Algebraic Structures and Their Applications Vol. 11 No. 1 (2024) pp 151-163.

Research Paper

SEMIHYPERGROUPS THAT EVERY HYPERPRODUCT ONLY CONTAINS SOME OF THE FACTORS

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Abstract. Breakable semihypergroups, defined by a simple property: every non-empty subset of them is a subsemihypergroup. In this paper, we introduce a class of semihypergroups, in which every hyperproduct of n elements is equal to a subset of the factors, called π_n-semihypergroups. Then, we prove that every semihypergroup of type π_{2k}, $(k \geq 2)$ is breakable and every semihypergroup of type π_{2k+1} is of type π_3. Furthermore, we obtain a decomposition of a semihypergroup of type π_n into the cyclic group of order 2 and a breakable semihypergroup. Finally, we give a characterization of semi-symmetric semihypergroups of type π_n.

1. Introduction

A natural extension of well-known group theory is introduced by Marty [21] which leads to begin the theory of algebraic hyperstructures. Furthermore, algebraic hyperstructure theory

DOI: 10.22034/as.2023.17037.1439

MSC(2010): Primary: 20N20.

Keywords: Breakable semihypergroup, Hypergroup, Hyperproduct, Semihypergroup of type π_n.

Received: 24 July 2021, Accepted: 21 July 2023.

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has various applications in other area of science such as probability theory, graphs, fuzzy sets, automata, cryptography, codes, chemistry, and artificial intelligence (See for example [8, 16, 17, 22, 23, 24, 27]). Some outline of applications can be found in the two important books: the first is "Applications of Hyperstructure Theory" by Corsini and Leoreanu [4] and the other is "Hyperring Theory and Application" by Davvaz and Leoreanu [6].

There has been numerous studies to investigate the concept of semihypergroup as a generalization of semigroup. Basic definitions and results about the semihypergroups are found in [7, 9, 10, 12, 15, 29].

This paper is in continuation with our earlier paper [13] where we extended the classical concepts of breakable semigroups to algebraic hypercompositional structures. Recall that a semigroup is called breakable if every subset is a subsemigroup. Rédei [26] has proved that they are semigroups with empty Frattini-substructure. One can see that a semigroup S is breakable if and only if $xy \in \{x, y\}$ for any $x, y \in S$.

Using the associated power semigroup, other researchers presented another characterization of above-mentioned semigroups [28], that is, a semigroup is breakable if and only if its power semigroup is idempotent. However, Rédei provided a complete description of breakable semigroups [26], stated in Theorem 2.1.

In the theory of algebraic hyperstructures, the notion of hyperoperation on the set S is defined as a mapping from $S \times S$ to $\mathcal{P}^*(S)$, where $\mathcal{P}^*(S)$ is the set of non-empty subsets of S. If the set S is equipped with a binary associative operation, i.e. $(S, \cdot)$ is a semigroup, then this operation can be extended also to $\mathcal{P}^*(S)$, in the most natural way: $A \star B = \{a \cdot b \mid a \in A, b \in B\}$. In this way, $(\mathcal{P}^*(S), \star)$ becomes a semigroup, called the power semigroup of S. In particular, if $(G, \cdot)$ is a group, then a non-empty subset $\mathcal{G}$ of $\mathcal{P}^*(G)$ is called an HX-group [18] on G. An overview on the links between HX-groups and hypergroups has been recently proposed by Cristea et al. [5].

Similarly, if $(S, \circ)$ is a semihypergroup, then we can again define an associative operation as follows:

$$A \star B = \bigcup_{a \in A, b \in B} a \circ b,$$

for all $A, B \in \mathcal{P}^*(S)$ so $(\mathcal{P}^*(S), \star)$ is a semigroup.

In [3] Corsini introduced the notion of Chinese hyperoperation associated with HX-groups defined by $a \widehat{\circ} b = \{a \cdot b \mid a \in A, b \in B\}$, where G is a group, $a, b \in G$ and $A, B \in \mathcal{P}^*(G)$. HX-groups and HX-hypergroups have an important role in applications to modeling, chaotic (hyperchaotic) systems and differential equations [31].

