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Research Paper

SOME RESULTS ON THE STRONGLY ANNIHILATING SUBMODULE GRAPH OF A MODULE

REZA BEYRANVAND* AND PARVIN KARIMI BEIRANVAND

Abstract. Let M be a module over a commutative ring R. We continue our study of strongly annihilating submodule graph $\mathbb{SAG}(M)$ introduced in [11]. In addition to providing the more properties of this graph, we introduce the subgraph $\mathbb{SAG}^*(M)$ of $\mathbb{SAG}(M)$ and compare the properties of $\mathbb{SAG}^*(M)$ with $\mathbb{SAG}(M)$ and $\mathbb{AG}(M)$ (the annihilating submodule graph of M introduced in [4]).

1. Introduction

Throughout this paper, R is a commutative ring with nonzero identity element and M is a unitary right R-module. By $N \leq M$ we means that N is a submodule of M. For any $N \leq M$, the ideal $\{r \in R \mid Mr \subseteq N\}$ is denoted by $(N :_R M)$ (briefly $(N : M)$). We denote $((0) : M)$ by $\text{ann}_R(M)$ or simply $\text{ann}(M)$. If $\text{ann}(M) = 0$, then M is said to be *faithful*.

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*Corresponding author

There are many papers on assigning graphs to rings, modules and groups (see for example [1, 2, 3, 5, 9]). *The annihilating ideal graph* $\mathbb{AG}(R)$ was introduced in [6]. $\mathbb{AG}(R)$ is a graph whose vertices are ideals of R with nonzero annihilators and in which two distinct vertices I and J are adjacent if and only if $IJ = 0$. In [4], the authors generalized the above idea to submodules of M and defined the graph $\mathbb{AG}(M)$, called *the annihilating submodule graph*, with vertices $\{0 \neq N \leq M \mid M(N : M)(K : M) = 0, \text{for some } 0 \neq K \leq M\}$, and two distinct vertices N and K are adjacent if and only if $M(N : M)(K : M) = 0$. In [8], *the strongly annihilating submodule graph*, denoted by $\mathbb{SAG}(M)$, introduced and studied. In fact $\mathbb{SAG}(M)$ is an undirected (simple) graph in which a nonzero submodule N of M is a vertex if $N(K : M) = 0$ or $K(N : M) = 0$, for some $0 \neq K \leq M$ and two distinct vertices N and K are adjacent if and only if $N(K : M) = 0$ or $K(N : M) = 0$. Clearly $\mathbb{SAG}(M)$ is a subgraph of $\mathbb{AG}(M)$ and $\mathbb{SAG}(R) = \mathbb{AG}(R)$. The notations of graph theory used in the sequel can be found in [10].

In this paper, we define the subgraph of $\mathbb{SAG}(M)$, denoted by $\mathbb{SAG}^*(M)$, with vertices $\{0 \neq N \leq M \mid (N : M) \neq 0 \text{ and } N(K : M) = 0 \text{ or } K(N : M) = 0, \text{for some } 0 \neq K \leq M \text{ with } (K : M) \neq 0\}$, and two distinct vertices N and K are adjacent if and only if $N - K$ is an edge in $\mathbb{SAG}(M)$. Among other results, in addition to comparing properties of $\mathbb{SAG}(M)$ with $\mathbb{SAG}^*(M)$ in Section 2, we prove that if $\text{ann}_R(M)$ is a nil ideal of R, then there exists a vertex in $\mathbb{AG}(M)$ that is joined to all other vertices if and only if there exists a vertex in $\mathbb{SAG}(M)$ that is joined to all other vertices (Theorem 2.5). Also for any faithful module M over a reduced ring R, it is shown that $\mathbb{SAG}^*(M)$ is a star graph if and only if $M = M_1 \oplus M_2$, where M_1 is simple and M_2 is a prime submodule of M (Corollary 2.9). We show that if R is an Artinian ring and M is a finitely generated faithful R-module, then any nonzero proper submodule of M is a vertex in $\mathbb{SAG}^*(M)$ (Proposition 2.17). Also the necessary and sufficient conditions for M, when $\mathbb{SAG}(M)$ has only one vertex, two vertices or three vertices are given (Theorem 2.18). In Section 3, the coloring of graph $\mathbb{SAG}^*(M)$ is considered. We compare the clique number and the chromatic number of $\mathbb{SAG}^*(M)$ with $\mathbb{AG}^*(M)$ (later defined), see Proposition 3.3 and Theorem 3.5. Also we show that for a semiprime module M, the clique number of $\mathbb{SAG}^*(M)$ is finite if and only if the chromatic number of $\mathbb{SAG}^*(M)$ is finite (Theorem 3.10).

