Algebraic Structures and Their Applications

Yazd University

Algebraic Structures and Their Applications Vol. 11 No. 2 (2024) pp 165-171.

Research Paper

ON THE REE GROUPS ${}^2G_2(q)$ CHARACTERIZED BY A SIZE OF A CONJUGACY CLASS

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Abstract. One of the important problem in finite groups theory is group characterization by specific property. Properties, such as element order, the set of element with the same order, etc. In this paper, we prove that Ree group ${}^2G_2(q)$, where $q \pm \sqrt{3q}+1$ is a prime number can be uniquely determined by its order and one conjugacy class size.

1. Introduction

One of the important problems in finite group theory is a characterization of a group by specific property. Properties often involve element orders, the set of elements with the same order, the largest elements order, their graphs and etc. Next, we say the group G is characterized by property M if every group fulfilling M is isomorphic to G.

Let G be a finite group, the set of all conjugacy class sizes of a group G will be denoted by $N(G)$. Also, we denote the conjugacy class of size of prime number p by $m_p(G)$. For every

DOI: 10.22034/as.2023.19014.1589

MSC(2010): Primary: 20D06.

Keywords: Conjugacy class, Element order, Prime graph, Ree group.

Received: 21 September 2022, Accepted: 21 July 2023.

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integer n denote by $\pi(n)$ the set of all prime divisors of n. The prime graph $\pi(G)$ of G is constructed upon the vertex set $\pi(|G|)$ in such a way that two distinct primes p and q are joined by an edge if and only if G has an element of order pq.

Let $t(G)$ be the number of connected components of $\pi(G)$. These components will be denoted by π_1, π_2,...,$\pi_{t(G)}$. If G is of even order, then π_1 is chosen to be the component in which 2 is a vertex. We denote m_1, m_2,...,$m_t(G)$ to be the integers such that $|G| = m_1, ..., m_t(G)$ and $\pi(m_i)$ is the vertex set of π_i. If m_i is odd, call π_i an odd order component [12]. The starting point for our discussion is from a conjecture of J. G. Thompson, which is Problem 12.38 in the Kourovka notebook [24] is as follows:

Thompson's conjecture. Let G be a group with trivial center. If M is a non-abelian simple group satisfying $N(G) = N(M)$, then $G \cong M$. Next, for example the authors in([2, 7, 4, 5, 6, 9, 10, 14, 18, 19]), proved that the sporadic simple groups, Alt_{10}, $PSL(4, 4)$ and $PSL(2, p)$, $PSL(n, 2)$, ${}^2D_n(2)$,${}^2D_{n+1}(2)$, $C_n(2)$, Alt_n where $n \in \{p, p+1, p+2\}$ and Sym_p where p is a prime number,the projective special linear groups $PSL(5, q)$ where $p = \frac{q^4+q^3+q^2+q+1}{(5,q-1)}$ be a prime number, where q is a prime power and the projective special unitary groups $PSU(5, q)$, where $p = \frac{q^4-q^3+q^2-q+1}{(5,q+1)}$ be a prime number are characterizable by using the order of the group and the conjugacy class of size. The group G is called a 2-Frobenius group if there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H respectively. In this paper, we prove that Ree groups ${}^2G_2(q)$, where $q \pm \sqrt{3q} + 1$ is a prime number can be uniquely determined by its order and one conjugacy class of size. In fact, we prove the following main theorem.

Main Theorem. Let G be a group such that $|G| = |{}^2G_2(q)|$. If $p = q \pm \sqrt{3q} + 1$ is a prime, then $G \cong {}^2G_2(q)$ if and only if G has a conjugacy class of size $m_p(G) = \frac{|{}^2G_2(q)|}{p}$.

2. Preliminaries

Lemma 2.1. [17] *Let G be a Frobenius group of even order with kernel K and complement H. Then*

(1) *$t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$,*
(2) *$|H|$ divides $|K| - 1$,*
(3) *K is nilpotent.*

Lemma 2.2. [8] *Let G be a 2-Frobenius group of even order. Then*

(1) *$t(G) = 2$, $\pi(H) \cup \pi(G/K) = \pi_1$ and $\pi(K/H) = \pi_2$,*
(2) *G/K and K/H are cyclic groups satisfying $|G/K|$ divides $|Aut(K/H)|$.*

Lemma 2.3. [29] *Let G be a finite group with $t(G) \geq 2$. Then one of the following statements holds:*

