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Research Paper

THE DUALS OF ANNIHILATOR CONDITIONS FOR MODULES

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ABSTRACT. Let R be a commutative ring with identity and let M be an R-module. The purpose of this paper is to introduce and investigate the submodules of an R-module M which satisfy the dual of Property $\mathcal{A}$, the dual of strong Property $\mathcal{A}$, and the dual of proper strong Property $\mathcal{A}$. Moreover, a submodule N of M which satisfy Property $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$ and Property $\mathcal{I}^{\mathcal{M}}_{\mathcal{J}}(\mathcal{N})$ will be introduced and investigated.

1. Introduction

Throughout this paper, R will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers.

Let M be an R-module. The set of zero divisors of R on M is $Z_R(M) = \{r \in R | rm = 0 \ for \ some \ nonzero \ m \in M\}$ and the set of torsion elements of M with respect to R is $T_R(M) = \{m \in M | rm = 0 \ for \ some \ 0 \neq r \in R\}$.

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An R-module M satisfies *Property* $\mathcal{A}$ (resp., *Property* $\mathcal{T}$) if for every finitely generated ideal I of R (resp., finitely generated submodule N of M) with $I \subseteq Z_R(M)$ (resp., $N \subseteq T_R(M)$), there exists a nonzero $m \in M$ (resp., $r \in R$) with $Im = 0$ (resp., $rN = 0$), or equivalently $(0 :_M I) \neq 0$ (resp., $Ann_R(N) \neq 0$) [2]. An R-module M satisfies *strong Property* $\mathcal{A}$ (resp., *strong Property* $\mathcal{T}$) if for any $r_1, ..., r_n \in Z_R(M)$ (resp., $m_1, ..., m_n \in T_R(M)$), there exists a non-zero $m \in M$ (resp., $r \in R$) with $r_1 m = \cdots = r_n m = 0$ (resp., $rm_1 = \cdots = rm_n = 0$) [2]. An R-module M satisfies *proper strong Property* $\mathcal{A}$ if for any proper finitely generated ideal $I = \langle a_1, a_2, ..., a_n \rangle$ of R such that $a_i \in Z_R(M)$ we have $(0 :_M I) \neq 0$ [1]. The class of modules satisfies proper strong Property $\mathcal{A}$ lying properly between the class of modules satisfies strong Property $\mathcal{A}$ and Property $\mathcal{A}$ [1, Corollary 2.12].

Let M be an R-module. The subset $W_R(M)$ of R (that is the dual notion of $Z_R(M)$) is defined by $\{r \in R | rM \neq M\}$ [19]. A non-zero submodule N of M is said to be *secondal* if $W_R(N)$ is an ideal of R. In this case, $W_R(N)$ is a prime ideal of R [6].

Recently, the annihilator conditions on modules over commutative rings have attracted the attention of several researchers. A brief history of this can be found in [2, 1]. The purpose of this paper is to introduce and study the dual of Property $\mathcal{A}$, the dual of strong Property $\mathcal{A}$, and the dual of proper strong Property $\mathcal{A}$ for modules over a commutative ring. Also, for a submodule N of an R-module M we introduce and investigate the Properties $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$ and $\mathcal{I}^{\mathcal{M}}_{\mathcal{J}}(\mathcal{N})$. Some of the results in this article are dual of the results for Property $\mathcal{A}$, strong Property $\mathcal{A}$, and proper strong Property $\mathcal{A}$ considered in [1] and [2].

2. The duals of Property $\mathcal{A}$ and strong Property $\mathcal{A}$ for modules

Definition 2.1. We say that an R-module M satisfies the *dual of Property* $\mathcal{A}$ if for each finitely generated ideal I of R with $I \subseteq W_R(M)$ we have $IM \neq M$.

A proper submodule N of an R-module M is said to be *completely irreducible* if $N = \bigcap_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of submodules of M, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of M is an intersection of completely irreducible submodules of M [13].

Definition 2.2. We say that an R-module M satisfies *the dual of strong Property* $\mathcal{A}$ if for any $a_1, ..., a_n \in W_R(M)$, there exists a completely irreducible submodule L of M such that $a_i M \subseteq L \neq M$ for $i = 1, 2, .., n$.

Clearly, if an R-module M satisfies the dual of strong Property $\mathcal{A}$, then M satisfies the dual of Property $\mathcal{A}$. Nevertheless, the following example shows that the converse is not true in general.

Example 2.3. The $\mathbb{Z}$-module $\mathbb{Z}$ satisfies the dual of Property $\mathcal{A}$ but does not satisfies the dual of strong Property $\mathcal{A}$.

Remark 2.4. Let N and K be two submodules of an R-module M. To prove $N \subseteq K$, it is enough to show that if L is a completely irreducible submodule of M such that $K \subseteq L$, then $N \subseteq L$ [6].

Theorem 2.5. *Let M be an R-module. Consider the conditions:*

(a) *M satisfies the dual of strong Property $\mathcal{A}$,*

(b) *M is a secondal R-module.*

Then $(a) \Rightarrow (b)$. If further R is a PID, then $(b) \Rightarrow (a)$.

Proof. $(a) \Rightarrow (b)$. Let $a, b \in W_R(M)$. Then by part (a), there exists a completely irreducible submodule L of M such that $aM \subseteq L \neq M$ and $bM \subseteq L \neq M$. Thus $(a-b)M \subseteq L \neq M$ and so $a - b \in W_R(M)$. This implies that M is a secondal R-module.