2. $\mathbb{SAG}(M)$ and $\mathbb{SAG}^*(M)$

Let M be an R-module. In [3], the authors defined the subgraph of $\mathbb{AG}(M)$ that vertices are proper submodules like N with $M(N :_R M) \neq 0$ such that there exists a proper submodule K with $M(K :_R M) \neq 0$ and $M(N :_R M)(K :_R M) = 0$. Also two vertices N and K are joined whenever $M(N :_R M)(K :_R M) = 0$. This subgraph is denoted by $\mathbb{AG}^*(M)$. Inspired by this definition and the definition of $\mathbb{SAG}(M)$ in [11], we define the graph $\mathbb{SAG}^*(M)$ as follows.

Definition 2.1. For an R-module M, $\mathbb{SAG}^*(M)$ is a simple graph with vertices $V(\mathbb{SAG}^*(M)) = \{0 \neq N \leq M \mid (N :_R M) \neq 0$ and there exists a nonzero submodule $K \leq M$ with $(K :_R M) \neq 0$ such that $N(K :_R M) = 0$ or $K(N :_R M) = 0\}$. In this graph, two distinct vertices N, K are adjacent if and only if $N(K :_R M) = 0$ or $K(N :_R M) = 0$.

Example 2.2. (a) Consider $\mathbb{Z}$-module $M = \mathbb{Z}_2 \oplus \mathbb{Z}_4$. A simple calculation shows that $\mathbb{SAG}^*(M) = \mathbb{SAG}(M)$.

(b) Let S_1 be a faithful simple R-module and S_2 be an unfaithful R-module. Setting $M = S_1 \oplus S_1 \oplus S_2$, the submodule $N = (0) \oplus (0) \oplus S_2$ is not a vertex in $\mathbb{SAG}^*(M)$, since $(N :_R M) = \text{ann}_R(S_1) = 0$. But for the nonzero submodule $K = (0) \oplus S_1 \oplus (0)$ we have $N \cap K = 0$ and hence N and K are adjacent in $\mathbb{SAG}(M)$.

(c) The submodule $N = \mathbb{Q} \oplus (0)$ of the $\mathbb{Q}$-module $M = \mathbb{Q} \oplus \mathbb{R}$ is simple and faithful. Since $(N :_R M) = 0$, N is not a vertex in $\mathbb{SAG}^*(M)$, however it is a vertex in $\mathbb{SAG}(M)$, because its intersection with $(0) \oplus \mathbb{R}$ is zero.

(d) Consider $M = \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}$ as a $\mathbb{Z}$-module. Then $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ is a submodule of M that is a vertex in $\mathbb{SAG}(M)$, while it is not a vertex in $\mathbb{SAG}^*(M)$.

(e) An R-module M is called *multiplication* if every submodule of it can be written in the form MI, where I is an ideal of R. It is easy to check that M is a multiplication module if and only if every submodule N of M can be written in the form $N = M(N :_R M)$. Clearly, if M is a multiplication module, then $\mathbb{SAG}(M) = \mathbb{SAG}^*(M)$. Also if M is an unfaithful R-module, then $\mathbb{SAG}(M) = \mathbb{SAG}^*(M)$, because for any $N \leq M$ we have $0 \neq \text{ann}_R(M) \subseteq (N :_R M)$.

Proposition 2.3. *If $SAG^*(M) \neq \emptyset$, then any minimal submodule N of M with $(N :_R M) \neq 0$ is a vertex in $SAG^*(M)$.*

Proof. Since $\mathbb{SAG}^*(M) \neq \emptyset$, there are nonzero submodules K and K' in M with $(K :_R M) \neq 0$, $(K' :_R M) \neq 0$ such that $K(K' :_R M) = 0$ or $K'(K :_R M) = 0$. Since N is a minimal submodule, we have $N \cap K = 0$ or $N \cap K = N$. If $K \cap N = 0$, then $K(N :_R M) \subseteq N \cap K = 0$ and so N is a vertex in $\mathbb{SAG}^*(M)$. If $N \cap K = N$, then $N \subseteq K$ and we have $N(K' :_R M) = 0$ or $K'(N :_R M) = 0$. Therefore in any case N is a vertex in $\mathbb{SAG}^*(M)$. □