In [13, 14] we have presented and discussed on an extended version of Rédei's theorem for semi-symmetric breakable semihypergroups. In this paper, we continue our study and generalize the property (A_n), that is considered by Pelicán [25] on semigroups, in order to investigate and characterize semihypergroups that every hyperproduct only contains some of the factors.

The rest of the paper is structured as follows. In Section 2, we recall some definitions and properties of semigroups and semihypergroups. Since the paper aims to be self-contained, some of the previous results of breakable semihypergroups are also included.

In Section 3, we define the concept of semihypergroups of type π_n, give some examples and prove some properties. In particular, we show that every semihypergroup of type π_{2k}, $(k \geq 2)$ is breakable and every semihypergroup of type π_{2k+1} is of type π_3.

The main part of the paper is covered by Section 4, where we present a characterization of semihypergroups of type π_n using the power set and the generalization of Rédei's theorem for semi-symmetric semihypergroups, that permits to decompose π_n-semihypergroups in a certain way.

2. **Preliminaries**

Let us begin with some basic definitions and notations that will be needed in this paper. The reader is referred to [25, 26, 28] for additional details of classical algebraic structures and to [2, 4, 30] for further discussions about algebraic hyperstructures.

Let $(S, \cdot)$ be a semigroup. Then S is called a *left (right) zero semigroup* (*l-semigroup (r-semigroup)* for short), if each element is a left (right) zero element, i.e. for any $x \in S$, we have $x \cdot y = x (x \cdot y = y)$ for all $y \in S$.

A semigroup S is *breakable* if every non-empty subset of S is a subsemigroup. *Breakable semigroups* are considered by Rédei [26] as a subclass of the semigroups having an empty Frattini-substructure. One can see that a semigroup $(S, \cdot)$ is breakable if and only if $x \cdot y \in \{x, y\}$ for any $x, y \in S$.

In [26] a complete description of the structure of a breakable semigroup is given by following Theorem.

Theorem 2.1. [26] *A semigroup S is breakable if and only if, it can be partitioned into classes and the set of classes can be ordered in such a way that every class constitutes an l-semigroup or an r-semigroup, and for any two elements $x \in C$ and $y \in C'$ of two different classes C, C', with $C < C'$, we have $x \cdot y = y \cdot x = y$.*

Pelicán in [25] generalized Redei's theorem in another direction and determined all semigroups with the following property:

$$(A_n) \qquad \text{for any } a_1, a_2, \ldots, a_n \in S, a_1 a_2 \cdots a_n = a_i,$$

for some $1 \leq i \leq n$. One can see that semigroups with the property (A_2) are exactly the breakable ones and every breakable semigroup satisfies $(A_n,)(n \geq 2)$. On the other hand, $\mathbb{Z}_2$ (the cyclic group oi order 2) satisfies (A_3) but not (A_2).

Tamura and Shafer characterized a breakable semigroup using properties of its power semigroup [28].

The fundamental relations defined on hyperstructures in order to obtain an equivalent classical structure from a given hyperstructure. More exactly, let $(S, \circ)$ be a semihypergroup and define the relation β and its transitive closure β^*. Then the quotient S/β^* is a semigroup with a suitable operation, called the *fundamental semigroup* related to S. Here below we recall the construction, introduced by Koskas [20] and studied mainly by Freni [11], who proved that $\beta = \beta^*$ on hypergroups. For all natural numbers $n > 1$, define the relation β_n on a semihypergroup $(S, \circ)$, as follows: $a\beta_n b$ if and only if there exist $x_1, \ldots, x_n \in S$ such that $\{a, b\} \subseteq \prod_{i=1}^{n} x_i$. Take $\beta = \bigcup_{n\geq 1} \beta_n$, where $\beta_1 = \{(x, x) \mid x \in S\}$ is the diagonal relation on S. Denote by β^* the transitive closure of β. The relation β^* is a strongly regular relation. On the quotient S/β^* define a binary operation as follows: $\beta^*(a) \odot \beta^*(b) = \beta^*(c)$ for all $c \in \beta^*(a) \circ \beta^*(b)$. Moreover, the relation β^* is the smallest equivalence relation on a semihypergroup S, such that the quotient S/β^* is a semigroup. The quotient S/β^* is called the *fundamental semigroup*.

Definition 2.2. [13] A semihypergroup $(S, \cdot)$ is called *semi-symmetric* if $|x \circ y| = |y \circ x|$ for every $x, y \in S$.