$(b) \Rightarrow (a)$. Let $a_1, ..., a_n \in W_R(M)$. Then by part (b), $W_R(M)$ is an ideal of R. As R is a PID, there exists an $a \in R$ such that $\langle a_1, ..., a_n \rangle = Ra$. Thus $a \in W_R(M)$. Hence there exists a completely irreducible submodule L of M such that $aM \subseteq L \neq M$ by Remark 2.4. This implies that $a_iM \subseteq L \neq M$ for $i = 1, 2, .., n$, as needed. □

Theorem 2.6. *Let $f : R \to \acute{R}$ be a homomorphism of commutative rings and let M be an $\acute{R}$-module. Consider M as an R-module with $rm := f(r)m$ for $r \in R$ and $m \in M$.*

(a) *Suppose for each (finitely generated) ideal I of R, $f(I)\acute{R} = \{f(i)\acute{r} | i \in I, \acute{r} \in \acute{R}\}$ (e.g., f is surjective or $f : R \to R_N$, $f(r) = r/1$, where N is a multiplicatively closed subset of R). Then M satisfies the dual of Property $\mathcal{A}$ as an $\acute{R}$-module implies M satisfies the dual of Property $\mathcal{A}$ as an R-module.*

(b) *Suppose that every (finitely generated) ideal J of $\acute{R}$ has the form $J = f(I)\acute{R}$ for some (finitely generated) ideal I of R (e.g., f is surjective or $f : R \to R_N$, $f(r) = r/1$, where N is a multiplicatively closed subset of R). Then M satisfies the dual of Property $\mathcal{A}$ as an R-module implies M satisfies the dual of Property $\mathcal{A}$ as an $\acute{R}$-module.*

Proof. (a) Suppose M satisfies the dual of Property $\mathcal{A}$ as an $\acute{R}$-module. Let I be an ideal of R with $I \subseteq W_R(M)$. So for $i \in I$, there is a $m \in M \setminus iM = M \setminus f(i)M$. Hence, $f(i) \in W_{\acute{R}}(M)$ and so $\acute{r}f(i) \in W_{\acute{R}}(M)$ for each $\acute{r} \in \acute{R}$. Thus $f(I)\acute{R} = \{f(i)\acute{r} | i \in I, \acute{r} \in \acute{R}\}$ is an ideal of $\acute{R}$ with $f(I)\acute{R} \subseteq W_{\acute{R}}(M)$. Suppose that I is finitely generated. Then $f(I)\acute{R}$ is finitely generated. Hence there is a $m \in M \setminus f(I)\acute{R}M$. This implies that $m \in M \setminus IM$. Thus M satisfies the dual of Property $\mathcal{A}$ as an R-module.

(b) Suppose that M satisfies the dual of Property $\mathcal{A}$ as an R-module. Let J be an ideal of $\acute{R}$ with $J \subseteq W_{\acute{R}}(M)$. Then there is an ideal I of R with $J = f(I)S$. For $i \in I$, $f(i) \in W_{\acute{R}}(M)$. So, there is a $m \in M \setminus f(i)M = M \setminus iM$. So, $I \subseteq W_R(M)$. If J is finitely generated, we can choose I to be finitely generated. Since M satisfies the dual of Property $\mathcal{A}$ as an R-module, there is a $m \in M \setminus IM$. It follows that $m \in M \setminus f(I)\acute{R}M = M \setminus JM$. So, M satisfies the dual of Property $\mathcal{A}$ as an $\acute{R}$-module. □

Corollary 2.7. *Let M be an R-module, $J \subseteq Ann_R(M)$ an ideal of R, and put $\bar{R} = R/J$. Then M satisfies the dual of Property $\mathcal{A}$ as an R-module if and only if M satisfies the dual of Property $\mathcal{A}$ as an $\bar{R}$-module. In particular, M satisfies the dual of Property $\mathcal{A}$ as an R-module if and only if M satisfies the dual of Property $\mathcal{A}$ as an $R/Ann_R(M)$-module.*

Proof. This follows from Theorem 2.6. □

Recall that an R-module M is said to be *Hopfian* (resp. *co-Hopfian*) if every surjective (resp. injective) endomorphism f of M is an isomorphism.

An R-module M is said to be a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$ [12].

A submodule N of an R-module M is said to be *idempotent* if $N = (N :_R M)^2M$. Also, M is said to be *fully idempotent* if every submodule of M is idempotent [7].

An R-module M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$ [3]. R is said to be a *comultiplication ring* if, as an R-module, R is a comultiplication R-module [4].

A submodule N of an R-module M is said to be *coidempotent* if $N = (0 :_M Ann_R(N)^2)$. Also, an R-module M is said to be *fully coidempotent* if every submodule of M is coidempotent [7].

Proposition 2.8. *Let M be an R-module. Then we have the following.*

(a) *If R is a comultiplication ring and M is a faithful R-module, then M satisfies Property $\mathcal{A}$ and the dual of Property $\mathcal{A}$.*
(b) *If M is a Hopfian comultiplication (in particular, M is a fully coidempotent) R-module and satisfies the dual of Property $\mathcal{A}$, then M satisfies Property $\mathcal{A}$.*
(c) *If M is a co-Hopfian multiplication (in particular, M is a fully idempotent) R-module and satisfies Property $\mathcal{A}$, then M satisfies the dual of Property $\mathcal{A}$.*
(d) *If R is a principal ideal ring, then M satisfies the dual of strong Property $\mathcal{A}$.*

Proof. (a) This follows from [4, Lemma 3.11].

(b) First note that every fully coidempotent R-module is a Hopfian comultiplication R-module by [7, Theorem 3.9 and Proposition 3.5]. As M is a Hopfian comultiplication R-module, $Z_R(M) = W_R(M)$. Now the result follows from [5, Proposition 3.1].

(c) First note that every fully idempotent R-module is a co-Hopfian multiplication R-module by [7, Proposition 2.7]. Since M is a co-Hopfian multiplication R-module, $Z_R(M) = W_R(M)$. Now the result follows from [18, Note 1.13].

(d) This is clear. □

Lemma 2.9. *Let S be a multiplicatively closed subset of R, I and ideal of R, and M be an R-module. Then we have the following.*

(a) *If $S^{-1}I \subseteq W_{S^{-1}R}(S^{-1}M)$, then $I \subseteq W_R(M)$.*

(b) *If $Z_R(M) \cap S = \emptyset$, $W_R(M) \cap S = \emptyset$, and $I \subseteq W_R(M)$, then $S^{-1}I \subseteq W_{S^{-1}R}(S^{-1}M)$.*

(c) *If M is an Hopfian module (in particular, M is a multiplication or coidempotent module), $W_R(M) \cap S = \emptyset$, and $I \subseteq W_R(M)$, then $S^{-1}I \subseteq W_{S^{-1}R}(S^{-1}M)$.*

Proof. (a) This is clear.