Lemma 2.4. *Let M be an R-module such that $\text{ann}_R(M)$ is a nil ideal of R. For any minimal submodule N of M, $N(N :_R M) = 0$ or $N = Me$, for some idempotent e in R.*

Proof. By [3, Lemma 2.4], we have $M(N :_R M)(N :_R M) = 0$ or $N = Me$, for some idempotent e in R. Now since N is minimal, $M(N :_R M)(N :_R M) = 0$, implies that $N(N :_R M) = 0$ and so we are done. □

An R-module M is called *prime* if the annihilatior of M is equal to the annihilator of any its nonzero submodule. A proper submodule N of M is called *prime submodule* if M/N is a prime module. One can easily check that a proper submodule N of M is prime if and only if for any $r \in R$ and any submodule K of M, the relation $Kr \subseteq N$ implies that $K \subseteq N$ or $Mr \subseteq N$. Also the set of all zero divisors of M is denoted by $Z(M) = \{r \in R \mid xr = 0$, for some $0 \neq x \in M\}$.

Theorem 2.5. *Let M be an R-module such that $\text{ann}_R(M)$ is a nil ideal of R. Then there exists a vertex in $\mathbb{AG}(M)$ that is joined to all other vertices if and only if there exists a vertex in $\mathbb{SAG}(M)$ that is joined to all other vertices.*

Proof. By [11, Lemma 2.2], $V(\mathbb{AG}(M)) = V(\mathbb{SAG}(M))$ and since $\mathbb{SAG}(M)$ is a subgraph of $\mathbb{AG}(M)$, the " if " part is clear. For the " only if " part, assume that there exists a vertex in $\mathbb{AG}(M)$ such that it is joined to all other vertices. By [3, Theorem 2.5], one of the following cases holds:

(1) There is $e^2 = e \in R$ such that $M = Me \oplus M(1-e)$, where Me is a simple module and $M(1-e)$ is a prime module. Suppose that N is a vertex in $\mathbb{AG}(M)$ that is adjacent to every other vertex. If $N \in \{Me, M(1-e)\}$, then clearly N is adjacent to every other vertex in $\mathbb{SAG}(M)$. Thus assume that $N \notin \{Me, M(1-e)\}$. Then since $Me(N :_R M) = M(Me :_R M)(N :_R M) = 0$ and $M(1-e)(N :_R M) = M(M(1-e) :_R M)(N :_R M) = 0$, we conclude that $M(N :_R M) = 0$. Therefore N is adjacent to every other vertex in $\mathbb{SAG}(M)$.

(2) There is a nonzero submodule N of M such that $Z(M) = \text{ann}_R(M(N :_R M))$. In this case if $M(N :_R M) = 0$, then $K(N :_R M) = 0$, for any $N \neq K \leq M$. This means that N is adjacent to any submodule K of M. Now we suppose that $M(N :_R M) \neq 0$, and K is an arbitrary nonzero vertex in $\mathbb{SAG}(M)$. Then since $\mathbb{SAG}(M)$ is connected, there exists $0 \neq L \leq M$ such that $K(L :_R M) = 0$ or $L(K :_R M) = 0$. In any case we have $M(L :_R M)(K :_R M) = 0$. If $M(L :_R M) = 0$, then L is joined to all other vertices in $\mathbb{SAG}(M)$. Otherwise $(K :_R M) \subseteq Z(M)$ and by the hypothesis, $M(N :_R M)(K :_R M) = 0$ and so K and $M(N :_R M)$ are adjacent. Therefore $M(N :_R M)$ is adjacent to every other vertex in $\mathbb{SAG}(M)$.

(3) M is a vertex in $\mathbb{AG}(M)$. Then there is a nonzero submodule K of M such that $M(M :_R M)(K :_R M) = 0$. Therefore $M(K :_R M) = 0$ and so K is joined to any vertex of $\mathbb{SAG}(M)$.

□

Example 2.6. Consider $M = \mathbb{Z}_2 \oplus \mathbb{Z}_3$ as a $\mathbb{Z}_{12}$-module. Then $\text{ann}_{\mathbb{Z}_{12}}(M)$ is a nilpotent ideal and $\mathbb{SAG}(M)$ is a star graph with two vertices $\mathbb{Z}_2 \oplus (0)$ and $(0) \oplus \mathbb{Z}_3$.

