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Research Paper

MODULAR GROUP ALGEBRA WITH UPPER LIE NILPOTENCY INDEX $11p-9$

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Abstract. Let KG be the modular group algebra of a group G over a field K of characteristic $p > 0$. Recently, we have seen the classification of group algebras KG with upper Lie nilpotency index $t^L(KG)$ up to $10p-8$. In this paper, our aim is to classify the modular group algebra KG with upper Lie nilpotency index $11p-9$, for $G' = \gamma_2(G)$ as an abelian group.

1. Introduction

Let KG be the group algebra of a group G over a field K of characteristic $p > 0$. The group algebra KG can be treated as a Lie algebra, by defining the Lie commutator as $[x, y] = xy - yx$, $\forall x, y \in KG$. By induction, we let $[x_1, x_2, ...x_n] = [[x_1, x_2, ...x_{n-1}], x_n]$, where $x_1, x_2, ...x_n \in KG$. The n^{th} lower Lie power $KG^{[n]}$ of KG is the associated ideal generated by the Lie commutators $[x_1, x_2, ...x_n]$, where $KG^{[1]} = KG$. Using induction, the n^{th}

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upper Lie power $KG^{(n)}$ of KG is the associated ideal generated by all the Lie commutators $[x, y]$, where $x \in KG^{(n-1)}$, $y \in KG$ and $KG^{(1)} = KG$. Now KG called upper Lie nilpotent (lower Lie nilpotent) if there exists n such that $KG^{(n)} = 0\,(KG^{[n]} = 0)$. The least positive integer n such that $KG^{(n)} = 0$ and $KG^{[n]} = 0$ is said to be upper Lie nilpotency index and lower Lie nilpotency index of KG, denoted by $t^L(KG)$ and $t_L(KG)$ respectively. Other basic notations and definitions can be seen in [1]. Shalev [14] initiated the study of group algebras with maximum Lie nilpotency index. This problem was completed in [5]. Some interesting results on the next smaller Lie nilpotency index can be easily seen in [3, 4, 5, 6]. In [2], Bovdi and Kurdics discussed the upper and lower Lie nilpotency index of a modular group algebra of metabelian group G and determine the nilpotency class of the group of units. Sharma, Srivastava and Bist [11, 12] proved a classical result which states that if G is a non-abelian nilpotent group with $|G'| = p^n$, then $p + 1 \leq t_L(KG) \leq t^L(KG) \leq p^n + 1$. Thus we can say that $p + 1$ is the minimal and $p^n + 1$ is the maximal Lie nilpotency index. Therefore, it is clear that $2p$, $3p - 1$ and $4p - 2$ are the next possible minimal Lie nilpotency indices. Shalev initiated the classification of Lie nilpotent group algebras whose Lie nilpotency indices are $t_L(KG) = 2p$ and $3p - 1$, for $p \geq 5$ and obtained certain interesting results (see [13]). Sahai [7] classified the group algebras KG which are Lie nilpotent having Lie nilpotency indices $2p$, $3p - 1$ and $4p - 2$, for all $p > 0$. A complete description of the Lie nilpotent group algebras with next possible Lie nilpotency indices $5p - 3$, $6p - 4$, $7p - 5$, $8p - 6$ and $9p - 7$ is given in [8, 9, 10]. Recently, Bhatt and Chandra in [1], classified the group algebra KG which are Lie nilpotent with upper Lie nilpotency index $10p - 8$.

In this paper, we have characterized the group algebras with upper Lie nilpotency index $11p - 9$, with G' as an abelian group.

2. Preliminaries

We have used the following Lemma in the characterization of group algebra with upper Lie nilpotency index $11p - 9$ for computations of $d_{(m)}$'s in each case with $G' = \gamma_2(G)$.

Lemma 2.1. ([14]) *Let K be a field with $Char K = p > 0$ and G be a nilpotent group such that $|G'| = p^n$ and $exp(G') = p^l$.*

(1) *If $d_{(l+1)} = 0$ for some $l < pm$, then $d_{(pm+1)} \leq d_{(m+1)}$.*
(2) *If $d_{(m+1)} = 0$, then $d_{(s+1)} = 0$ for all $s \geq m$ with $\vartheta_{p'}(s) \geq \vartheta_{p'}(m)$, where $\vartheta_{p'}(x)$ is the maximal divisor of x which is relatively prime to p.*

3. Main Result

Theorem 3.1. *Let G be a group and K be a field of characteristics $p > 0$ such that KG is Lie nilpotent. Then $t^L(KG) = 11p - 9$ if and only if one of the following condition satisfied:*

(1) $G' \cong (C_7)^5$, $\gamma_3(G) = 1$;

(2) (a) $G' \cong C_{5^2} \times (C_5)^2$, $G'^5 \subseteq \gamma_3(G) \cong (C_5)^3$, $\gamma_4(G) \cong (C_5)^2$, $\gamma_5(G) \cong C_5$, $\gamma_6(G) = 1$;
(b) $G' \cong (C_5)^4$, $|G'^5 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_5)^3$, $\gamma_4(G) \cong C_5 \times C_5$, $\gamma_5(G) \cong C_5$, $\gamma_6(G) = 1$;

(3) (a) $G' \cong C_{25} \times (C_5)^3$, $|G'^5 \cap \gamma_3(G)| = 1$, $\gamma_4(G) \cong C_5$, $\gamma_3(G) \cong (C_5)^2$, $\gamma_5(G) = 1$ *or* $|G'^5 \cap \gamma_3(G)| = 5$, $\gamma_4(G) \cong C_5$, $\gamma_3(G) \cong (C_5)^3$, $\gamma_5(G) = 1$;
(b) $G' \cong (C_5)^5$, $|G'^5 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_5)^3$, $\gamma_4(G) \cong C_5$, $\gamma_5(G) = 1$;

(4) (a) $G' \cong (C_p)^5$, $|G'^p \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_p)^3$, $\gamma_4(G) \cong (C_p)^2$, $\gamma_5(G) \cong C_p$, $\gamma_6(G) = 1$, *for* $p \geq 5$;
(b) $G' \cong C_9 \times (C_3)^3$, $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong (C_3)^2$, $\gamma_5(G) \cong C_3$, $\gamma_6(G) = 1$;
(c) $G' \cong (C_3)^5$, $|G'^3 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong (C_3)^2$, $\gamma_5(G) \cong C_3$, $\gamma_6(G) = 1$;

(5) (a) $G' \cong C_8 \times C_2 \times C_2$, $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$, $\gamma_4(G) \cong (C_2)^2$, $\gamma_5(G) \cong C_2$, $\gamma_6(G) = 1$;
(b) $G' \cong C_4 \times C_4 \times C_2$, $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$, $\gamma_4(G) \cong (C_2)^2$, $\gamma_5(G) \cong C_2$, $\gamma_6(G) = 1$;
(c) $G' \cong C_4 \times (C_2)^3$, $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$, $\gamma_4(G) \cong (C_2)^2$, $\gamma_5(G) \cong C_2$, $\gamma_6(G) = 1$;
(d) $G' \cong (C_2)^5$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_4 \times C_2$, $\gamma_4(G) \cong (C_2)^2$, $\gamma_5(G) \cong C_2$, $\gamma_6(G) = 1$;

(6) (a) $G' \cong C_8 \times (C_2)^4$, $G'^2 \subseteq \gamma_3(G) \cong C_4$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$;
(b) $G' \cong C_4 \times (C_2)^5$, $G'^2 \subseteq \gamma_3(G) \cong C_4$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$;
(c) $G' \cong (C_2)^7$, $|G'^2 \cap \gamma_3(G)| = 1$,$\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$;
(d) $G' \cong C_4 \times C_4 \times (C_2)^3$, $G'^2 \subseteq \gamma_3(G) \cong C_4$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$;

(7) (a) $G' \cong C_4 \times (C_2)^4$, $|\gamma_3(G)| = 2^3$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong C_4 \times C_2$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$ *or* $|\gamma_3(G)| = 2^2$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_4$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$;
(b) $G' \cong C_8 \times (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 4$, $\gamma_3(G) \cong C_4 \times C_2$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$ *or* $|\gamma_3(G)| = 2^2$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong C_4$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$;
(c) $G' \cong C_4 \times C_4 \times C_2 \times C_2$, $|\gamma_3(G)| = 2^3$, $|G'^2 \cap \gamma_3(G)| = 4$, $\gamma_3(G) \cong C_4 \times C_2$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$ *or* $|\gamma_3(G)| = 2^2$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong C_4$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$;
(d) $G' \cong (C_2)^6$, $|\gamma_3(G)| = 2^3$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_4 \times C_2$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$;