(b) Suppose that $S^{-1}I \not\subseteq W_{S^{-1}R}(S^{-1}M)$ and seek for a contradiction. Then $S^{-1}(aM) = S^{-1}M$ for some $a \in I$. As $I \subseteq W_R(M)$, there exists $m \in M \setminus IM$. Now we have $stm = sam_1$ for some $s, t \in S$ and $m_1 \in M$. Since $W_R(M) \cap S = \emptyset$, $tM = M$ and so $m_1 = tm_2$ for some $m_2 \in M$. Hence, $st(m - am_2) = 0$. Now $Z_R(M) \cap S = \emptyset$ implies that $m = am_2$, which is a contradiction.

(c) This follows from the fact that $Z_R(M) \subseteq W_R(M)$ and part (b). □

Corollary 2.10. *Let S be a multiplicatively closed subset of R and M be an R-module. Consider the conditions:*

(a) *M_S satisfies the dual of Property $\mathcal{A}$ as an R-module,*

(b) *M_S satisfies the dual of Property $\mathcal{A}$ as an R_S-module,*

(c) *M satisfies the dual of Property $\mathcal{A}$ as an R-module.*

Then (a) $\Leftrightarrow$ (b). If further $S \cap W_R(M) = \emptyset$ and $S \cap Z_R(M) = \emptyset$, (a), (b) and (c) are equivalent.

Proof. The equivalence of (a) and (b) follows from Theorem 2.6. Now assume that $S \cap W_R(M) = \emptyset$ and $S \cap Z_R(M) = \emptyset$. Then $(b) \Leftrightarrow (c)$ from Lemma 2.9 (b). □

Proposition 2.11. *Let X be an indeterminate over R, M be an R-module, and $M[X]$ satisfies the dual of strong Property $\mathcal{A}$ over $R[X]$. Then M satisfies the dual of Property $\mathcal{A}$.*

Proof. Let $I = \langle a_1, ..., a_n \rangle$ be a finitely generated ideal of R such that $a_i \in W_R(M)$ for $i = 1, ..., n$. Then $I[X] = \langle a_1, ..., a_n \rangle R[X]$ is a finitely generated ideal of $R[X]$ such that $a_i \in W_{R[X]}(M[X])$ for $i = 1, ..., n$. Since $M[X]$ satisfies the dual of strong Property $\mathcal{A}$ over $R[X]$, we get $I[X]M[X] \neq M[X]$. This implies that $IM \neq M$. Thus M satisfies the dual of Property $\mathcal{A}$. □

Recall that a ring R is called *Bézout* if every finitely generated ideal I of R is principal.

A submodule N of an R-module M is *small* if for any submodule X of M, $X + N = M$ implies that $X = M$.

A prime ideal P of R is said to be a *coassociated prime ideal* of an R-module M if there exists a cocyclic homomorphic image T of M such that $Ann_R(T) = P$. The set of coassociated prime ideals of M is denoted by $Coass(M)$ [20].

Theorem 2.12. (a) *The trivial R-module vacuously satisfies the dual of Property $\mathcal{A}$.*

(b) *Every module over a Bézout ring satisfies the dual of Property $\mathcal{A}$.*

(c) *Let R be a zero-dimensional commutative ring (e.g., R is Artinian). Then every R-module satisfies the dual of Property $\mathcal{A}$.*

(d) *Let M be a finitely generated R-module. Then M satisfies the dual of Property $\mathcal{A}$.*

(e) *Let M be an Artinian R-module. Then M satisfies the dual of Property $\mathcal{A}$. In fact, for any ideal I of R with $I \subseteq W_R(M)$, $IM \neq M$.*

(f) *Let M and $\acute{M}$ be R-modules with $W_R(M) \subseteq W_R(\acute{M})$. If $\acute{M}$ satisfies the dual of Property $\mathcal{A}$ (respectively, the dual of strong Property A), then $M \oplus \acute{M}$ satisfies the dual of Property $\mathcal{A}$ (respectively, the dual of strong Property $\mathcal{A}$).*

(g) *Let N be a small submodule of M. Then M satisfies the dual of Property $\mathcal{A}$ (resp. M is a secondal module) if and only if M/N satisfies the dual of Property $\mathcal{A}$ (resp. M/N is a secondal module).*

Proof. (a) Note that $W_R(0) = \emptyset$.

(b) This is clear.

(c) Suppose $dimR = 0$ and M is an R-module. We can assume $M \neq 0$. Let I be a finitely generated ideal of R with $I \subseteq W_R(M)$. So, $I \subseteq P \subseteq W_R(M)$ for some prime ideal P of R by using [20, Theorem 2.15]. Since $htP = 0$ and I is finitely generated, $I_P^n = 0$ for some $n \geq 1$. Hence there is an $s \in R \setminus P$ with $I^n s = 0$. Since $s \in R \setminus P$, $s \notin Ann_R(M)$. Thus $sM \neq 0$. We have $I^n sM = 0$. Suppose $I^t sM \neq 0$, but $I^{t+1}sM = 0$. Then $IM \subseteq (0 :_M I^t s) \neq M$. Therefore, $IM \neq M$.

(d) Let M be a finitely generated R-module and I be an ideal of R with $I \subseteq W_R(M)$. Assume contrary that $IM = M$. Then $(1 + a)M = 0$ by [14, Theorem 76]. As $a \in I \subseteq W_R(M)$, there exists an $m \in M \setminus aM$. Now, $(1 + a)m = 0$ implies that $m \in aM$, which is a contradiction.

(e) As M is an Artinian R-module, $W_R(M) = \cup_{i=1}^{n} P_i$ by using [20, Theorem 2.10 (c), Theorem 2.15, Corollary 3.2], where $P_i \in Coass(M)$. Now let $I \subseteq W_R(M)$ be an ideal. Then $I \subseteq P_i$ for some $P_i \in Coass(M)$. Hence for some completely irreducible submodule L of M with $L \neq M$, we have $I \subseteq P_i = (L :_R M)$. This implies that $IM \neq M$.

(f) Let $\acute{M}$ satisfies the dual of Property $\mathcal{A}$. It is easy to see that $W_R(M \oplus \acute{M}) = W_R(M) \cup W_R(\acute{M}) = W_R(\acute{M})$. Let I be a finitely generated ideal of R with $I \subseteq W_R(M) \cup W_R(\acute{M}) = W_R(\acute{M})$. Then $I\acute{M} \neq \acute{M}$. Thus there exists an $x \in I\acute{M} \setminus \acute{M}$. This implies that $(0, x) \notin I(M \oplus \acute{M})$ and so $I(M \oplus \acute{M}) \neq M \oplus \acute{M}$, as needed.