It is clear that any commutative semihypergroup is also semi-symmetric.

Definition 2.3. [13] A semihypergroup S is called breakable if every non-empty subset of S is a subsemihypergroup.

For example, l-semigroups and r-semigroups are breakable semihypergroups. If the property $x, y \in x \circ y$ holds for all elements $x, y \in S$, then the hyperoperation "$\circ$" is called *extensive* or closed [1, 22].

Example 2.4. Consider $S = (\{1, 2, 3, 4, 5\}, \circ)$ defined by the following Cayley table

$\circ$	1	2	3	4	5
1	1	2	3	4	5
2	2	2	$\{2,3\}$	2	$\{2,5\}$
3	3	$\{2,3\}$	3	3	$\{3,5\}$
4	4	2	3	4	5
5	5	$\{2,5\}$	$\{3,5\}$	5	5

Then S is a breakable semihypergroup.

The most simple hyperoperation of this type was defined by the first time by Konguetsof [19] around 70's as $x \circ y = \{x, y\}$ for all $x, y \in S$ and re-considered by Massouros [23, 24] in the framework of automata theory, proving the following result.

Theorem 2.5. [24] *Let H be a non-empty set. For every $x, y \in H$ define $x \star_B y = \{x, y\}$. Then $(H, \star_B)$ is a join hypergroup.*

G. Massouros called this hyperstructure a *B-hypergroup*, after the binary result that the hyperoperation gives.

Proposition 2.6. [13] *The fundamental semigroup of a breakable semihypergroup is breakable, too.*

Theorem 2.7. [13] *A semi-symmetric semihypergroup $(S, \circ)$ is breakable if and only if it can be partitioned into classes, i.e. $S = \bigcup_{\gamma \in \Gamma} S_\gamma$, where Γ is a chain and all S_γ are pairwise disjoint l-semigroups, r-semigroups or B-hypergroups. Moreover, for every $x \in S_\alpha$ and $y \in S_\beta$, with $\alpha < \beta$, we have $x \circ y = y \circ x = y$.*

3. Semihypergroups of type π_n

In this section we continue to study the notion of breakable semihypergroups based on new point of view that is called π_n-semihypergroups or semihypergroups of type π_n.

Let n be a positive integer greater than 1. In this section, we study semihypergroups in which every hyperproduct of n elements of it is equal to a subset of factors. First, we adapt the property A_n for semihypergroups as follows:

Definition 3.1. Let $(S, \circ)$ be a semihypergroup and $n \geq 2$. Then, we define the property π_n as follows:

$$(\pi_n) \qquad \text{for any } a_1, a_2, \ldots, a_n \in S,\ a_1 \circ a_2 \circ \cdots \circ a_n \subseteq \{a_1, a_2, \ldots, a_n\}.$$

Moreover, S is called a π_n-semihypergroup or a semihypergroup ot type π_n, whenever satisfies the property (π_n).

Every semigroup with the property (A_n) satisfies the property (π_n) this means that (π_n) is a suitable generalization of (A_n).

Remark 3.2. Semihypergroups of type π_2 are exactly the breakable ones and every breakable semihypergroup satisfies π_n for every $n \geq 2$. On the other hand, in addition to the group $\mathbb{Z}_2$, there exist semihypergroups of type π_3 but not π_2 (see Example 3.3). It will turn out that in a certain sense this is the only essentially new type of semihypergroups (that are not semigroups) with the property (π_n).

Example 3.3. (a) Consider the set $S_1 = \{e, a, b, c, d, f\}$ and define on it the hyperproduct given by Table 3.3. Then, $(S_1, \circ_1)$ is a non-breakable π_3-semihypergroup.

$\circ_1$	e	a	b	c	d	f
e	e	a	b	c	d	f
a	e	e	b	c	d	f
b	b	b	b	b	d	f
c	c	c	c	c	d	f
d	d	d	d	d	d	f
f	f	f	f	f	{d,f}	f

TABLE 1. A non-breakable π_3-semihypergroup on 6 elements

(b) Consider the set $S_2 = \{e, a, b, c, d\}$ and define on it the hyperproduct given by Table 3.3. Then, $(S_2, \circ_2)$ is a non-breakable π_3-semihypergroup.