(8) (a) $G' \cong C_9 \times C_9 \times C_3$, $|G'^3 \cap \gamma_3(G)| = 3$, $\gamma_3(G) \cong C_3 \times C_3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$ *or* $G'^3 \subseteq \gamma_3(G) \cong C_3^3$,$\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$;
(b) $G' \cong C_9 \times C_9 \times C_3$, $G'^3 \subseteq \gamma_3(G) \cong C_3^3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$;
(c) $G' \cong C_9 \times (C_3)^3$, $|G'^3 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_3 \times C_3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$, $G'^3 \subseteq \gamma_3(G) \cong C_3^3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$;
(d) $G' \cong (C_3)^5$, $G'^3 \subseteq \gamma_3(G) \cong C_3^3$, $\gamma_4(G) \cong C_3$ $\gamma_5(G) = 1$;

(9) $G' \cong (C_p)^5$, $|G'^p \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_p)^4$, $\gamma_4(G) \cong (C_p)^2$, $\gamma_5(G) = 1$, *for* $p \geq 5$*;*

(10) *(a)* $G' \cong C_9 \times C_9 \times C_3$, $|G'^3 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_3 \times C_3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$ *or* $|G'^3 \cap \gamma_3(G)| = 3$, $\gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$ *or* $G'^3 \subseteq \gamma_3(G) \cong C_3^4$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$*;*

(b) $G' \cong C_9 \times C_9 \times C_3$, $|G'^3 \cap \gamma_3(G)| = 3$, $\gamma_3(G) \cong C_3^3$, $\gamma_4(G) \cong (C_3)^2$, $\gamma_5(G) = 1$ *or* $G'^3 \subseteq \gamma_3(G) \cong (C_3)^4$, $\gamma_4(G) \cong (C_3)^2$, $\gamma_5(G) = 1$*;*

(c) $G' \cong C_9 \times (C_3)^3$, $|G'^3 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$ *or* $G'^3 \subseteq \gamma_3(G) \cong (C_3)^4$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$*;*

(d) $G' \cong C_9 \times (C_3)^3$, $|G'^3 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong (C_3)^2$, $\gamma_5(G) = 1$ *or* $G'^3 \subseteq \gamma_3(G) \cong (C_3)^4$, $\gamma_4(G) \cong (C_3)^2$, $\gamma_5(G) = 1$*;*

(e) $G' \cong (C_3)^5$, $G'^3 \subseteq \gamma_3(G) \cong (C_3)^4$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$*;*

(f) $G' \cong (C_3)^5$, $G'^3 \subseteq \gamma_3(G) \cong (C_3)^4$, $\gamma_4(G) \cong (C_3)^2$, $\gamma_5(G) = 1$*;*

(11) *(a)* $G' \cong C_4 \times C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$ *or* $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$ *or* $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$*;*

(b) $G' \cong C_4 \times C_4 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong (C_2)^2$, $\gamma_5(G) = 1$ *or* $G'^3 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong (C_2)^2$, $\gamma_5(G) = 1$*;*

(c) $G' \cong C_4 \times (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$ *or* $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$*;*

(d) $G' \cong C_4 \times (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong (C_2)^2$, $\gamma_5(G) = 1$ *or* $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong (C_2)^2$, $\gamma_5(G) = 1$*;*

(e) $G' \cong (C_2)^5$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$*;*

(f) $G' \cong (C_2)^5$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong (C_2)^2$, $\gamma_5(G) = 1$*;*

(12) $G' \cong (C_p)^6$, $|G'^p \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_p)^3$, $\gamma_4(G) \cong (C_p)^2$, $\gamma_5(G) = 1$, *for* $p \geq 5$*;*

(13) *(a)* $G' \cong C_9 \times C_9 \times C_3 \times C_3$, $|G'^3 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_3 \times C_3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$ *or* $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$*;*

(b) $G' \cong C_9 \times C_9 \times C_3 \times C_3$, $|G'^3 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_3 \times C_3$, $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong C_3^2$, $\gamma_5(G) = 1$*;*

(c) $G' \cong C_9 \times (C_3)^4$, $|G'^3 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_3 \times C_3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$ *or* $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$*;*

(d) $G' \cong C_9 \times (C_3)^4$, $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong (C_3)^2$, $\gamma_5(G) = 1$*;*

(e) $G' \cong (C_3)^6$, $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$*;*

(f) $G' \cong (C_3)^6$, $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong (C_3)^2$, $\gamma_5(G) = 1$*;*

(14) *(a)* $G' \cong C_4 \times C_4 \times C_2 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong C_2^2$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$ *or* $G'^3 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$*;*

(b) $G' \cong C_4 \times C_4 \times C_2 \times C_2$, $G'^3 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong (C_2)^2$, $\gamma_5(G) = 1$*;*

(c) $G' \cong C_4 \times (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_2^2$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$ *or* $G'^3 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$;

(d) $G' \cong C_4 \times (C_2)^4$, $G'^3 \subseteq \gamma_3(G) \cong C_2^3$, $\gamma_4(G) \cong C_2^2$, $\gamma_5(G) = 1$;

(e) $G' \cong (C_2)^6$, $G'^3 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$;

(f) $G' \cong (C_2)^6$, $G'^3 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong (C_2)^2$, $\gamma_5(G) = 1$;

(15) *(a)* $G' \cong (C_9)^2 \times (C_3)^3$, $G'^3 \subseteq \gamma_3(G) \cong (C_3)^2$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$;

(b) $G' \cong C_9 \times (C_3)^5$, $G'^3 \subseteq \gamma_3(G) \cong (C_3)^2$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$;

(c) $G' \cong (C_3)^7$, $G'^3 \subseteq \gamma_3(G) \cong (C_3)^2$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$;

(16) $G' \cong (C_p)^6$, $|G'^p \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_p)^4$, $\gamma_4(G) \cong C_p$, $\gamma_5(G) = 1$, *for* $p \geq 5$;

(17) *(a)* $G' \cong C_9 \times (C_3)^4$, $|G'^3 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$ *or* $G'^3 \subseteq \gamma_3(G) \cong (C_3)^4$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$;

(b) $G' \cong (C_3)^6$, $G'^3 \subseteq \gamma_3(G) \cong C_3^4$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$;

(18) *(a)* $G' \cong C_4 \times C_4 \times C_2 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$ *or* $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$ *or* $G'^2 \subseteq \gamma_3(G) \cong C_2^4$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$;

(b) $G' \cong C_4 \times (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$ *or* $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$;

(c) $G' \cong (C_2)^6$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$;

(19) $G' \cong (C_p)^7$, $|G'^p \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_p)^3$, $\gamma_4(G) \cong C_p$, $\gamma_5(G) = 1$, *for* $p \geq 5$;

(20) *(a)* $G' \cong C_9 \times (C_3)^5$, $|G'^3 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_3 \times C_3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$ *or* $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$;

(b) $G' \cong (C_3)^7$, $G'^3 \subseteq \gamma_3(G) \cong C_3^3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$;

(21) *(a)* $G' \cong (C_4)^2 \times (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong C_2 \times C_2$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$ *or* $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$;

(b) $G' \cong C_4 \times (C_2)^5$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_2 \times C_2$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$ *or* $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$;

(c) $G' \cong (C_2)^7$, $G'^2 \subseteq \gamma_3(G) \cong C_2^3$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$;

(22) $G' \cong (C_p)^8$, $|G'^p \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_p)^2$, $\gamma_4(G) \cong C_p$, $\gamma_5(G) = 1$, *for* $p \geq 5$;

(23) *(a)* $G' \cong C_9 \times (C_3)^6$, $G'^3 \subseteq \gamma_3(G) \cong (C_3)^2$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$;

(b) $G' \cong (C_3)^8$, $G'^3 \subseteq \gamma_3(G) \cong C_3^2$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$;

(24) *(a)* $G' \cong (C_4)^2 \times (C_2)^4$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) \cong C_2$ $\gamma_5(G) = 1$;

(b) $G' \cong C_4 \times (C_2)^6$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$;

(c) $G' \cong (C_2)^8$, $G'^2 \subseteq \gamma_3(G) \cong C_2^2$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$;

(25) *(a)* $G' \cong C_9 \times (C_3)^7$, $G'^3 \subseteq \gamma_3(G) \cong C_3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$;

(b) $G' \cong (C_3)^9$, $G'^3 \subseteq \gamma_3(G) \cong C_3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$;

(26) $G' \cong (C_p)^6$, $|G'^p \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_p)^5$, $\gamma_4(G) = 1$, $p \geq 3$;