(g) We always have $W_R(M/N) \subseteq W_R(M)$. As N is small we get that $W_R(M) \subseteq W_R(M/N)$. Now the result is straightforward. □

Let M be an R-module. The idealization $R(+)M = \{(a, m) : a \in R, m \in M\}$ of M is a commutative ring whose addition is component-wise and whose multiplication is defined as $(a, m)(b, \acute{m}) = (ab, a\acute{m} + bm)$ for each $a, b \in R$, $m, \acute{m} \in M$ [17].

Proposition 2.13. *Let M be an R-module. Then we have*

$$W_{R(+)M}(R(+)M) = W_R(R)(+)M.$$

Proof. First note that $W_R(M) \subseteq W_R(R)$. Let $(a, x) \in W_{R(+)M}(R(+)M)$. Then there exists $(b, y) \in R(+)M \setminus (a, x)(R(+)M)$. This implies that for each $(c, z) \in R(+)M$, $(b, y) \neq (a, x)(c, z)$. Hence, $b \neq ac$ or $y \neq az + cx$. If $b \neq ac$, then $a \in W_R(R)$ and we are done. If $y \neq az + cx$. Then by setting $c = 0$, we have $y \neq az$. Thus $a \in W_R(M) \subseteq W_R(R)$ and so $W_{R(+)M}(R(+)M) \subseteq W_R(R)(+)M$. Now let $(a, x) \in W_R(R)(+)M$. Then $a \in W_R(R)$. Thus there exist $r \in R$ such that $r \in R \setminus aR$. Assume contrary that $(a, x)(R(+)M) = R(+)M$. Then $(r, 0) = (a, x)(c, y)$ for some $(c, y) \in R(+)M$. Thus $r = ac$, which is a contradiction. Hence $(a, x)(R(+)M) \neq R(+)M$, as needed. □

Example 2.14. Let $M = \oplus R/I$, where the sum runs over all proper finitely generated ideals of R. Then for a proper finitely generated ideal I of R, $I \subseteq W_R(M)$ and $I(R/I) = \bar{0} \neq R/I$ implies that M satisfies the dual of Property $\mathcal{A}$. As R is a submodule of M, we have R is a submodule of an R-module satisfying the dual of Property $\mathcal{A}$. Let $\acute{M}$ be any R-module. Then $M \oplus \acute{M}$ again, satisfies the dual of Property $\mathcal{A}$. Thus, any R-module is a submodule, homomorphic image, or direct factor of a module satisfying the dual of Property $\mathcal{A}$.

3. The dual of proper strong Property $\mathcal{A}$ for modules

Lemma 3.1. *Let M be an R-module and $S = R \setminus W_R(M)$. Then $S^{-1}R = R$ if and only if $R = U(R) \cup W_R(M)$, where $U(R)$ is the set of all invertible elements of R.*

Proof. Assume that $S^{-1}R = R$ and $s \in S$. Then $1/s \in S^{-1}R = R$ implies that s is invertible in R. Hence any element of S is invertible in R. The converse is clear. □

Definition 3.2. We say that an R-module M satisfies *the dual of proper strong Property $\mathcal{A}$* if for any proper finitely generated ideal $I = \langle a_1, a_2, ..., a_n \rangle$ of R such that $a_i \in W_R(M)$ we have $IM \neq M$.

Theorem 3.3. *Let M be an R-module. Then the following assertions are equivalent:*

(a) *M satisfies the dual of proper strong Property $\mathcal{A}$,*

(b) *M satisfies the dual of Property $\mathcal{A}$ and $\mathfrak{m} \cap W_R(M)$ is an ideal of R for each maximal ideal $\mathfrak{m}$ of R.*

Proof. $(a) \Rightarrow (b)$ Assume that M satisfies the dual of proper strong Property $\mathcal{A}$. Clearly, M satisfies the dual of Property $\mathcal{A}$. Let $\mathfrak{m}$ be a maximal ideal of R. Let $a, b \in \mathfrak{m} \cap W_R(M)$ and put $I = \langle a, b \rangle$ the ideal generated by a and b. Then $I \subseteq \mathfrak{m}$ and $a, b \in W_R(M)$. Since M satisfies the dual of proper strong Property $\mathcal{A}$, we get that $IM \neq M$. It follows that $I \subseteq \mathfrak{m} \cap W_R(M)$ and thus $\mathfrak{m} \cap W_R(M)$ is an ideal of R.

$(b) \Rightarrow (a)$ Let $I = \langle a_1, a_2, ..., a_n \rangle$ be a proper finitely generated ideal of R such that $a_i \in W_R(M)$ for $i = 1, ..., n$. Let $\mathfrak{m}$ be a maximal ideal of R with $I \subseteq \mathfrak{m}$. Then $a_1, a_2, ..., a_n \in \mathfrak{m} \cap W_R(M)$. As, by hypotheses, $\mathfrak{m} \cap W_R(M)$ is an ideal of R, it follows that $I \subseteq \mathfrak{m} \cap W_R(M)$. Now, since M satisfies the dual of Property $\mathcal{A}$, we get $IM \neq M$. Hence M satisfies the dual of proper strong Property $\mathcal{A}$. □

Theorem 3.4. *Let M be an R-module. Then*

$$M \ satisfies \ the \ dual \ of \ strong \ Property \ \mathcal{A} \ \Rightarrow$$

$$M \ satisfies \ the \ dual \ of \ proper \ strong \ Property \ \mathcal{A} \ \Rightarrow$$

$$M \ satisfies \ the \ dual \ of \ Property \ \mathcal{A}.$$

Proof. The proof is clear from the definitions. □

The Examples 3.5 and 3.12 show that the converse of Theorem 3.4 is not true in general.

Example 3.5. The $\mathbb{Z}$-module $\mathbb{Z}$ satisfies the dual of proper strong Property $\mathcal{A}$ but does not satisfies the dual of strong Property $\mathcal{A}$.

Let R_i be a commutative ring with identity and M_i be an R_i-module for each $i = 1, 2$. Assume that $M = M_1 \times M_2$ and $R = R_1 \times R_2$. Then M is clearly an R-module with component-wise addition and scalar multiplication. Also, each submodule N of M is of the form $N = N_1 \times N_2$, where N_i is a submodule of M_i for each $i = 1, 2$.