(1) *G is a Frobenius group,*
(2) *G is a 2-Frobenius group,*
(3) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|Out(K/H)|$.*

Lemma 2.4. [27] *Let G be a non-abelian simple group such that $(5, |G|) = 1$. Then G is isomorphic to one of the following groups:*

(1) $A_n(q')$, $n = 1, 2$, $q' \equiv \pm 2 (mod\ 5)$,
(2) $G_2(q')$, $q' \equiv \pm 2 (mod\ 5)$,
(3) ${}^2A_2(q')$, $q' \equiv \pm 2 (mod\ 5)$,
(4) ${}^3D_4(q')$,$q' \equiv \pm 2 (mod\ 5)$,
(5) ${}^2G_2(q')$, $q' = 3^{2m+1}$, $m \geq 1$.

Lemma 2.5. [30] *Let q, k, l be natural numbers. Then*

(1) $(q^k - 1, q^l - 1) = q^{(k,l)} - 1$.
(2) $(q^k + 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if both } \frac{k}{(k,l)} \text{ and } \frac{l}{(k,l)} \text{ are odd,} \\ (2, q+1) & \text{otherwise.} \end{cases}$
(3) $(q^k - 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if } \frac{k}{(k,l)} \text{ is even and } \frac{l}{(k,l)} \text{ is odd,} \\ (2, q+1) & \text{otherwise.} \end{cases}$

In particular, for every $q \geq 2$ and $k \geq 1$ the inequality $(q^k - 1, q^k + 1) \leq 2$ holds.

Notation: We note that X_i the set of all cyclic subgroups of G order t_i where $t_1 = \frac{q-1}{2}$, $t_2 = \frac{q+1}{4}$, $t_3 = q - \sqrt{3q} + 1$, $t_4 = q + \sqrt{3q} + 1$, $t_5 = q^3$.

Lemma 2.6. [25] *Assume X_i be as above and $B_i \in X_i$ for $1 \leq i \leq 5$ and $x \in G$ is non-trivial. Then*

(1) *if $x \in B_1$ then $|x^G| = q^3(q^3 + 1)$,*
(2) *if $x \in B_2$ then $|x^G| = q^3(q^2 - q + 1)(q - 1)$,*
(3) *if $x \in B_3$ then $|x^G| = q^3(q + 1)(q + \sqrt{3q} + 1)(q - 1)$,*
(4) *if $x \in B_4$ then $|x^G| = q^3(q + 1)(q - \sqrt{3q} + 1)(q - 1)$,*
(5) *if $x \in B_5$ we have if $x \in Z(B_5)$ then $|x^G| = (q^3 + 1)(q - 1)$.*

3. Proof of the Main Theorem

In this section, we prove the main theorem in the following lemmas. For this purpose, we denote the Ree groups ${}^2G_2(q)$ and prime number $q \pm \sqrt{3q} + 1$ by R and p respectively. Furthermore by [28], ${}^2G_2(q)$ has conjugacy class of size $m_p(G) = \frac{|{}^2G_2(q)|}{p}$. First we denote that if $G \cong {}^2G_2(q)$, then $m_p(G) = m_p({}^2G_2(q))$ and $|G| = |{}^2G_2(q)|$. Now, assume $m_p(G) =$

$m_p({}^2G_2(q))$ and $|G| = |{}^2G_2(q)|$. The aim is to prove $G \cong {}^2G_2(q)$. By the assumption on q, there exists an element α of order p in G such that $C_G(\alpha) =< \alpha >$ and $C_G(\alpha)$ is a sylow p-subgroup of G. By the sylow's theorem, we have that $C_G(\beta) =< \beta >$ for any element β in G of order p. In the following we prove p is an isolated vertex in $\Gamma(G)$. We note that $|{}^2G_2(q)| = q^3(q^3+1)(q-1)$ and $m_p({}^2G_2(q)) = \frac{|{}^2G_2(q)|}{p}$.

Lemma 3.1. *p is an isolated vertex in $\Gamma(G)$.*

Proof. We shall prove that p is an isolated vertex of Γ(G). Suppose to contrary. Then there is $t \in \pi(G) - \{p\}$ such that $tp \in \pi_e(G)$. So $tp \geq 2p = 2(q \pm \sqrt{3q}+1) > q + \sqrt{3q}+1$, thus $k(G) > q + \sqrt{3q} + 1$. As a result $t(G) \geq 2$. □

So by lemma 2.3 we have the following lemmas.