(27) *(a)* $G' \cong (C_4)^3$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) = 1$ *or* $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) = 1$ *or* $|G'^2 \cap \gamma_3(G)| = 2^2$, $\gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) = 1$ *or* $G'^2 \subseteq \gamma_3(G) \cong (C_2)^5$, $\gamma_4(G) = 1$;
(b) $G' \cong C_4 \times C_4 \times C_2 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_2^3$, $\gamma_4(G) = 1$ *or* $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) = 1$ *or* $G'^2 \subseteq \gamma_3(G) \cong (C_2)^5$, $\gamma_4(G) = 1$;
(c) $G' \cong C_4 \times (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) = 1$ *or* $G'^2 \subseteq \gamma_3(G) \cong (C_2)^5$, $\gamma_4(G) = 1$;
(d) $G' \cong (C_2)^6$, $G'^2 \subseteq \gamma_3(G) \cong C_2^5$, $\gamma_4(G) = 1$;

(28) $G' \cong (C_p)^7$, $|G'^p \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_p)^4$, $\gamma_4(G) = 1$, *for* $p \geq 3$;

(29) *(a)* $G' \cong (C_4)^3 \times C_2$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_2$, $\gamma_4(G) = 1$ *or* $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) = 1$ *or* $|G'^2 \cap \gamma_3(G)| = 2^2$, $\gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) = 1$ *or* $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) = 1$;
(b) $G' \cong (C_4)^2 \times (C_2)^3$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) = 1$ *or* $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) = 1$ *or* $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) = 1$;
(c) $G' \cong C_4 \times (C_2)^5$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) = 1$ *or* $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) = 1$;
(d) $G' \cong (C_2)^7$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) = 1$;

(30) $G' \cong (C_p)^8$, $|G'^p \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_p)^3$, $\gamma_4(G) = 1$, $p \geq 3$;

(31) *(a)* $G' \cong (C_4)^2 \times (C_2)^4$, $G'^2 \subseteq \gamma_3(G) \cong C_2$, $\gamma_4(G) = 1$ *or* $G'^2 \subseteq \gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) = 1$ *or* $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) = 1$;
(b) $G' \cong (C_4)^2 \times (C_2)^4$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_2$, $\gamma_4(G) = 1$ *or* $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) = 1$ *or* $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) = 1$;
(c) $G' \cong C_4 \times (C_2)^6$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) = 1$ *or* $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) = 1$;
(d) $G' \cong (C_2)^8$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) = 1$;

(32) *(a)* $G' \cong (C_4)^2 \times (C_2)^5$, $G'^2 \subseteq \gamma_3(G) \cong C_2$, $\gamma_4(G) = 1$ *or* $G'^2 \subseteq \gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) = 1$;
(b) $G' \cong C_4 \times (C_2)^7$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_2$, $\gamma_4(G) = 1$ *or* $G'^2 \subseteq \gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) = 1$;
(c) $G' \cong (C_2)^9$, $G'^2 \subseteq \gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) = 1$;

(33) $G' \cong (C_p)^{10}$, $|G'^p \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_p$, $\gamma_4(G) = 1$, *for* $p \geq 3$;

(34) *(a)* $G' \cong C_4 \times (C_2)^8$, $G'^2 \subseteq \gamma_3(G) \cong C_2$, $\gamma_4(G) = 1$; *(b)* $G' \cong (C_2)^{10}$, $G'^2 \subseteq \gamma_3(G) \cong C_2$, $\gamma_4(G) = 1$;

(35) $G' \cong (C_p)^{11}$, $|G'^p \cap \gamma_3(G)| = 1$, $\gamma_3(G) = 1$, *for all* $p > 0$.

Proof. Let $t^L(KG) = 11p-9$, thus $l = \frac{t^L(KG)-2}{p-1} = 11$ and therefore $d_{(2)}+2d_{(3)}+3d_{(4)}+4d_{(5)}+5d_{(6)}+6d_{(7)}+7d_{(8)}+8d_{(9)}+9d_{(10)}+10d_{(11)}+11d_{(12)} = 11$. Now from [8], $d_{(12)} = 0$, $d_{(11)} = 0$,

$d_{(10)} = 0$ and $d_{(9)} = 0$. If $d_{(8)} \neq 0$, then we have the following possibilities: $d_{(2)} = 4, d_{(8)} = 1$ or $d_{(2)} = 1$, $d_{(4)} = 1$, $d_{(8)} = 1$ or $d_{(2)} = 2$, $d_{(3)} = 1$, $d_{(8)} = 1$. Now $d_{(2)} = 4$, $d_{(8)} = 1$ is possible if and only if $p = 7$, $G' \cong (C_7)^5$ and $\gamma_3(G) = 1$. The remaining cases are discarded by Lemma 2.1. If $d_{(8)} = 0$, then we have $d_{(2)} + 2d_{(3)} + 3d_{(4)} + 4d_{(5)} + 5d_{(6)} + 6d_{(7)} = 11$. Let $d_{(7)} \neq 0$, then we have the following cases: $d_{(2)} = d_{(5)} = d_{(7)} = 1$, $d_{(3)} = d_{(4)} = d_{(6)} = 0$ or $d_{(2)} = d_{(7)} = 1$, $d_{(3)} = 2$, $d_{(4)} = d_{(5)} = d_{(6)} = 0$ or $d_{(2)} = 2$, $d_{(3)} = d_{(5)} = d_{(6)} = 0$, $d_{(4)} = d_{(7)} = 1$ or $d_{(2)} = 3$, $d_{(3)} = d_{(7)} = 1$, $d_{(4)} = d_{(5)} = d_{(6)} = 0$ or $d_{(2)} = 5$, $d_{(3)} = d_{(4)} = d_{(5)} = d_{(6)} = 0$, $d_{(7)} = 1$.

Let $d_{(2)} = d_{(5)} = d_{(7)} = 1$, $d_{(3)} = d_{(4)} = d_{(6)} = 0$. If $p = 2$, then by Lemma 2.1(1), $d_{(7)} \leq d_{(4)}$, a contradiction. If $p \neq 2$, then by Lemma 2.1(2), $\vartheta_{p'}(4) \geq \vartheta_{p'}(3)$ implies that $d_{(5)} = 0$, a contradiction. Hence this case is not possible. Using similar arguments the other cases can be easily discarded.

So let $d_{(7)} = 0$, then we have $d_{(2)} + 2d_{(3)} + 3d_{(4)} + 4d_{(5)} + 5d_{(6)} = 11$ and the possibilities for $d_{(6)} = 0$ or 1 or 2. If $d_{(6)} = 2$, then we have the case $d_{(2)} = 1$, $d_{(3)} = d_{(4)} = d_{(5)} = 0$ and in view of Lemma 2.1, this is not possible.

Now if $d_{(6)} = 1$, then we have the following cases: $d_{(2)} = d_{(3)} = d_{(4)} = 1$, $d_{(5)} = 0$ or $d_{(2)} = 2$, $d_{(3)} = d_{(4)} = 0$, $d_{(5)} = 1$ or $d_{(2)} = d_{(3)} = 2$, $d_{(4)} = d_{(5)} = 0$ or $d_{(2)} = 3$, $d_{(3)} = d_{(5)} = 0$, $d_{(4)} = 1$ or $d_{(2)} = 6$, $d_{(3)} = d_{(4)} = d_{(5)} = 0$.

Let $d_{(2)} = d_{(3)} = d_{(4)} = d_{(6)} = 1$, $d_{(5)} = 0$. If $p \neq 5$, then by Lemma 2.1(2), $\vartheta_{p'}(5) \geq \vartheta_{p'}(4)$ implies $d_{(6)} = 0$, which is not possible. Now if $p = 5$, then $|G'| = 5^4$, $|D_{(6),K}(G)| = 5$, $|D_{(5),K}(G)| = 5$, $|D_{(4),K}(G)| = 5^2$, $|D_{(3),K}(G)| = 5^3$. Let G' be an abelian group, then the possibilities for G' are: (a) $G' \cong C_{25} \times C_{25}$ (b) $G' \cong C_{25} \times C_5 \times C_5$ (c) $G' \cong (C_5)^4$. If $G' \cong C_{25} \times C_{25}$, then $G'^5 \cong C_5 \times C_5$, which is not possible as $|G'^5| \leq 5$. If $G' \cong C_{25} \times C_5 \times C_5$, then $G'^5 \cong C_5$ and $G' \cong C_{25} \times C_5 \times C_5$, $G'^5 \subseteq \gamma_3(G) \cong (C_5)^3$, $\gamma_4(G) \cong C_5 \times C_5$, $\gamma_5(G) \cong C_5$ and $\gamma_6(G) = 1$. If $G' \cong (C_5)^4$, then $|G'^5 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_5)^3$, $\gamma_4(G) \cong C_5 \times C_5$, $\gamma_5(G) \cong C_5$ and $\gamma_6(G) = 1$.