Proposition 3.6. *Let R_i be a commutative ring with identity and M_i be an R_i-module for each $i = 1, 2$. Then*

$$W_{R_1\times R_2}(M_1 \times M_2) = (W_{R_1}(M_1) \times R_2) \cup (R_1 \times W_{R_2}(M_2)).$$

Proof. This is straightforward. □

Theorem 3.7. *Let R_i be a commutative ring with identity and M_i be an R_i-module for each $i = 1, 2$. Let $M = M_1 \times M_2$, $R = R_1 \times R_2$, and $S_i = R_i \setminus W_{R_i}(M_i)$. Then the following assertions are equivalent:*

(a) *M satisfies the dual of proper strong Property $\mathcal{A}$,*

(b) *M_i satisfies the dual of proper strong Property $\mathcal{A}$ and $S_i^{-1}R_i = R_i$ for each $i = 1, 2$.*

Proof. $(a) \Rightarrow (b)$ Assume that M satisfies the dual of proper strong Property $\mathcal{A}$ and $I_1 = \langle a_1, a_2, ..., a_n\rangle$ is a finitely generated ideal of R_1 such that $I_1 \subseteq W_{R_1}(M_1)$. Set

$$I = \langle (a_1, 0), (a_2, 0), ..., (a_n, 0), (0, 1)\rangle.$$

Then $(0, 1), (a_i, 0) \in W_{R_1\times R_2}(M_1 \times M_2)$ for $i = 1, 2, ..., n$. By part (a), $I(M_1 \times M_2) \neq M_1 \times M_2$. Thus there exists $(x_1, x_2) \in M_1 \times M_2 \setminus I(M_1 \times M_2)$. This implies that $x_1 \notin I_1M_1$. Thus $I_1M_1 \neq M_1$ and M_1 satisfies the dual of proper strong Property $\mathcal{A}$. Now let $r_1 \in R_1 \setminus U(R_1)$. Clearly $(r_1, 0) \in W_{R_1\times R_2}(M_1 \times M_2)$. Set $J = \langle (r_1, 0), (0, 1)\rangle$. Thus by part (a), $J(M_1 \times M_2) \neq M_1 \times M_2$. This implies that $r_1M_1 \neq M_1$ and hence $r_1 \in W_{R_1}(M_1)$. Now by Lemma 3.1, $S_1^{-1}R_1 = R_1$. Similarly, one can see that M_2 satisfies the dual of proper strong Property $\mathcal{A}$ and $S_2^{-1}R_2 = R_2$.

$(b) \Rightarrow (a)$ Let $I = \langle (a_1, b_1), (a_2, b_2), ..., (a_n, b_n)\rangle$ be a proper finitely generated ideal of R such that $(a_i, b_i) \in W_{R_1\times R_2}(M_1 \times M_2)$ for each $i = 1, 2, ..., n$. Set $I_1 = \langle a_1, a_2, ..., a_n\rangle$ and $I_2 = \langle b_1, b_2, ..., b_n\rangle$. Then as I is proper, I_1 or I_2 is proper. Assume that I_1 is proper. Then by Lemma 3.1, $I_1 \subseteq W_{R_1}(M_1)$. Thus by part (b), $I_1M_1 \neq M_1$. Hence there exists $x_1 \in M_1 \setminus I_1M_1$. Now $(x_1, 0) \in (M_1 \times M_2) \setminus I(M_1 \times M_2)$ implies that M satisfies the dual of proper strong Property $\mathcal{A}$. □

Theorem 3.8. *Let M be an R-module and $S = R \setminus W_R(M)$. Then we have the following.*

(a) *If $S^{-1}R = R$, then M satisfies the dual of proper strong Property $\mathcal{A}$ if and only if M satisfies the dual of Property $\mathcal{A}$.*

(b) *If $S^{-1}R \neq R$, then M satisfies the dual of proper strong Property $\mathcal{A}$ if and only if M satisfies the dual of strong Property $\mathcal{A}$.*

Proof. (a) Since $S^{-1}R = R$, we have $W_R(M) = \cup_{\mathfrak{m} \in max(R)} \mathfrak{m}$ by Lemma 3.1. Thus far each maximal ideal $\mathfrak{m}$ of R, we have $\mathfrak{m} \cap W_R(M) = \mathfrak{m}$ is always an ideal of R. Now the result follows from Theorem 3.3. The reverse implication is clear.

(b) If M satisfies the dual of strong Property $\mathcal{A}$, then clearly, M satisfies the dual of proper strong Property $\mathcal{A}$. Conversely, assume that M satisfies the dual of proper strong Property $\mathcal{A}$. As $S^{-1}R \neq R$, there exists $x \in R$ such that x is not invertible and $x \notin W_R(M)$ by Lemma 3.1. Let $\mathfrak{m}$ be a maximal ideal of R such that $x \in m$. Let $I = \langle a_1, a_2, ..., a_n \rangle$ be a proper ideal of R such that $a_i \in W_R(M)$ for $i = 1, ..., n$. Then $xI = \langle xa_1, ..., xa_n \rangle \subseteq \mathfrak{m}$ is a proper ideal of R and $xa_i \in W_R(M)$ for $i = 1, ..., n$. Since M satisfies the dual of proper strong Property $\mathcal{A}$, $xIM \neq M$. Thus there exists a completely irreducible submodule L of M such that $xIM \subseteq L \neq M$ by Remark 2.4. Now, as $x \notin W_R(M)$, it follows that $IM \subseteq L \neq M$. This implies that $a_iM \subseteq L \neq M$ for $i = 1, ..., n$. Hence M satisfies the dual of strong Property $\mathcal{A}$. □

Proposition 3.9. *Let R be a zero-dimensional ring. Then any faithful R-module M satisfies the dual of proper strong Property $\mathcal{A}$. In particular, any R-module M satisfies the dual of proper strong Property $\mathcal{A}$ over $R/Ann_R(M)$.*

Proof. By Theorem 2.12 (c), M satisfies the dual of Property $\mathcal{A}$. We have $W_R(M) \subseteq Z_R(M) = Z_R(R)$ by using the proof of [1, Corollary 2.20]. Therefore, $W_R(M) = Z_R(R)$ because the inverse inclusion is clear. Thus $S^{-1}R = R$, where $S = R \setminus W_R(M) = R \setminus Z_R(R)$. This implies that M satisfies the dual of proper strong Property $\mathcal{A}$ by Theorem 3.8 (a). □