Lemma 3.2. *The group G is neither a Frobenius group and a 2-Frobenius group.*

Proof. (You can see lemma 3.2 of [13]) □

Lemma 3.3. *The group G is isomorphic to the group R.*

Proof. By lemma 3.1, p is an isolated vertex of $\Gamma(G)$. Thus $t(G) > 1$ and G satisfies one of the cases of lemma 2.3. At the moment by lemma 3.2 and lemma 2.2 implies that G is neither a Frobenius group and a 2-Frobenius group. Thus only the case (c) of lemma 2.3 occure. So G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group. Since p is an isolated vertex of $\Gamma(G)$, we have $p \mid |K/H|$. On the other hand, $5 \nmid |G|$, so K/H is isomorphic one of the groups lemma 2.4.

Step 1. Suppose that $K/H \cong A_1(q')$, $q' \equiv \pm 2 (\text{mod } 5)$. On the other hand, by[29], $\pi(A_1(q')) = q' \pm 1$ or $\frac{q' \pm 1}{2}$. We know that $|A_1(q')|$ divide $|G|$, $q'(q'^2-1) \mid q^3(q^3+1)(q-1)$. Now, we consider $p = q' \pm 1$, so $q \pm \sqrt{3q}+1 = q' \pm 1$. As aresult $3^{2m+1} \pm 3^{m+1} + 1 = q' \pm 1$. So we deduce $q' = 3^{2m+1} \pm 3^{m+1}$, and also $q' = 3^{2m+1} \pm 3^{m+1} + 2$. Since that $|A_1(q')| \nmid |G|$, where this is a contradiction. Now, if $p = \frac{q' \pm 1}{2}$, then $q \pm \sqrt{3q} + 1 = \frac{q' \pm 1}{2}$. Since that $q' = p'^m$, where this is a contradiction. If $K/H \ncong {}^2A_2(q')$ and $K/H \ncong A_2(q')$, then we have a contradiction, similarily.

Step 2. Suppose that $K/H \cong G_2(q')$ where $q' \equiv \pm 2 (\text{mod } 5)$. On the other hand, by [29], $\pi(G_2(q') = q'^2 \pm q' + 1$. We know that $|G_2(q')|$ divide $|G|$, so $q'^6(q'^6-1)(q'^2-1) \mid q^3(q^3+1)(q-1)$. Now, we consider $p = q'^2 \pm q' + 1$, so $q \pm \sqrt{3q} + 1 = q'^2 \pm q' + 1$. It follows that $3^{2m+1} \pm 3^{m+1} + 1 = q'^2 \pm q' + 1$, so $3^{m+1}(3^m+1) = q'(q'+1)$. Since that $(3^{m+1}, 3^m+1) = 1$, so $q' = 3^m + 1$ and $q' + 1 = 3^{m+1}$, where this is a contradiction.

Step 3. Suppose that $K/H \cong {}^3D_4(q')$, $q \equiv \pm 2 (\text{mod } 5)$. On the other hand, by [29],

$\pi(^3D_4(q')) = q'^4 - q'^2 + 1$. We know that $|^3D_4(q')|$ divided $|G|$, so $q'^{12}(q'^8 + q'^4 + 1)(q'^6 - 1)(q'^2 - 1) \mid q^3(q^3+1)(q-1)$. Now, we consider $p = q'^4 - q'^2 + 1$, so $q \pm \sqrt{3q} + 1 = q'^4 - q'^2 + 1$. It follows that $3^{2m+1} \pm 3^{m+1} + 1 = q'^4 - q'^2 + 1$, thus $3^{m+1}(3^m \pm 1) = q'^2(q'^2 - 1)$. Since that $(3^{m+1}, 3^m \pm 1) = 1$ so $3^{m+1} = q'^2$ and $3^m + 1 = q'^2 - 1$. Since that $|^3D_4(q')| \nmid |G|$, where this is a contradiction.

Hence, $K/H \cong {}^2G_2(q')$. Now since that $|K/H| = |R| = |G|$ and also $p \in \pi(K/H)$ so $p = p'$. So $q \pm \sqrt{3q} + 1 = q' \pm \sqrt{3q'} + 1$. Thus $q = q'$. On the other hand, $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, thus $H = 1$, $G = K \cong R$. $\square$

4. Acknowledgments

The authors wish to sincerely thank the referees for several useful comments.

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