Let $d_{(2)} = 2$, $d_{(3)} = d_{(4)} = 0$, $d_{(5)} = d_{(6)} = 1$. Then again in view of Lemma 2.1 this case is not possible.

Let $d_{(2)} = d_{(3)} = 2$ $d_{(4)} = d_{(5)} = 0$, $d_{(6)} = 1$. If $p \neq 5$, and $d_{(5)} = 0$, then by Lemma 2.1(2), $\vartheta_{p'}(5) \geq \vartheta_{p'}(4)$ implies $d_{(6)} = 0$, which is a contradiction. If $p = 5$, then $|G'| = 5^5$, $|D_{(6),K}(G)| = 5$, $|D_{(5),K}(G)| = 5$, $|D_{(4),K}(G)| = 5$ and $|D_{(3),K}(G)| = 5^3$. Let G' is abelian, then we have the following possibilities for G': (a) $G' \cong C_{25} \times C_{25} \times C_5$ (b) $G' \cong C_{25} \times (C_5)^3$ (c) $G' \cong (C_5)^5$. Clearly $G' \cong C_{25} \times C_{25} \times C_5$ is not possible as $|G'^5| \leq 5$. If $G' \cong C_{25} \times (C_5)^3$, then

$|G'^5 \cap \gamma_3(G)| = 1$, $\gamma_4(G) \cong C_5$, $\gamma_3(G) \cong (C_5)^2$ and $\gamma_5(G) = 1$ or $|G'^5 \cap \gamma_3(G)| = 5$, $\gamma_4(G) \cong C_5$, $\gamma_3(G) \cong (C_5)^3$ and $\gamma_5(G) = 1$. If $G' \cong (C_5)^5$, then $|G'^5 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_5)^3$, $\gamma_4(G) \cong C_5$ and $\gamma_5(G) = 1$.
Let $d_{(2)} = 3$, $d_{(3)} = d_{(5)} = 0$, $d_{(4)} = d_{(6)} = 1$ and $d_{(2)} = 6$, $d_{(3)} = d_{(4)} = d_{(5)} = 0$, $d_{(6)} = 1$. Again these two cases are discarded by Lemma 2.1.

If $d_{(6)} = 0$, then $d_{(2)} + 2d_{(3)} + 3d_{(4)} + 4d_{(5)} = 11$ and $d_{(5)} = 0$ or 1 or 2. Let $d_{(5)} = 2$, then we have the following cases $d_{(2)} = 1$, $d_{(3)} = 1$, $d_{(4)} = 0$ or $d_{(2)} = 3$, $d_{(3)} = 0$, $d_{(4)} = 0$. These cases are not possible by Lemma 2.1.
Let $d_{(5)} = 1$, then $d_{(2)} + 2d_{(3)} + 3d_{(4)} = 7$ and we have the following possibilities: $d_{(2)} = 1$, $d_{(3)} = 3$, $d_{(4)} = 0$ or $d_{(2)} = 1$, $d_{(3)} = 0$, $d_{(4)} = 2$ or $d_{(2)} = 2$, $d_{(3)} = 1$, $d_{(4)} = 1$ or $d_{(2)} = 5$, $d_{(3)} = 1$, $d_{(4)} = 0$ or $d_{(2)} = 3$, $d_{(3)} = 2$, $d_{(4)} = 0$ or $d_{(2)} = 7$, $d_{(3)} = 0$, $d_{(4)} = 0$.

Let $d_{(2)} = 1$, $d_{(3)} = 3$, $d_{(4)} = 0$, $d_{(5)} = 1$ or $d_{(2)} = 1$, $d_{(3)} = 0$, $d_{(4)} = 2$, $d_{(5)} = 1$, then by Lemma 2.1 these cases are not possible for any p.
Let $d_{(2)} = 2$, $d_{(3)} = 1$, $d_{(4)} = 1$, $d_{(5)} = 1$. Then $|G'| = p^5$, for every $p > 0$ and $|D_{(3),K}(G)| = p^3$, $|D_{(4),K}(G)| = p^2$, $|D_{(5),K}(G)| = p$ and $D_{(6),K}(G) = 1$. Let G' be an abelian group. If $p \geq 5$, then $G' \cong (C_p)^5$, $|G'^p \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_p)^3$, $\gamma_4(G) \cong (C_p)^2$ and $\gamma_5(G) \cong C_p$, $\gamma_6(G) = 1$. If $p = 3$, then $|D_{(6),K}(G)| = 1$, $|D_{(5),K}(G)| = 3$, $|D_{(4),K}(G)| = 3^2$, $|D_{(3),K}(G)| = 3^3$ and $|G'| = 3^5$. Hence we have the following possibilities: (a) $G' \cong C_9 \times C_9 \times C_3$ (b) $G' \cong C_9 \times (C_3)^3$ (c) $G' \cong (C_3)^5$. If $G' \cong C_9 \times C_9 \times C_3$, then this case is not possible as $G'^9 = 1$. If $G' \cong C_9 \times (C_3)^3$, then $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong (C_3)^2$, $\gamma_5(G) \cong C_3$, $\gamma_6(G) = 1$. If $G' \cong \times (C_3)^5$, then $|G'^3 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong (C_3)^2$, $\gamma_5(G) \cong C_3$, $\gamma_6(G) = 1$. If $p = 2$, then $|D_{(6),K}(G)| = 1$, $|D_{(5),K}(G)| = 2$, $|D_{(4),K}(G)| = 4$, $|D_{(3),K}(G)| = 8$ and $|G'| = 2^5$. Let G' be an abelian group. We have the following possibilities: (a) $G' \cong C_8 \times C_4$ (b) $G' \cong C_8 \times C_2 \times C_2$ (c) $G' \cong C_4 \times C_4 \times C_2$ (d) $G' \cong C_4 \times (C_2)^3$ (e) $G' \cong (C_2)^5$. Since $G' \cong C_8 \times C_4$ is not possible as $|G'^2| = 8$. If $G' \cong C_8 \times C_2 \times C_2$, then $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$, $\gamma_4(G) \cong (C_2)^2$, $\gamma_5(G) \cong C_2$, $\gamma_6(G) = 1$. If $G' \cong C_4 \times C_4 \times C_2$, then $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$, $\gamma_4(G) \cong (C_2)^2$, $\gamma_5(G) \cong C_2$, $\gamma_6(G) = 1$. If $G' \cong C_4 \times (C_2)^3$, then $G'^2 \subseteq \gamma_3(G) \cong C_4 \times C_2$, $\gamma_4(G) \cong (C_2)^2$, $\gamma_5(G) \cong C_2$, $\gamma_6(G) = 1$. If $G' \cong (C_2)^5$, then $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_4 \times C_2$, $\gamma_4(G) \cong (C_2)^2$, $\gamma_5(G) \cong C_2$, $\gamma_6(G) = 1$.

Let $d_{(2)} = 5$, $d_{(3)} = 1$, $d_{(4)} = 0$, $d_{(5)} = 1$, then by Lemma 2.1, this case is not possible for all $p \neq 2$. If $p = 2$, then $|D_{(6),K}(G)| = 1$, $|D_{(5),K}(G)| = 2$, $|D_{(4),K}(G)| = 2$, $|D_{(3),K}(G)| = 4$ and $|G'| = 2^7$. Let G' be an abelian group, then we have the following possibilities: (a) $G' \cong C_8 \times C_8 \times C_2$ (b) $G' \cong C_8 \times C_4 \times C_2 \times C_2$ (c) $G' \cong C_8 \times (C_2)^4$ (d) $G' \cong C_8 \times C_4 \times C_4$ (e)

$G' \cong C_4 \times (C_2)^5$ (f) $G' \cong (C_4)^3 \times C_2$ (g) $G' \cong C_4 \times C_4 \times (C_2)^3$ (h) $G' \cong (C_2)^7$. Now clearly $G' \cong C_8 \times C_8 \times C_2$ or $C_8 \times C_4 \times C_2 \times C_2$ or $C_8 \times C_4 \times C_4$ or $(C_4)^3 \times C_2$ are not possible as $|G'^4| \leq 2$. If $G' \cong C_8 \times (C_2)^4$, then $G'^2 \subseteq \gamma_3(G) \cong C_4$, $\gamma_4(G) \cong C_2$ and $\gamma_5(G) = 1$. If $G' \cong C_4 \times (C_2)^5$, then $G'^2 \subseteq \gamma_3(G) \cong C_4$, $\gamma_4(G) \cong C_2$ and $\gamma_5(G) = 1$. If $G' \cong (C_2)^7$, then $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_4(G) \cong C_2$ and $\gamma_5(G) = 1$. If $G' \cong C_4 \times C_4 \times (C_2)^3$, then $G'^2 \subseteq \gamma_3(G) \cong C_4$, $\gamma_4(G) \cong C_2$ and $\gamma_5(G) = 1$.