Theorem 3.10. *Let R_i be a commutative ring with identity and M_i be an R_i-module for each $i = 1, 2$. Let $M = M_1 \times M_2$, $R = R_1 \times R_2$, and $S_i = R_i \setminus Z_{R_i}(M_i)$. Then the following assertions are equivalent:*

(a) *M satisfies the proper strong Property $\mathcal{A}$,*
(b) *M_i satisfies the proper strong Property $\mathcal{A}$ and $S_i^{-1}R_i = R_i$ for each $i = 1, 2$.*

Proof. $(a) \Rightarrow (b)$ Assume that M satisfies the proper strong Property $\mathcal{A}$ and $I_1 = \langle a_1, a_2, ..., a_n \rangle$ is a finitely generated ideal of R_1 such that $I_1 \subseteq Z_{R_1}(M_1)$. Set

$$I = \langle (a_1, 0), (a_2, 0), ..., (a_n, 0), (0, 1) \rangle.$$

Then $(0, 1), (a_i, 0) \in Z_{R_1 \times R_2}(M_1 \times M_2)$ for $i = 1, 2, ..., n$. By part (a), $(0 :_{M_1 \times M_2} I) \neq 0$. Thus there exists $0 \neq (x_1, x_2) \in M_1 \times M_2$ such that $I(x_1, x_2) = 0$. This implies that $0 \neq x_1 \subseteq$

$(0 :_{M_1} I_1)$ and M_1 satisfies the proper strong Property $\mathcal{A}$. Now let $r_1 \in R_1 \setminus U(R_1)$. Clearly $(r_1, 0) \in Z_{R_1 \times R_2}(M_1 \times M_2)$. Set $J = \langle (r_1, 0), (0, 1) \rangle$. Thus by part (a), $(0 :_{M_1 \times M_2} J) \neq 0$. This implies that $r_1 \in Z_{R_1}(M_1)$. Now by [1, Lemma 2.1], $S_1^{-1} R_1 = R_1$. Similarly, one can see that M_2 satisfies the proper strong Property $\mathcal{A}$ and $S_2^{-1} R_2 = R_2$.

$(b) \Rightarrow (a)$ Let $I = \langle (a_1, b_1), (a_2, b_2), ..., (a_n, b_n) \rangle$ be a proper finitely generated ideal of R such that $(a_i, b_i) \in Z_{R_1 \times R_2}(M_1 \times M_2)$ for each $i = 1, 2, ..., n$. Set $I_1 = \langle a_1, a_2, ..., a_n \rangle$ and $I_2 = \langle b_1, b_2, ..., b_n \rangle$. Then as I is proper, I_1 or I_2 is proper. Assume that I_1 is proper. Then $I_1 \subseteq Z_{R_1}(M_1)$. Thus by part (b), $(0 :_{M_1} I_1) \neq 0$. Hence there exists $0 \neq x_1 \in M_1$ such that $I_1 x_1 = 0$. Now $(0, 0) \neq (x_1, 0) \in (0 :_{M_1 \times M_2} I)$ implies that M satisfies the proper strong Property $\mathcal{A}$. □

The following examples show that the concepts of proper strong Property $\mathcal{A}$ and the dual of proper strong Property $\mathcal{A}$ are different in general.

Example 3.11. Consider the $\mathbb{Z} \times \mathbb{Z}$-module $\mathbb{Z} \times \mathbb{Z}$. As $W_{\mathbb{Z}}(\mathbb{Z}) = \mathbb{Z} \setminus \{1, -1\}$, we have $S = \mathbb{Z} \setminus W_{\mathbb{Z}}(\mathbb{Z}) = \{1, -1\}$ and so $S^{-1}\mathbb{Z} = \mathbb{Z}$. Now Theorem 3.7 implies that $\mathbb{Z} \times \mathbb{Z}$ satisfies the dual of proper strong Property $\mathcal{A}$. But by [1, Example 2.13 (1)], $\mathbb{Z} \times \mathbb{Z}$ not satisfies the proper strong Property $\mathcal{A}$.

Example 3.12. Consider the $\mathbb{Z} \times \mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z}$. Since $W_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}) = \{0\}$, we have $S = \mathbb{Z} \setminus W_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}) = \mathbb{Z} \setminus \{0\}$ and so $\mathbb{Q} = S^{-1}\mathbb{Z} \neq \mathbb{Z}$. Now Theorem 3.7 implies that $\mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z}$ not satisfies the dual of proper strong Property $\mathcal{A}$. On the other hand since $Z_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}) = \mathbb{Z} \setminus \{1, -1\}$, we have $S = \mathbb{Z} \setminus W_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}) = \{1, -1, \}$ and so $S^{-1}\mathbb{Z} = \mathbb{Z}$. Clearly, the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ satisfies the proper strong Property $\mathcal{A}$. Now Theorem 3.10, implies that $\mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z}$ satisfies the proper strong Property $\mathcal{A}$.

Theorem 3.13. *Let $M = R/\mathfrak{m}_1 \oplus R/\mathfrak{m}_2 \oplus ... \oplus R/\mathfrak{m}_n$ be an R-module, where $\mathfrak{m}_i \in max(R)$ for $i = 1, ..., n$. Then M satisfies the dual of proper strong Property $\mathcal{A}$ if and only if either $max(R) = \{\mathfrak{m}_1, \mathfrak{m}_2, ..., \mathfrak{m}_n\}$ or $\mathfrak{m}_1 = \mathfrak{m}_2 = ... = \mathfrak{m}_n$.*