$\circ_2$	e	a	b	c	d
e	e	a	b	c	d
a	e	e	b	c	d
b	b	b	b	b	{b,d}
c	c	c	{b,c}	c	{c,d}
d	d	d	{b,d}	d	d

TABLE 2. A non-breakable π_3-semihypergroup on 5 elements

In the following theorem we prove that a π_n-semihypergroup is breakable if and only if n is an even integer.

Theorem 3.4. *If $(S, \circ)$ is a semihypergroup of type π_n for some $n = 2k$ $(k \geq 2)$, then it is breakable.*

Proof. Let $k \geq 2$ and S be a semihypergroup of type π_{2k}. Suppose that $a \in S$ and $b \in a^2$. Then, $b \circ a^{2k-2} \subseteq a^{2k} = a$ so we have

(1) $$b \circ a^{2k-2} = a.$$

Also, the property (π_{2k}) implies

(2) $$b^2 \circ a^{2k-2} \subseteq \{a, b\}.$$

Since $b \in a^2$ we have

$$b^2 \circ a^{2k-2} \subseteq a^{2k+2} = a^3. \tag{3}$$

From the equations (2) and (3) we can conclude that $a \in a^3$ or $b \in a^3$. In the former case, we have $a^2 \subseteq a^4 \subseteq \cdots \subseteq a^{2k} = a$ thus,

$$a^2 = a. \tag{4}$$

Now, we consider the case $b \in a^3$. Then, the equation (1) concludes that

$$a = b \circ a^{2k-2} \subseteq a^3 \circ a^{2k-2} = a^2$$

hence, by the property (π_{2k})

$$a^k \subseteq a^{2k} = a. \tag{5}$$

Hence, the equations (4) and (5) imply that

$$a^k = a. \tag{6}$$

Now, let $x, y \in S$ and $z \in x \circ y$. Then, from the equation (6) and the property (π_{2k}) we can conclude that

$$z = z^k \in (x \circ y)^k \subseteq \{x, y\}, \tag{7}$$

therefore, $x \circ y \subseteq \{x, y\}$ and the proof is complete. □

In what follows, we focus on non-breakable semihypergroups of type π_n where n is an odd integer greater than 1.

Lemma 3.5. *Let $(S, \circ)$ be a non-breakable semihypergroup of type π_{2k+1}. Then, there exist $e, u \in S$ such that $u^2 = e \neq u$. Furthermore, e is an identity element.*

Proof. Let $(S, \circ)$ be a non-breakable semihypergroup of type π_{2k+1}, where $k \geq 1$. Let us first point out that if $x \in x^2$ for every $x \in S$, then

$$x \in x^2 \subseteq x^3 \subseteq \cdots \subseteq x^{2k+1} \subseteq \{x\},$$

thus $x = x^n$ for every $n \geq 1$ hence, $x \circ y = x^{2k} \circ y \subseteq \{x, y\}$ so S is breakable that is a contradiction. It follows that there exists $u \in S$ such that $u \notin u^2$. Now, take $c \in u^2$ and $e \in c^{2k}$. Then, $c^k \circ u \subseteq u^{2k} \circ u = u$ hence $c^k \circ u = u$. Similarly $u \circ c^k = u$. Furthermore,

$$c^{2k} \circ u = c^k \circ (c^k \circ u) = c^k \circ u = u.$$

and similarly we have $u \circ c^{2k} = u$. Also, since $e \neq u$ and

$$e \in c^{2k} = c^{2k-1} \circ c \subseteq c^{2k-1} \circ a^2 \subseteq \{c, u\}.$$

We conclude that $e = c$. Therefore, $e^2 \subseteq c \circ c^{2k} = c = e$ hence $e^2 = e$. Now, we show that $e \circ x = x = x \circ e$ for every $x \in S \setminus \{e, u\}$. It is sufficient to prove that $e \notin e \circ x$. If $e \in e \circ x$, then

$$e = e^k \subseteq e^{k-1} \circ (e \circ x) = e^k \circ x \subseteq u^{2k} \circ x \subseteq \{u, x\},$$

a contradiction. On the other hand, $e \circ x = e^{2k} \circ x \subseteq \{e, x\}$. Thus, $e \circ x = x$, and similarly we get $x \circ e = x$.