Let $d_{(2)} = 3$, $d_{(3)} = 2$, $d_{(4)} = 0$, $d_{(5)} = 1$, then by Lemma 2.1 this case is not possible for all $p \neq 2$. If $p = 2$, then $|D_{(6),K}(G)| = 1$, $|D_{(5),K}(G)| = 2$, $|D_{(4),K}(G)| = 2$, $|D_{(3),K}(G)| = 8$ and $|G'| = 2^6$. Let G' be an abelian group, then we have the following possibilities: (a) $G' \cong C_8 \times C_8$ (b) $G' \cong C_8 \times C_4 \times C_2$ (c) $G' \cong C_8 \times (C_2)^3$ (d) $G' \cong C_4 \times (C_2)^4$ (e) $G' \cong (C_4)^3$ (f) $G' \cong C_4 \times C_4 \times C_2 \times C_2$ (g) $G' \cong (C_2)^6$. Since $|G'^2| \leq 4$, therefore $G' \cong C_8 \times C_8$ or $C_8 \times C_4 \times C_2$ or $(C_4)^3$ are not possible. If $G' \cong C_4 \times (C_2)^4$, $|\gamma_3(G)| = 2^3$ then $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong C_4 \times C_2$, $\gamma_4(G) \cong C_2$ and $\gamma_5(G) = 1$. If $|\gamma_3(G)| = 2^2$ then $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_4$, $\gamma_4(G) \cong C_2$ and $\gamma_5(G) = 1$. If $G' \cong C_8 \times (C_2)^3$, then we have for $|\gamma_3(G)| = 2^3$, $|G'^2 \cap \gamma_3(G)| = 4$, $\gamma_3(G) \cong C_4 \times C_2$, $\gamma_4(G) \cong C_2$ and $\gamma_5(G) = 1$ and for $|\gamma_3(G)| = 2^2$, $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong C_4$, $\gamma_4(G) \cong C_2$ and $\gamma_5(G) = 1$. If $G' \cong C_4 \times C_4 \times C_2 \times C_2$, then we have for $|\gamma_3(G)| = 2^3$, $|G'^2 \cap \gamma_3(G)| = 4$, $\gamma_3(G) \cong C_4 \times C_2$, $\gamma_4(G) \cong C_2$ and $\gamma_5(G) = 1$ and if $|\gamma_3(G)| = 2^2$, then $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong C_4$, $\gamma_4(G) \cong C_2$ and $\gamma_5(G) = 1$. If $G' \cong (C_2)^6$, then we have for $|\gamma_3(G)| = 2^3$, $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_4 \times C_2$, $\gamma_4(G) \cong C_2$ and $\gamma_5(G) = 1$.

Let $d_5 = 0$, then $d_{(2)} + 2d_{(3)} + 3d_{(4)} = 11$ and we have $d_{(4)} = 0$ or 1 or 2 or 3. Let $d_{(4)} = 3$, then we have only one case $d_{(2)} = 2$, $d_{(3)} = 0$, $d_{(4)} = 3$ and by Lemma 2.1 this case is not possible for all $p \neq 3$. If $p = 3$, then $|D_{(5),K}(G)| = 1$, $|D_{(4),K}(G)| = 3^3$, $|D_{(3),K}(G)| = 3^3$ and $|G'| = 3^5$. Let G' be an abelian group hence we have the following possibilities: (a) $G' \cong C_9 \times C_9 \times C_3$ (b) $G' \cong C_9 \times (C_3)^3$ (c) $G' \cong (C_3)^5$. Let $G' \cong C_9 \times C_9 \times C_3$. If $|\gamma_4(G)| = 3$, then $|\gamma_3(G)| = 3^2, 3^3$, then $|G'^3 \cap \gamma_3(G)| = 3$, $\gamma_3(G) \cong C_3 \times C_3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$ and $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong C_3$ and $\gamma_5(G) = 1$. If $|\gamma_4(G)| = 3^2$, then $|\gamma_3(G)| = 3^3$ and $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong C_3$ and $\gamma_5(G) = 1$. Let $G' \cong C_9 \times (C_3)^3$, then $|G'^3 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_3 \times C_3$, $\gamma_4(G) \cong C_3$ $\gamma_5(G) = 1$ and $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong C_3$ $\gamma_5(G) = 1$. Let $G' \cong (C_3)^5$, then $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong C_3$ $\gamma_5(G) = 1$.

Let $d_{(4)} = 2$, then $d_{(2)} + 2d_{(3)} = 5$. Now we have the following cases: $d_{(2)} = 1$, $d_{(3)} = 2$, $d_{(4)} = 2$ or $d_{(2)} = 3$, $d_{(3)} = 1$, $d_{(4)} = 2$ or $d_{(2)} = 2$, $d_{(3)} = 0$, $d_{(4)} = 2$.

Let $d_{(2)} = 1$, $d_{(3)} = 2$, $d_{(4)} = 2$, this case is possible for all $p > 0$ and $|G'| = p^5$, for every $p > 0$. Thus $|D_{(3),K}(G)| = p^4$, $|D_{(4),K}(G)| = p^2$, $|D_{(5),K}(G)| = 1$. Let G' be an abelian group. Let $p \geq 5$, then $G' \cong (C_p)^5$, $|G'^p \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_p)^4$, $\gamma_4(G) \cong (C_p)^2$ and $\gamma_5(G) = 1$. If $p = 3$, then $|D_{(5),K}(G)| = 1$, $|D_{(4),K}(G)| = 3^2$, $|D_{(3),K}(G)| = 3^4$ and $|G'| = 3^5$ hence we have the following possibilities: (a) $G' \cong C_9 \times C_9 \times C_3$ (b) $G' \cong C_9 \times (C_3)^3$ (c) $G' \cong (C_3)^5$. Let $G' \cong C_9 \times C_9 \times C_3$. If $|\gamma_4(G)| = 3$, then $|\gamma_3(G)| = 3^2$ or 3^3 or 3^4, $|G'^3 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_3 \times C_3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$ or $|G'^3 \cap \gamma_3(G)| = 3$, $\gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$ or $G'^3 \subseteq \gamma_3(G) \cong (C_3)^4$, $\gamma_4(G) \cong C_3$ and $\gamma_5(G) = 1$. If $|\gamma_4(G)| = 3^2$, then $|\gamma_3(G)| = 3^3$ or 3^4 thus $|G'^3 \cap \gamma_3(G)| = 3$, $\gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong (C_3)^2$, $\gamma_5(G) = 1$ or $G'^3 \subseteq \gamma_3(G) \cong (C_3)^4$, $\gamma_4(G) \cong (C_3)^2$, $\gamma_5(G) = 1$. Now Let $G' \cong C_9 \times (C_3)^3$. If $|\gamma_4(G)| = 3$, then $|\gamma_3(G)| = 3^2$ or 3^3 or 3^4 thus $|G'^3 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$ or $G'^3 \subseteq \gamma_3(G) \cong (C_3)^4$, $\gamma_4(G) \cong C_3$ and $\gamma_5(G) = 1$. If $|\gamma_4(G)| = 3^2$, then $|\gamma_3(G)| = 3^3$ or 3^4 thus $|G'^3 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong (C_3)^2$, $\gamma_5(G) = 1$ or $G'^3 \subseteq \gamma_3(G) \cong (C_3)^4$, $\gamma_4(G) \cong (C_3)^2$ and $\gamma_5(G) = 1$. Let $G' \cong (C_3)^5$. If $|\gamma_4(G)| = 3$, then $|\gamma_3(G)| = 3^2$ or 3^3 or 3^4 thus $G'^3 \subseteq \gamma_3(G) \cong (C_3)^4$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$ and if $|\gamma_4(G)| = 3^2$, then $|\gamma_3(G)| = 3^3$ or 3^4 thus $G'^3 \subseteq \gamma_3(G) \cong (C_3)^4$, $\gamma_4(G) \cong (C_3)^2$, $\gamma_5(G) = 1$. If $p = 2$, then $|D_{(5),K}(G)| = 1$, $|D_{(4),K}(G)| = 4$, $|D_{(3),K}(G)| = 2^4$ and $|G'| = 2^5$. Let G' be an abelian group, then we have the following possibilities: (a) $G' \cong C_4 \times C_4 \times C_2$ (b) $G' \cong C_4 \times (C_2)^3$ (c) $G' \cong (C_2)^5$. Let $G' \cong C_4 \times C_4 \times C_2$. If $|\gamma_4(G)| = 2$, then $|\gamma_3(G)| = 2^2$ or 2^3 or 4^4 and thus $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$ or $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$ or $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong C_2$ and $\gamma_5(G) = 1$. If $|\gamma_4(G)| = 2^2$, then $|\gamma_3(G)| = 2^3$ or 2^4 and thus $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong (C_2)^2$, $\gamma_5(G) = 1$ or $G'^3 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong C_2^2$, $\gamma_5(G) = 1$. Let $G' \cong C_4 \times (C_2)^3$. If $|\gamma_4(G)| = 2$, then $|\gamma_3(G)| = 2^2$ or 2^3 or 4^4 and thus $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$ or $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong C_2$ and $\gamma_5(G) = 1$. If $|\gamma_4(G)| = 2^2$, then $|\gamma_3(G)| = 2^3$ or 2^4 and thus $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong (C_2)^2$, $\gamma_5(G) = 1$ or $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong (C_2)^2$ and $\gamma_5(G) = 1$. Let $G' \cong (C_2)^5$. If $|\gamma_4(G)| = 2$, then $|\gamma_3(G)| = 2^2$ or 2^3 or 4^4 and thus $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong C_2$ and $\gamma_5(G) = 1$. If $|\gamma_4(G)| = 2^2$, then $|\gamma_3(G)| = 2^3$ or 2^4 and thus $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong (C_2)^2$ and $\gamma_5(G) = 1$.