Proof. First, note that $W_R(M) = \mathfrak{m}_1 \cup \mathfrak{m}_2 \cup ... \cup \mathfrak{m}_n$. So, it is easy to see that any ideal I contained in $W_R(M)$ is contained in some maximal ideal m_j. By Theorem 2.12 (d), M satisfies the dual of Property $\mathcal{A}$. Now, assume that $max(R) = \{\mathfrak{m}_1, \mathfrak{m}_2, ..., \mathfrak{m}_n\}$. Then, we get $S^{-1}R = R$. Then, using Theorem 3.8 (a), M satisfies the dual of proper strong Property $\mathcal{A}$. On the other hand, suppose that $\mathfrak{m}_1 = \mathfrak{m}_2 = ... = \mathfrak{m}_n$. Then $W_R(M) = \mathfrak{m}$ is an ideal of R and hence M satisfies the dual of strong Property $\mathcal{A}$. It follows that M satisfies the dual of proper strong Property $\mathcal{A}$. Conversely, assume that $\{\mathfrak{m}_1, \mathfrak{m}_2, ..., \mathfrak{m}_n\} \subset max(R)$ and that $card(\{\mathfrak{m}_1, \mathfrak{m}_2, ..., \mathfrak{m}_n\}) \geq 2$. Then, by Lemma 3.1, $S^{-1}R \neq R$ as there exists a maximal ideal $\mathfrak{m} \not\subseteq \mathfrak{m}_1 \cup \mathfrak{m}_2 \cup ... \cup \mathfrak{m}_n = W_R(M)$ and thus there exists an element $x \in \mathfrak{m}$ which is neither invertible nor $x \in W_R(M)$. Suppose contrary that $W_R(M)$ is an ideal. Then as $W_R(M) = \mathfrak{m}_1 \cup \mathfrak{m}_2 \cup ... \cup \mathfrak{m}_n$, there exists $j \in \{1, ..., n\}$ such that $W_R(M) = m_j$. Hence $\mathfrak{m}_1 = \mathfrak{m}_2 = ... = \mathfrak{m}_n$, which is a contradiction. Therefore, $W_R(M)$ is not an ideal and so M not satisfies the dual of strong Property $\mathcal{A}$ by Theorem 2.5. Therefore, by Theorem 3.8 (b), M not satisfies the dual of proper strong Property $\mathcal{A}$, as needed. □

4. Properties $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$ and $\mathcal{I}^{\mathcal{M}}_{\mathcal{J}}(\mathcal{N})$

Let J be an ideal of R and let N be a submodule of an R-module M. Set

$$\mathcal{S}_{\mathcal{J}}(\mathcal{N}) = \{m \in M \mid rm \in N \ \mathit{for\ some}\ r \in R - J\}.$$

When J is a prime ideal of R, then $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$ is called *the saturation of N with respect to J or J-closure of N* [11, 15, 16].

Set

$$\mathcal{I}^{\mathcal{M}}_{\mathcal{J}}(\mathcal{N}) = \cap\{L \mid L \ \mathit{is\ a\ completely\ irreducible\ submodule\ of}\ M\ \mathit{and}$$
$$rN \subseteq L \ \mathit{for\ some}\ r \in R - J\}.$$

When J is a prime ideal of R, then $\mathcal{I}^{\mathcal{M}}_{\mathcal{J}}(\mathcal{N})$ is called *the J-interior of N relative to M* [8, 9, 10].

Definition 4.1. Let J be an ideal of R. We say that a submodule N of an R-module M satisfies *Property* $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$ if for each finitely generated submodule K of M with $K \subseteq \mathcal{S}_{\mathcal{J}}(\mathcal{N})$ there exists a $r \in R \setminus J$ with $rK \subseteq N$.

Definition 4.2. Let J be an ideal of R. We say that a submodule N of an R-module M satisfies *Property* $\mathcal{I}^{\mathcal{M}}_{\mathcal{J}}(\mathcal{N})$ (that is the dual of Property $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$) if for each submodule K of M with M/K is finitely cogenerated and $\mathcal{I}^{\mathcal{M}}_{\mathcal{J}}(\mathcal{N}) \subseteq K$ there exists a $r \in R \setminus J$ with $rN \subseteq K$.

Definition 4.3. Let J be an ideal of R. We say that a submodule N of an R-module M satisfies *the strong Property* $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$ if for any $m_1, ..., m_n \in \mathcal{S}_{\mathcal{J}}(\mathcal{N})$ there exists a $r \in R \setminus J$ with $rm_1 \in N$, $...rm_n \in N$.

Example 4.4. Let J be a prime ideal of R and N be a submodule of an R-module M such that $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$ is a finitely generated submodule of M. Then one can see that there exists a $r \in R \setminus J$ with $rN \subseteq \mathcal{S}_{\mathcal{J}}(\mathcal{N})$. This implies that N satisfies the strong Property $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$.

Example 4.5. Let J be a prime ideal of R and N be a submodule of an R-module M such that $M/\mathcal{I}^{\mathcal{M}}_{\mathcal{J}}(\mathcal{N})$ is a finitely cogenerated R-module. Then N satisfies Property $\mathcal{I}^{\mathcal{M}}_{\mathcal{J}}(\mathcal{N})$ by using [10, Lemma 2.3].

If $J = 0$ and $N = 0$ in Definition 4.1 (resp. Definition 4.3), then M under the name Property T (resp. strong Property T) was studied in [2].

Theorem 4.6. *Let M be an R-module and J be an ideal of R. Then we have the following.*

(a) *A submodule N of M satisfies the strong Property $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$ if and only if N satisfies Property $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$ and $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$ is a submodule of M.*

(b) *The zero submodule of M satisfies the strong Property $\mathcal{S}_{\mathcal{J}}(0)$.*

(c) *If a submodule N of M satisfies Property $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$ (respectively, strong Property $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$) and $N \subseteq K \subseteq \mathcal{S}_{\mathcal{J}}(\mathcal{N})$, then K satisfies Property $\mathcal{S}_{\mathcal{J}}(\mathcal{K})$ (respectively, strong Property $\mathcal{S}_{\mathcal{J}}(\mathcal{K})$).*

(d) *Let $\psi = \{N_\lambda\}_{\lambda\in\Lambda}$ be a chain of submodules of M with $N \subseteq N_\lambda \subseteq \mathcal{S}_{\mathcal{J}}(\mathcal{N})$ for each $\lambda \in \Lambda$. Then $\cup_{\lambda\in\Lambda} N_\lambda$ satisfies Property $\mathcal{S}_{\mathcal{J}}(\cup_{\lambda\in *}\mathcal{N}_\lambda)$ (respectively, strong Property $\mathcal{S}_{\mathcal{J}}(\cup_{\lambda\in *}\mathcal{N}_\lambda)$) if and only if each N_λ satisfies Property $\mathcal{S}_{\mathcal{J}}(\mathcal{N}_\lambda)$ (respectively, strong Property $\mathcal{S}_{\mathcal{J}}(\mathcal{N}_\lambda)$).*

(e) *If for a submodule N of M we have $Ann_R(M/N) \not\subseteq J$, or more generally, $Ann_R(\mathcal{S}_{\mathcal{J}}(\mathcal{N})/\mathcal{N}) \not\subseteq J$, then N satisfies strong Property $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$.*

(f) *If J is an irreducible ideal of R (e.g., R/J is an integral domain), then every submodule N of M satisfies strong Property $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$.*

(g) *If a submodule N of M is a Bezout module (respectively, chained module), then N satisfies Property $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$ (respectively, strong Property $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$).*

Proof. (a), (b), (e), and (g) are straightforward.