Consequently e is an identity element of S. Hence, uniqueness of the identity element implies $u^2 = e$. □

Theorem 3.6. *If $(S, \circ)$ is a semihypergroup of type π_{2k+1}, ($k \geq 2$), then $(S, \circ)$ is also of type π_3.*

Proof. Let $(S, \circ)$ be a semihypergroup of type (π_{2k+1}), $(k \geq 2)$. If S is breakable, then the conclusion follows from Remark 3.2. Now, suppose that S is non-breakable and $x, y, z \in S$. Then, Lemma 3.5 implies S has the identity element e and we can consider the following two cases.

Case 1. Let $e \in \{x, y, z\}$. Then, take $x = e$ we may write

$$x \circ y \circ z = e^{2k-1} \circ y \circ z \subseteq \{e, y, z\},$$

as required.

Case 2. Let $e \notin \{x, y, z\}$. Then, suppose by contradiction that $x \circ y \circ z \nsubseteq \{x, y, z\}$. Since

$$x \circ y \circ z = x \circ y \circ z \circ e^{2k-2} \subseteq \{x, y, z, e\}.$$

We obtain $e \in x \circ y \circ z$ and claim that

$$e \notin (x \circ y) \cap (y \circ z). \tag{8}$$

If $e \in x \circ y$, then $e \in (x \circ y)^{k-1} \circ (x \circ y \circ z) \subseteq \{x, y, z\}$ a contradiction. Similarly, we can see that $e \notin y \circ z$.

On the other hand, $x \circ y = x \circ y \circ e^{2k-1} \subseteq \{x, y, e\}$ implies $x \circ y \subseteq \{x, y\}$ thus we have

$$e \in x \circ y \circ z = (x \circ z) \cup (y \circ z),$$

hence, Equation 8 concludes that $e \in x \circ z$. Then, we obtain

$$e \in (x \circ y \circ z) \circ (x \circ z)^{k-1} \subseteq \{x, y, z\}.$$

Thus, this contraction completes the proof. □

Lemma 3.7. *Let $(S, \circ)$ be a non-breakable semihypergroup of type π_3 and $x, y \in S$. Then, the following assertions hold:*

(i) $|x^2| = 1$*;*

(ii) *If* $x \neq y$*, then* $x \circ y \subseteq \{x, y\}$*;*

(iii) *If* $x \neq x^2 = y^2 \neq y$*, then* $x = y$*.*

Proof. Let $(S, \circ)$ be a non-breakable semihypergroup of type π_3.

(i) For every $x \in S$, we can consider two cases:

Case 1. Let $x \in x^2$. Then, $x^2 \subseteq x^3 = x$ so $x^2 = x$.

Case 2. Let $x \notin x^2$. Then, there exists $a \in x^2$ such that $a \neq x$. Thus, $ax \subseteq x^3 = x$ hence the property (π_3) implies $x^2 = ax^2 \subseteq \{a, x\}$ thus $x^2 = a$. Therefore, in any cases we have $|x^2| = 1$.

(ii) Suppose by contradiction that $x \circ y \nsubseteq \{x, y\}$ for two distinct elements $x, y \in S$. Then, there exists $z \in S \setminus \{x, y\}$ such that $z \in x \circ y$. Hence, part (i) and the property (π_3) imply

$$x \circ z \circ y \subseteq x^2 \circ y^2 \subseteq \{x^2, y\} \cap \{x, y^2\}.$$

So, if $x \in x \circ z \circ y$, then $x^2 = x$ thus $z \in x \circ y = x^2 \circ y \subseteq \{x, y\}$, contradiction. Similarly, if $y \in x \circ z \circ y$ we have $z \in \{x, y\}$ is a contradiction. It follows that $z = x \circ z \circ y = x^2 = y^2$ thus $x^2 = z \in x \circ y$ and hence $x = x^3 \in x^2 \circ y = y^3 = y$ is a contradiction. Therefore, $x \circ y \subseteq \{x, y\}$.

(iii) Let $x, y \in S$ such that $x \neq x^2 = y^2 \neq y$. Then, by Lemma 3.5 and part (i) we conclude that $e = x^2 = y^2$, where e is the identity element of S. Suppose $x \neq y$ and $z \in x \circ y$, then $z \circ y \subseteq x \circ y^2 = x$ so part (ii) implies $x = z$ hence,

$$y^2 = e = x \circ x = x \circ (x \circ y) = x^2 \circ y = y.$$

This contradiction completes the proof. □

The following theorem shows that the minimum order of non-breakable semihypergroups of type π_3 which are not semigroups is 4.