Let $d_{(2)} = 3$, $d_{(3)} = 1$, $d_{(4)} = 2$, then this case is possible for all $p > 0$ and $|G'| = p^6$, for every $p > 0$. Thus $|D_{(3),K}(G)| = p^3$, $|D_{(4),K}(G)| = p^2$, $|D_{(5),K}(G)| = 1$. Let G' be an abelian group. If $p \geq 5$, then $G' \cong (C_p)^6$, $|G'^p \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_p)^3$, $\gamma_4(G) \cong (C_p)^2$ and $\gamma_5(G) = 1$. If $p = 3$, then $|D_{(5),K}(G)| = 1$, $|D_{(4),K}(G)| = 3^2$, $|D_{(3),K}(G)| = 3^3$ and $|G'| = 3^6$, hence we have the following possibilities: (a) $G' \cong (C_9)^3$ (b) $G' \cong C_9 \times C_9 \times C_3 \times C_3$ (c) $G' \cong C_9 \times (C_3)^4$ (d) $G' \cong (C_3)^6$. Clearly $G' \cong (C_9)^3$ is not possible. Now let $G' \cong C_9 \times C_9 \times C_3 \times C_3$. If

$|\gamma_4(G)| = 3$, then $|\gamma_3(G)| = 3^2$ or 3^3 and thus $|G'^3 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_3 \times C_3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$ or $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong C_3$ $\gamma_5(G) = 1$. If $|\gamma_4(G)| = 3^2$, then $|\gamma_3(G)| = 3^3$, $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong (C_3)^2$, $\gamma_5(G) = 1$. Let $G' \cong C_9 \times (C_3)^4$. If $|\gamma_4(G)| = 3$, then $|\gamma_3(G)| = 3^2$ or 3^3 and thus $|G'^3 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_3 \times C_3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$ or $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$. If $|\gamma_4(G)| = 3^2$, then $|\gamma_3(G)| = 3^3$ and thus $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong (C_3)^2$, $\gamma_5(G) = 1$. Now, let $G' \cong (C_3)^6$. If $|\gamma_4(G)| = 3$, then $|\gamma_3(G)| = 3^2$ or 3^3 and thus $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$. If $|\gamma_4(G)| = 3^2$, then $|\gamma_3(G)| = 3^3$ and thus $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong (C_3)^2$, $\gamma_5(G) = 1$. Now let $p = 2$, then $|D_{(5),K}(G)| = 1$, $|D_{(4),K}(G)| = 4$, $|D_{(3),K}(G)| = 2^3$ and $|G'| = 2^6$. If G' be an abelian group, then we have the following possibilities: (a) $G' \cong (C_4)^3$ (b) $G' \cong C_4 \times C_4 \times C_2 \times C_2$ (c) $G' \cong C_4 \times (C_2)^4$ (d) $G' \cong (C_2)^6$. Let $G' \cong (C_4)^3$, then clearly this case is not possible. Now let $G' \cong C_4 \times C_4 \times C_2 \times C_2$. If $|\gamma_4(G)| = 2$, then $|\gamma_3(G)| = 2^2$ or 2^3 and thus $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$ or $G'^3 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$. If $|\gamma_4(G)| = 2^2$, then $|\gamma_3(G)| = 2^3$ and thus $G'^3 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong (C_2)^2$, $\gamma_5(G) = 1$. Let $G' \cong C_4 \times (C_2)^4$. If $|\gamma_4(G)| = 2$, then $|\gamma_3(G)| = 2^2$ or 2^3 and thus $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$ or $G'^3 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$. If $|\gamma_4(G)| = 2^2$, then $|\gamma_3(G)| = 2^3$ and thus $G'^3 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong (C_2)^2$, $\gamma_5(G) = 1$. Let $G' \cong (C_2)^6$. If $|\gamma_4(G)| = 2$, then $|\gamma_3(G)| = 2^2$ or 2^3 and thus $G'^3 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$. If $|\gamma_4(G)| = 2^2$, then $|\gamma_3(G)| = 2^3$ and thus $G'^3 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong (C_2)^2$, $\gamma_5(G) = 1$.

Let $d_{(2)} = 5$, $d_{(3)} = 0$, $d_{(4)} = 2$, then by Lemma 2.1 this case is possible only when $p = 3$. Thus $|D_{(5),K}(G)| = 1$, $|D_{(4),K}(G)| = 3^2$, $|D_{(3),K}(G)| = 3^2$ and $|G'| = 3^7$. Let G' be an abelian group, hence we have the following possibilities: (a) $G' \cong (C_9)^3 \times C_3$ (b) $G' \cong (C_9)^2 \times (C_3)^3$ (c) $G' \cong C_9 \times (C_3)^5$ (d) $G' \cong (C_3)^7$. Clearly $G' \cong (C_9)^3 \times C_3$ is not possible. If $G' \cong (C_9)^2 \times (C_3)^3$, then $G'^3 \subseteq \gamma_3(G) \cong (C_3)^2$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$. If $G' \cong C_9 \times (C_3)^5$, then $G'^3 \subseteq \gamma_3(G) \cong (C_3)^2$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$. If $G' \cong (C_3)^7$, then $G'^3 \subseteq \gamma_3(G) \cong (C_3)^2$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$.

Let $d_{(4)} = 1$, then $d_{(2)} + 2d_{(3)} = 8$ and we have the following cases $d_{(2)} = 2$, $d_{(3)} = 3$, $d_{(4)} = 1$ or $d_{(2)} = 4$, $d_{(3)} = 2$, $d_{(4)} = 1$ or $d_{(2)} = 6$, $d_{(3)} = 1$, $d_{(4)} = 1$ or $d_{(2)} = 8$, $d_{(3)} = 0$, $d_{(4)} = 1$.