(c) Let T be a finitely generated submodule of M with $T \subseteq \mathcal{S}_{\mathcal{J}}(\mathcal{K})$. Then $T \subseteq \mathcal{S}_{\mathcal{J}}(\mathcal{K}) \subseteq \mathcal{S}_{\mathcal{J}}(\mathcal{S}_{\mathcal{J}}(\mathcal{N}))$. Clearly, $\mathcal{S}_{\mathcal{J}}(\mathcal{S}_{\mathcal{J}}(\mathcal{N})) = \mathcal{S}_{\mathcal{J}}(\mathcal{N})$. Hence $T \subseteq \mathcal{S}_{\mathcal{J}}(\mathcal{N})$. Now, by assumption, there exists a $r \in R \setminus J$ with $rT \subseteq N$ and so $rT \subseteq K$.

(d) First note that, if $\cup_{\lambda\in\Lambda} N_\lambda$ satisfies Property $\mathcal{S}_{\mathcal{J}}(\cup_{\lambda\in *}\mathcal{N}_\lambda)$ (respectively, strong Property $\mathcal{S}_{\mathcal{J}}(\cup_{\lambda\in *}\mathcal{N}_\lambda)$), then each N_λ satisfies Property $\mathcal{S}_{\mathcal{J}}(\mathcal{N}_\lambda)$ (respectively, strong Property $\mathcal{S}_{\mathcal{J}}(\mathcal{N}_\lambda)$). Conversely, suppose that each N_λ satisfies Property $\mathcal{S}_{\mathcal{J}}(\mathcal{N}_\lambda)$ (respectively, strong Property $\mathcal{S}_{\mathcal{J}}(\mathcal{N}_\lambda)$) and K is a finitely generated submodule of $\mathcal{S}_{\mathcal{J}}(\cup_{\lambda\in *}\mathcal{N}_\lambda)$. Then K is a submodule of $\mathcal{S}_{\mathcal{J}}(\mathcal{N}_\alpha)$ for some $\alpha \in \Lambda$ and hence $(N_\alpha :_R K) \not\subseteq I$. So $(\cup_{\lambda\in\Lambda} N_\lambda :_R K) \not\subseteq I$, as desired.

(f) Let J be an irreducible ideal of R and $m_1, ..., m_n \in \mathcal{S}_{\mathcal{J}}(\mathcal{N})$, where $r_i m_i \in N$ with $r_i \in R \setminus J$ for $i = 1, 2, .., n$. Since J is irreducible, $Rr_1 \cap ... \cap Rr_n \neq J$. Hence there exists $r \in (Rr_1 \cap ... \cap Rr_n) \setminus J$. Thus $rm_i \in N$ for $i = 1, 2, .., n$, as needed. □

Theorem 4.7. *Let M be an R-module and J be an ideal of R. Then we have the following.*

(a) *Let J be a prime ideal of R. A submodule N of M satisfies Property $\mathcal{I}^{\mathcal{M}}_{\mathcal{J}}(\mathcal{N})$ if and only if for any completely irreducible submodules L_1, ..., L_n of M with $\mathcal{I}^{\mathcal{M}}_{\mathcal{J}}(\mathcal{N}) \subseteq L_1$, ..., $\mathcal{I}^{\mathcal{M}}_{\mathcal{J}}(\mathcal{N}) \subseteq L_n$ there exists a $r \in R \setminus J$ with $rN \subseteq L_1$,..., $rN \subseteq L_n$.*

(b) *M satisfies Property $\mathcal{I}^{\mathcal{M}}_{0}(\mathcal{M})$ if and only if every submodule K of M with M/K is finitely cogenerated M/K satisfies Property $\mathcal{I}^{\mathcal{M}/\mathcal{K}}_{0}(\mathcal{M}/\mathcal{K})$.*

(c) *If for a submodule N of M we have $Ann_R(N) \not\subseteq J$, or more generally, $Ann_R(N/\mathcal{I}^{\mathcal{M}}_{\mathcal{J}}(\mathcal{N})) \not\subseteq J$, then N satisfies Property $\mathcal{I}^{\mathcal{M}}_{\mathcal{J}}(\mathcal{N})$.*

(d) *If J is an irreducible ideal of R (e.g., R/J is an integral domain), then every submodule N of M satisfies Property $\mathcal{I}^{\mathcal{M}}_{\mathcal{J}}(\mathcal{N})$.*

Proof. (a) The necessity is clear. For the sufficiency assume that for a submodule K of M with M/K is finitely cogenerated we have $\mathcal{I}^{\mathcal{M}}_{\mathcal{J}}(\mathcal{N}) \subseteq K$. As M/K is finitely cogenerated, there exist completely irreducible submodules $L_1, ..., L_n$ of M such that $K = \cap_{i=1}^{n} L_i$. Now by assumption, there exist $r_1, ..., r_n \in R \setminus J$ such that $r_i N \subseteq L_i$ for $i = 1, 2, .., n$. Set $r = r_1 r_2 ... r_n$. As J is prime, $r \in R \setminus J$. Now $rN \subseteq K$, as needed.

(b) and (c) are straightforward.

(d) Let J be an irreducible ideal of R and $\mathcal{I}^{\mathcal{M}}_{\mathcal{J}}(\mathcal{N}) \subseteq L_i$, where L_i is a completely irreducible submodule of M, $r_i N \subseteq L_i$, and $r_i \in R \setminus J$ for $i = 1, 2, ..., n$. As J is irreducible, $Rr_1 \cap ... \cap Rr_n \neq J$. Hence there exists $r \in (Rr_1 \cap ... \cap Rr_n) \setminus J$. Thus $rN \subseteq L_i$ for $i = 1, 2, .., n$, as needed. □

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