Proposition 3.8. *Every non-breakable* π_3*-semihypergroup having cardinality less than 4 is a semigroup. Moreover, there exists a non-breakable* π_3*-semihypergroup having cardinality 4.*

Proof. Let $(S, \circ)$ be a non-breakable π_3-semihypergroup. Then, by Lemma 3.7 there exist $e, u \in S$ such that e is the identity element of S and $e = u^2 \neq e$. So, if $|S| = 2$, then S is isomorphic to the cyclic groups of order 2. Let $S = \{e, u, a\}$. Then, Lemma 3.7(ii) implies $a^2 \neq e$. Also, if $a^2 \neq u$, then $e = a^2 \circ u \subseteq a, u$ is impossible. Thus, $a^2 = a$. Moreover, $a \circ u$, since $u \in a \circ u$ implies $e \in u \circ a \circ u$ contradiction. Similarly, $u \circ a = a$. So, S is a semigroup.

An example having order 4. Consider the set $S = \{e, u, a, b\}$ and define on it the hyperproduct given by Table 3. Then, $(S, \circ)$ is a non-breakable π_3-semihypergroup. □

$\circ$	e	u	a	b
e	e	u	a	b
u	u	e	a	b
a	a	a	a	{a,b}
b	b	b	{a,b}	b

TABLE 3. A non-breakable π_3-semihypergroup on 4 elements

4. Characterization of π_n-semihypergroups

In what follows, we define a hyperoperation on the disjoint union of two distinct hypergroupoids in order to construct a new hypergroupoid containing each of them.

Definition 4.1. Let $(A, \circ)$ and $(B, *)$ be two hypergroupoids such that $A \cap B = \emptyset$. Then, we define the hyperoperation $\circledast$ on the set $S = A \cup B$ as follows: let $x, y \in S$

$$x \circledast y = \begin{cases} x \circ y; & \text{if } x, y \in A, \\ y; & \text{if } x \in A, y \in B, \\ x; & \text{if } x \in B, y \in A, \\ x * y; & \text{if } x, y \in B. \end{cases}$$

The hypergroupoiud $(S, \circledast)$ is called the disjoint extension of A by B and denoted by $S = A \uplus B$.

The following elementary properties can be checked directly using the structure of $A \uplus B$ considered in Definition 4.1.

Lemma 4.2. *The following assertions hold:*

(i) *If $(A, \circ)$ and $(B, *)$ are two semihypergroups, then $A \uplus B$ is a semihypergroup, too.*

(ii) *If $(A, \circ)$ and $(B, *)$ are two semihypergroups, then*

$$\frac{A \uplus B}{\beta^*} \cong \frac{A}{\beta^*} \uplus \frac{B}{\beta^*},$$

where β^* is the fundamental relation.

Proof. (i) Let $x, y, z \in A \cup B$. If $x \in A$ and $y, z \in B$, then by Definition 4.1, we have $x \circledast (y \circledast z) = y * z = (x \circledast y) \circledast z$. The proofs of other cases are similar.

(ii) It is concluded by considering the isomorphism $\beta^*(x) \mapsto \beta_A^*(x)$ $(\beta^*(x) \mapsto \beta_B^*(x))$ for every $x \in A (x \in B)$. $\square$

The next theorem shows that every semihypergroup of type π_n is breakable or disjoint extension of the cyclic group of order 2, $\mathbb{Z}_2$, by a breakable semihypergroup.