Let $d_{(2)} = 2$, $d_{(3)} = 3$, $d_{(4)} = 1$, this case is possible for all $p > 0$ and $|G'| = p^6$ for every $p > 0$. Thus $|D_{(3),K}(G)| = p^4$, $|D_{(4),K}(G)| = p$, $|D_{(5),K}(G)| = 1$. Let G' be an abelian group. If $p \geq 5$, then $G' \cong (C_p)^6$, $|G'^p \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_p)^4$, $\gamma_4(G) \cong C_p$ and $\gamma_5(G) = 1$. If $p = 3$, then $|D_{(5),K}(G)| = 1$, $|D_{(4),K}(G)| = 3$, $|D_{(3),K}(G)| = 3^4$ and $|G'| = 3^6$, hence we have the following possibilities: (a) $G' \cong (C_9)^3$ (b) $G' \cong C_9 \times C_9 \times C_3 \times C_3$ (c) $G' \cong C_9 \times (C_3)^4$ (d) $G' \cong (C_3)^6$. Clearly $G' \cong (C_9)^3$ and $G' \cong C_9 \times C_9 \times C_3 \times C_3$ are not possible. Let $G' \cong C_9 \times (C_3)^4$. If $|\gamma_4(G)| = 3$, then $|\gamma_3(G)| = 3^2$ or 3^3 or 3^4 and therefore $|G'^3 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$ or $G'^3 \subseteq \gamma_3(G) \cong (C_3)^4$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$. Let $G' \cong (C_3)^6$, $G'^3 \subseteq \gamma_3(G) \cong (C_3)^4$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$. If $p = 2$, then $|D_{(5),K}(G)| = 1$, $|D_{(4),K}(G)| = 2$, $|D_{(3),K}(G)| = 2^4$ and $|G'| = 2^6$ hence we have the following possibilities: (a) $G' \cong (C_4)^3$ (b) $G' \cong C_4 \times C_4 \times C_2 \times C_2$ (c) $G' \cong C_4 \times (C_2)^4$ (d) $G' \cong (C_2)^6$. Clearly $G' \cong (C_4)^3$ this case is not possible. Let $G' \cong C_4 \times C_4 \times C_2 \times C_2$. If $|\gamma_4(G)| = 2$, then $|\gamma_3(G)| = 2^2$ or 2^3 or 2^4 and thus $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_2^2$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$ or $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$ or $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$. Let $G' \cong C_4 \times (C_2)^4$. If $|\gamma_4(G)| = 2$, then $|\gamma_3(G)| = 2^2$ or 2^3 or 2^4 and thus $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$ or $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$. Let $G' \cong (C_2)^6$. If $|\gamma_4(G)| = 2$, then $|\gamma_3(G)| = 2^2$ or 2^3 or 2^4 and thus $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$.

Let $d_{(2)} = 4$, $d_{(3)} = 2$, $d_{(4)} = 1$, then this case is possible for all $p > 0$ and $|G'| = p^7$ for every $p > 0$. Thus $|D_{(3),K}(G)| = p^3$, $|D_{(4),K}(G)| = p$, $|D_{(5),K}(G)| = 1$. Let G' be an abelian group. If $p \geq 5$, then $G' \cong (C_p)^7$, $|G'^p \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_p)^3$, $\gamma_4(G) \cong C_p$ and $\gamma_5(G) = 1$. If $p = 3$, then $|D_{(5),K}(G)| = 1$, $|D_{(4),K}(G)| = 3$, $|D_{(3),K}(G)| = 3^3$ and $|G'| = 3^7$, hence we have the following possibilities: (a) $G' \cong (C_9)^3 \times C_3$ (b) $G' \cong (C_9)^2 \times (C_3)^3$ (c) $G' \cong C_9 \times (C_3)^5$ (d) $G' \cong (C_3)^7$. Clearly $G' \cong (C_9)^3 \times C_3$ and $G' \cong (C_9)^2 \times (C_3)^3$ are not possible. Let $G' \cong C_9 \times (C_3)^5$. If $|\gamma_4(G)| = 3$, then $|\gamma_3(G)| = 3^2$ or 3^3 and thus $|G'^3 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_3 \times C_3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$ or $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$. Let $G' \cong (C_3)^7$. If $|\gamma_4(G)| = 3$, then $|\gamma_3(G)| = 3^2$ or 3^3 and thus $G'^3 \subseteq \gamma_3(G) \cong (C_3)^3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$. If $p = 2$, then $|D_{(5),K}(G)| = 1$, $|D_{(4),K}(G)| = 2$, $|D_{(3),K}(G)| = 2^3$ and $|G'| = 2^7$, hence we have the following possibilities: (a) $G' \cong (C_4)^3 \times C_2$ (b) $G' \cong (C_4)^2 \times C_2^3$ (c) $G' \cong C_4 \times (C_2)^5$ (d) $G' \cong (C_2)^7$. Clearly $G' \cong (C_4)^3 \times C_2$ is not possible. Let $G' \cong (C_4)^2 \times (C_2)^3$. If $|\gamma_4(G)| = 2$, then $|\gamma_3(G)| = 2^2$ or 2^3 and thus $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong C_2 \times C_2$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$ or $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$. Let $G' \cong C_4 \times (C_2)^5$. If $|\gamma_4(G)| = 2$, then $|\gamma_3(G)| = 2^2$ or 2^3 and thus $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_2 \times C_2$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$ or $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$. Let $G' \cong (C_2)^7$. If $|\gamma_4(G)| = 2$, then $|\gamma_3(G)| = 2^2$ or 2^3 and thus $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) \cong C_2$ $\gamma_5(G) = 1$.

Let $d_{(2)} = 6$, $d_{(3)} = 1$, $d_{(4)} = 1$, then this case is possible for all $p > 0$ and $|G'| = p^8$, for every $p > 0$. Thus $|D_{(3),K}(G)| = p^2$, $|D_{(4),K}(G)| = p$, $|D_{(5),K}(G)| = 1$. Let G' be an abelian group. If $p \geq 5$, then $G' \cong (C_p)^8$, $|G'^p \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_p)^2$, $\gamma_4(G) \cong C_p$ and $\gamma_5(G) = 1$. If $p = 3$, then $|D_{(5),K}(G)| = 1$, $|D_{(4),K}(G)| = 3$, $|D_{(3),K}(G)| = 3^2$ and $|G'| = 3^8$ hence we have the following possibilities: (a) $G' \cong (C_9)^4$ (b) $G' \cong (C_9)^3 \times (C_3)^2$ (c) $G' \cong (C_9)^2 \times (C_3)^4$ (d) $G' \cong C_9 \times (C_3)^6$ (e) $G' \cong (C_3)^8$. Clearly $G' \cong (C_9)^4$, $G' \cong (C_9)^3 \times (C_3)^2$ and $G' \cong (C_9)^2 \times (C_3)^4$ are not possible. If $G' \cong C_9 \times (C_3)^6$, then $G'^3 \subseteq \gamma_3(G) \cong (C_3)^2$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$. If $G' \cong (C_3)^8$, then $G'^3 \subseteq \gamma_3(G) \cong (C_3)^2$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$. Let $p = 2$, then $|D_{(5),K}(G)| = 1$, $|D_{(4),K}(G)| = 2$, $|D_{(3),K}(G)| = 2^2$ and $|G'| = 2^8$. Hence we have the following possibilities: (a) $G' \cong (C_4)^4$ (b) $G' \cong (C_4)^3 \times (C_2)^2$ (c) $G' \cong (C_4)^2 \times (C_2)^4$ (d) $G' \cong C_4 \times (C_2)^6$ (e) $G' \cong (C_2)^8$. Clearly $G' \cong (C_4)^4$ and $G' \cong (C_4)^3 \times (C_2)^2$ are not possible. If $G' \cong (C_4)^2 \times (C_2)^4$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$. If $G' \cong C_4 \times (C_2)^6$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) \cong C_2$ $\gamma_5(G) = 1$. If $G' \cong (C_2)^8$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) \cong C_2$, $\gamma_5(G) = 1$.

Let $d_{(2)} = 8$, $d_{(3)} = 0$, $d_{(4)} = 1$, then by Lemma 2.1 this case is possible only when $p = 3$. Thus $|D_{(3),K}(G)| = 3$, $|D_{(4),K}(G)| = 3$, $|D_{(5),K}(G)| = 1$ and $|G'| = 2^9$. Let G' be an abelian group then we have the following possibilities: (a) $G' \cong (C_9)^4 \times C_3$ (b) $G' \cong (C_9)^3 \times (C_3)^3$ (c) $G' \cong (C_9)^2 \times (C_3)^5$ (d) $G' \cong C_9 \times (C_3)^7$ (e) $G' \cong (C_3)^9$. Clearly $G' \cong (C_9)^4 \times C_3$, $G' \cong (C_9)^3 \times (C_3)^3$ and $G' \cong (C_9)^2 \times (C_3)^5$ are not possible. If $G' \cong C_9 \times (C_3)^7$, then $G'^3 \subseteq \gamma_3(G) \cong C_3$, $\gamma_4(G) \cong C_3$, $\gamma_5(G) = 1$. If $G' \cong (C_3)^9$, then $G'^3 \subseteq \gamma_3(G) \cong C_3$, $\gamma_4(G) \cong C_3$ $\gamma_5(G) = 1$.

Let $d_{(4)} = 0$, then $d_{(2)} + 2d_{(3)} = 11$ and we have the following cases: $d_{(2)} = 1$, $d_{(3)} = 5$ or $d_{(2)} = 3$, $d_{(3)} = 4$ or $d_{(2)} = 5$, $d_{(3)} = 3$ or $d_{(2)} = 7$, $d_{(3)} = 2$ or $d_{(2)} = 9$, $d_{(3)} = 1$ or $d_{(2)} = 11$, $d_{(3)} = 0$.

