \in H$. □

Corollary 3.44. *Let $G = (H, E)$ and $\Gamma = G$ be a disconnected hypergraph with at least 3 connected components, then $\mathfrak{H} = (H, \circ)$ is a transposition H_v-group.*

Proof. Clearly $a \circ d \cap b \circ c \neq \emptyset$ for every $a, b, c, d \in H$. □

4. An application to marketing in social networks

Social networks are being constructed whenever a set of related nodes are formed. These nodes might be of the type of individuals, groups or organizations. Investigating the relationship between nodes and studying the flow of knowledge and information among these nodes are important issues about social networks.

One of the specialized fields related to social networks is Social Network Analysis or SNA which is referred to sociology in last centuries. Sociologists are interested in using mathematical achievements in graph theory and network analysis to recognize and analyze the connections in human societies. Here we aim to use theoretical concepts and give an example related to marketing in social networks to show how we can use our results to analyze the connections in network graphs.

Now consider one of the most popular social medias which is used in specific area(such as a small town etc). Let $G = (V, E)$ be a subgraph of main network graph of this social media, V be the set of its users in that specific area and E includes all edges among the members of V which shows the connection among them. Now let $x, y \in V$ be two nodes belonging to two small and same businesses with similar products or services and similar prices without any competitive advantage. These two try to attract more clients in related same market. There are some factors that can help us to use graph theory and hyperoperation defined on the vertices and compare the possibility of their success. Let $\Gamma = G$, then:

A. Here $E(x)$ denotes all contacts of x and they can be considered as potential clients. $|E(x)|$ can show us the number of people whom x attracts. $E(x) \cap E(y)$ can give us more information.

B. $E^c(x)$ denotes people who are not in the list of contacts of x, they should get attracted. They can get attracted by y as well.

C. $diam(\Gamma)$ can show us the expansion of graph, contacts and clients. Let for instance $diam(\Gamma) = 1$, this means x—y. Moreover $|E(x)| = |E(y)| = |V|$ which shows x and y are absolute competitors for each other. The more it increases, the less the burden of competition annoys them.

D. $d(x, y)$ is another factor that can help us to analyze connection between x and y. This factor beside diameter of graph can help us to get some results about their connection. Let for instance $d(x, y) = 1$, this can create situation similar to when $diam(\Gamma) = 1$.

Now let define hyperoperation as follows:

$$x \circ y = E(x) \cup E^c(y).$$

This hyperoperation can show the potential market for x. It includes two groups of people, some who are connected to x and some who are not connected to y yet and x can attract them to its own businesses.

Considering all the factors above, we can compare $x \circ y$ and $y \circ x$ and get some results. Also comparing $|x \circ y|$ and $|y \circ x|$ can be useful.

Example 4.1. (1) Let G be an $(a_1, a_2)-$friendship graph. Hence $3 \leq diam(\Gamma) \leq 5$ and $x \circ y \cap y \circ x = \emptyset$, they do not have common potential clients. This can be ideal condition for them. Comparing $|x \circ y|$ and $|y \circ x|$ in real examples can give us more information.

(2) Let $diam(\Gamma) \geq 6$. In this situation $x \circ y \cap y \circ x \neq \emptyset$, but the number of non shared contacts gets more and more. It can help them to attract their own clients instead of focusing on the competition.

Now all materials are provided to give an example of real world:

Example 4.2. Let $G = (V, E)$ in Figure 7 be a graph which shows connections among finite set of people in a specific social media such that $V = \{a, b, c, d, e, f, g, h\}$. In this graph the nodes

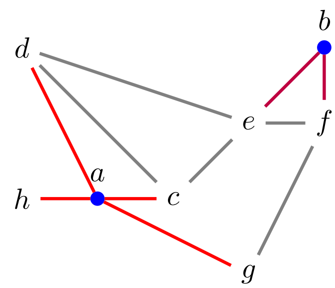


FIGURE 7. The connections among contacts of a and b.

a and b belong to two confectioners that attract the majority of their customers in this social media. It is easy to see that G is an $(a, b)-$friendship graph, $d(a, b) = 3$ and $a \circ b \cap b \circ a = \emptyset$. It means a and b do not have any common potential clients and there is no competition. But we should consider that G is a graph with diameter 4 and circumstances may change soon. Further, $|E(a)| \geq |E(b)|$, hence $|a \circ b| \geq |b \circ a|$ which means a has been more successful so far. Now consider the graph G is changed as is shown in Figure 8. In this case $a \circ b \cap b \circ a = \{d\}$. We can consider them business competitors now.

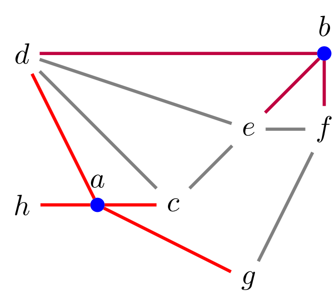


FIGURE 8. The graph in new situation.

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