Theorem 4.3. *Let $(S, \circ)$ be a non-breakable π_3-semihypergroup. Then, there exists a breakable semihypergroup $(B, *)$ such that*

$$S \cong \mathbb{Z}_2 \uplus B.$$

Proof. Let $(S, \circ)$ be a non-breakable π_3-semihypergroup. Then, by Lemma 3.5, there exist $e, u \in S$ such that $e = u^2 \neq u$ and e is the identity element of S. Hence, $\{e, u\}$ is a subgroup of S isomorphic to $\mathbb{Z}_2$. Also, we see that in the subset $B = S \setminus \{e, u\}$, the property (π_2) holds so B is a breakable sub-semihypergroup of S moreover, for every $x \in B$ we have $x \circ e = x = x \circ x$ and $x \circ u = x = u \circ x$. Consequently, we have $S \cong \mathbb{Z}_2 \uplus B$. □

Proposition 4.4. *The fundamental semigroup of a π_n-semihypergroup satisfies the property A_n.*

Proof. The result holds for the case that S is a breakable semihypergroup, by Theorem 2.7. Now, let S be a non-breakable semihypergroup of type π_n, then Theorem 4.3 and Lemma 4.2 concludes that

$$\frac{S}{\beta^*} \cong \frac{\mathbb{Z}_2 \uplus B}{\beta^*} \cong \frac{\mathbb{Z}_2}{\beta^*} \uplus \frac{B}{\beta^*} \cong \mathbb{Z}_2 \uplus \frac{B}{\beta^*},$$

where B is a breakable semihypergroup and Proposition 2.6 implies $\frac{B}{\beta^*}$ is a breakable semigroup. □

Theorem 4.5. *Let $(S, \circ)$ be a semi-symmetric semihypergroup satisfying π_3. Then, either S is breakable or S can be partitioned into classes, i.e. $S = \bigcup_{\gamma \in \Gamma} S_\gamma$, where Γ is a chain with minimal $\varepsilon \in \Gamma$ and all S_γ, $\gamma \neq \varepsilon$, are pairwise disjoint l-semigroups, r-semigroups or B-hypergroups and S_ε is isomorphic to $\mathbb{Z}_2$. Moreover, for every $x \in S_\alpha$ and $y \in S_\beta$, with $\alpha < \beta$, we have $x \circ y = y \circ x = y$.*

Proof. Let $(S, \circ)$ be a semi-symmetric non-breakable π_3-semihypergroup. Then, by Theorem 4.3, there exists a breakable semihypergroup B such that $S \cong \mathbb{Z}_2 \uplus B$. Let $e, u \in s$ such that $S_\varepsilon = \{e, u\} \cong \mathbb{Z}_2$. Therefore, Proposition 2.6 completes the proof. □

Theorem 4.6. *A π_n-semihypergroup $(H, \circ)$ is a hypergroup if and only if $(H, \circ)$ is a B-hypergroup or isomorphic to $\mathbb{Z}_2$.*

Proof. First, suppose that $(H, \circ)$ is a breakable hypergroup. Then, for any two distinct elements x and y of H, by left reproducibility, there exists $z \in H$ such that $y \in x \circ z$. Since H is breakable, it follows that $\{x, z\}$ is a subsemihypergroup, so $x \circ z \subseteq \{x, z\}$. It follows that $y \in \{x, z\}$ and thus $y = z$ hence $y \in x \circ y$. Similarly, using the right reproducibility, one proves that $x \in x \circ y$. So, we obtain $x \circ y = \{x, y\}$, i.e. $(H, \circ)$ is a B-hypergroup.

Now, let $(H, \circ)$ be a non-breakable hypergroup of type π_{2k+1}. Then, by Lemma 3.5 there exist $e, u \in H$ such that $H_\varepsilon = \{e, u\}$ is a subgroup of H. Suppose $x \in H \setminus H_\varepsilon$ then by using the right reproducibility, there exists $y \in H$ such that $e \in x \circ y$. So, if $x \neq y$, then by the property π_{2k+1}, $e = y$ hence $e = x$ is a contradiction. Thus, $e \in x^2 = x$ is impossible. Therefore, $H = H_\varepsilon \cong \mathbb{Z}_2$.

The converse implication is obvious. □

5. Conclusions

We have addressed the problem of breakable semihypergroups, based on the classical concept of breakable semigroups. We have introduced and characterized semihypergroups, in which every product of n elements is equal to a subset of factors, called π_n-semihypergroups or semihypergroups of type π_n. Then, it was proved that every semihypergroup of type π_{2k}, $(k \geq 2)$ is breakable and every semihypergroup of type π_{2k+1} is of type π_3. Also, we have characterized semi-symmetric semihypergroups of type π_n.

6. Acknowledgments

The authors wish to sincerely thank the referees for several useful comments.

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