Let $d_{(2)} = 1$, $d_{(3)} = 5$, then this case is possible for all $p > 0$ and $|G'| = p^6$, for every $p > 0$. Thus $|D_{(3),K}(G)| = p^5$, $|D_{(4),K}(G)| = 1$. Let G' be an abelian group. If $p \geq 3$, then $G' \cong (C_p)^6$, $|G'^p \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_p)^5$ and $\gamma_4(G) = 1$. If $p = 2$, then $|D_{(4),K}(G)| = 1$, $|D_{(3),K}(G)| = 2^5$ and $|G'| = 2^6$ hence we have the following possibilities: (a) $G' \cong (C_4)^3$ (b) $G' \cong C_4 \times C_4 \times C_2 \times C_2$ (c) $G' \cong C_4 \times (C_2)^4$ (d) $G' \cong (C_2)^6$. If $G' \cong (C_4)$, then $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) = 1$ or $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) = 1$ or $|G'^2 \cap \gamma_3(G)| = 2^2$, $\gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) = 1$ or $G'^2 \subseteq \gamma_3(G) \cong (C_2)^5$, $\gamma_4(G) = 1$. If $G' \cong C_4 \times C_4 \times C_2 \times C_2$, then $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) = 1$ or $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) = 1$ or $G'^2 \subseteq \gamma_3(G) \cong (C_2)^5$, $\gamma_4(G) = 1$. If $G' \cong C_4 \times (C_2)^4$, then $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) = 1$ or $G'^2 \subseteq \gamma_3(G) \cong (C_2)^5$, $\gamma_4(G) = 1$. If $G' \cong (C_2)^6$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^5$,

$\gamma_4(G) = 1$.

Let $d_{(2)} = 3$, $d_{(3)} = 4$, then this case is possible for all $p > 0$ and $|G'| = p^7$, for every $p > 0$. Thus $|D_{(3),K}(G)| = p^4$, $|D_{(4),K}(G)| = 1$. Let G' be an abelian group. If $p \geq 3$, then $G' \cong (C_p)^7$, $|G'^p \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_p)^4$ and $\gamma_4(G) = 1$. If $p = 2$, then $|D_{(4),K}(G)| = 1$, $|D_{(3),K}(G)| = 2^4$ and $|G'| = 2^7$, hence we have the following possibilities: (a) $G' \cong (C_4)^3 \times C_2$ (b) $G' \cong (C_4)^2 \times (C_2)^3$ (c) $G' \cong C_4 \times (C_2)^5$ (d) $G' \cong (C_2)^7$. If $G' \cong (C_4)^3 \times C_2$, then $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_2$, $\gamma_4(G) = 1$ or $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) = 1$ or $|G'^2 \cap \gamma_3(G)| = 2^2$, $\gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) = 1$ or $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) = 1$. If $G' \cong (C_4)^2 \times (C_2)^3$, then $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) = 1$ or $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) = 1$ or $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) = 1$. If $G' \cong C_4 \times (C_2)^5$, then $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) = 1$ or $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) = 1$. If $G' \cong (C_2)^7$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^4$, $\gamma_4(G) = 1$.

Let $d_{(2)} = 5$, $d_{(3)} = 3$, then this case is possible for all $p > 0$ and $|G'| = p^8$ for every $p > 0$. Thus $|D_{(3),K}(G)| = p^3$, $|D_{(4),K}(G)| = 1$. Let G' be an abelian group. If $p \geq 3$, then $G' \cong (C_p)^8$, $|G'^p \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_p)^3$ and $\gamma_4(G) = 1$. If $p = 2$, then $|D_{(4),K}(G)| = 1$, $|D_{(3),K}(G)| = 2^3$ and $|G'| = 2^8$, hence we have the following possibilities: (a) $G' \cong (C_4)^4$ (b) $G' \cong (C_4)^3 \times (C_2)^2$ (c) $G' \cong (C_4)^2 \times (C_2)^4$ (d) $G' \cong C_4 \times (C_2)^6$ (e) $G' \cong (C_2)^8$. Clearly $G' \cong (C_4)^4$ and $G' \cong (C_4)^3 \times (C_2)^2$ is not possible. If $G' \cong (C_4)^2 \times (C_2)^4$, then $G'^2 \subseteq \gamma_3(G) \cong C_2$, $\gamma_4(G) = 1$ or $G'^2 \subseteq \gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) = 1$ or $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) = 1$. If $G' \cong (C_4)^2 \times (C_2)^4$, then $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_2$, $\gamma_4(G) = 1$ or $|G'^2 \cap \gamma_3(G)| = 2$, $\gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) = 1$ or $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) = 1$. If $G' \cong C_4 \times (C_2)^6$, then $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) = 1$ or $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) = 1$. If $G' \cong (C_2)^8$ this is possible, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^3$, $\gamma_4(G) = 1$.

Let $d_{(2)} = 7$, $d_{(3)} = 2$, this case is possible for all $p > 0$ and $|G'| = p^9$, for every $p > 0$. Thus $|D_{(3),K}(G)| = p^2$, $|D_{(4),K}(G)| = 1$. Let G' be an abelian group. If $p \geq 3$, then $G' \cong (C_p)^9$, $|G'^p \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong (C_p)^2$ and $\gamma_4(G) = 1$. If $p = 2$, then $|D_{(4),K}(G)| = 1$, $|D_{(3),K}(G)| = 2^2$ and $|G'| = 2^9$ hence we have the following possibilities: (a) $G' \cong (C_4)^4 \times C_2$ (b) $G' \cong (C_4)^3 \times (C_2)^3$ (c) $G' \cong (C_4)^2 \times (C_2)^5$ (d) $G' \cong C_4 \times (C_2)^7$ (e) $G' \cong (C_2)^9$. If $G' \cong (C_4)^4 \times C_2$ and $G' \cong (C_4)^3 \times (C_2)^3$, then these cases are not possible. If $G' \cong (C_4)^2 \times (C_2)^5$, then $G'^2 \subseteq \gamma_3(G) \cong C_2$, $\gamma_4(G) = 1$ or $G'^2 \subseteq \gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) = 1$. If $G' \cong C_4 \times (C_2)^7$, then $|G'^2 \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_2$, $\gamma_4(G) = 1$ or $G'^2 \subseteq \gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) = 1$. If $G' \cong (C_2)^9$, then $G'^2 \subseteq \gamma_3(G) \cong (C_2)^2$, $\gamma_4(G) = 1$.

Let $d_{(2)} = 9$, $d_{(3)} = 1$, then this case is not possible for all $p > 0$. Now $|G'| = p^{10}$, for every $p > 0$. Thus $|D_{(3),K}(G)| = p$, $|D_{(4),K}(G)| = 1$. Let G' be an abelian group. If $p \geq 3$, then $G' \cong (C_p)^{10}$, $|G'^p \cap \gamma_3(G)| = 1$, $\gamma_3(G) \cong C_p$ and $\gamma_4(G) = 1$. If $p = 2$, then $|D_{(4),K}(G)| = 1$, $|D_{(3),K}(G)| = 2$ and $|G'| = 2^{10}$, hence we have the following possibilities: (a) $G' \cong (C_4)^5$ (b) $G' \cong (C_4)^4 \times (C_2)^2$ (c) $G' \cong (C_4)^3 \times (C_2)^4$ (d) $G' \cong (C_4)^2 \times (C_2)^6$ (e) $G' \cong C_4 \times (C_2)^8$ (f) $G' \cong (C_2)^{10}$. It is clear that $G' \cong (C_4)^5$, $G' \cong (C_4)^4 \times (C_2)^2$, $G' \cong (C_4)^3 \times (C_2)^4$ and $G' \cong (C_4)^2 \times (C_2)^6$ are not possible. If $G' \cong C_4 \times (C_2)^8$, $G'^2 \subseteq \gamma_3(G) \cong C_2$, $\gamma_4(G) = 1$. If $G' \cong (C_2)^{10}$, $G'^2 \subseteq \gamma_3(G) \cong C_2$, $\gamma_4(G) = 1$.

Let $d_{(2)} = 11$, $d_{(3)} = 0$, then this case is possible for all $p > 0$. Now $|G'| = p^{11}$, for every $p > 0$. Thus $|D_{(3),K}(G)| = 1$ and G' be abelian group. Thus $G' \cong (C_p)^{11}$, $|G'^p \cap \gamma_3(G)| = 1$ and $\gamma_3(G) = 1$, for all $p > 0$.

Converse can be easily done by computing $d_{(m)}$'s in each case. □

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