Theorem 2.7. *Let M be a faithful module. Then there exists a vertex in $\mathbb{SAG}^*(M)$ that is joined to all other vertices if and only if M can be written as $M = M_1 \oplus M_2$, where M_1 is a simple submodule and M_2 is a prime submodule of M, or $Z(R)$ is a nil ideal of R.*

Proof. Suppose that N is a vertex in $\mathbb{SAG}^*(M)$ that is joined to all other vertices. Since M is faithful, $V(\mathbb{AG}^*(M)) = V(\mathbb{SAG}^*(M))$. Thus N is joined to all other vertices in $\mathbb{AG}^*(M)$ too. Therefore by [3, Theorem 2.7], we have $M = M_1 \oplus M_2$, where M_1 is a simple submodule and M_2 is a prime submodule of M or $Z(R)$ is a nil ideal of R. Conversely, assume that $M = M_1 \oplus M_2$, where M_1 is a simple submodule and M_2 is a prime submodule or $Z(R)$ is a nil ideal of R. Again by [3, Theorem 2.7], there exists a vertex N in $\mathbb{AG}^*(M)$ that is joined to all other vertices, i.e., $M(N :_R M)(K :_R M) = 0$ for every other vertex K. Set $N' = M(N :_R M)$. Since M is faithful, N' is a vertex in $\mathbb{SAG}^*(M)$ that is joined to all other vertices. □

Example 2.8. $\mathbb{Q} \oplus \mathbb{Q}$ as a $\mathbb{Q} \oplus \mathbb{Z}$-module is faithful and $\mathbb{SAG}^*(\mathbb{Q} \oplus \mathbb{Q})$ is a star graph with two adjacent vertices $\mathbb{Q} \oplus (0)$ and $(0) \oplus \mathbb{Q}$.

Recall that a ring is called *reduced* if it has no nonzero nilpotent element.

Corollary 2.9. *Let R be a reduced ring and M be a faithful R-module. The following statements are equivalent:*

(1) *There exists a vertex in $\mathbb{SAG}^*(M)$ that is adjacent to every other vertex.*
(2) *$\mathbb{SAG}^*(M)$ is a star graph.*
(3) *$M = M_1 \oplus M_2$, where M_1 is a simple submodule and M_2 is a prime submodule of M.*

Proof. (1) $\Leftrightarrow$ (3) follows from Theorem 2.7.
(2) $\Rightarrow$ (1) is clear.
(1) $\Rightarrow$ (2). Since M is faithful, the set of vertices of $\mathbb{SAG}^*(M)$ and $\mathbb{AG}^*(M)$ are the same. Therefore there exists a vertex in $\mathbb{AG}^*(M)$ that is adjacent to every other vertex. By [3, Corollary 2.9], $\mathbb{AG}^*(M)$ is a star graph. Assume that N is the central vertex in $\mathbb{AG}^*(M)$. If there exists a vertex K in $\mathbb{SAG}^*(M)$ such that it is not adjacent to N, then $N(K :_R M) \neq 0$ and $K(N :_R M) \neq 0$. On the other hand $M(N :_R M)(K :_R M) = 0$ and we conclude that

$M(N :_R M) \neq N$. It is clear that $M(N :_R M) \neq 0$ and $0 \neq (N :_R M) \subseteq (M(N :_R M) :_R M)$. Thus $M(N :_R M)$ is a vertex in $\mathbb{SAG}^*(M)$ that is joined to K and so this vertex is joined to K in $\mathbb{AG}^*(M)$, contradicting the fact that $\mathbb{AG}^*(M)$ is a star graph. □

Corollary 2.10. *Let R be an Artinian ring and $\mathrm{ann}_R(M)$ be a nil ideal of R. Then there exists a vertex in $\mathbb{SAG}(M)$ that is adjacent to every other vertex if and only if $M = M_1 \oplus M_2$ where M_1 is simple and M_2 is prime semisimple or R is a local ring with nonzero maximal ideal or M is a vertex in $\mathbb{SAG}(M)$.*

Proof. It follows from [3, Corollary 2.10] and Theorem 2.7. □

Example 2.11. Consider $M = \mathbb{Z}_3 \oplus \mathbb{Z}_8$ as a $\mathbb{Z}_{48}$-module. One can easily check that $\mathrm{ann}_{\mathbb{Z}_{48}}(M) = \{0, 24\}$ is a nil ideal and $\mathbb{SAG}(M)$ is a star graph whose the set of vertices is $V(\mathbb{SAG}(M)) = \{\mathbb{Z}_3 \oplus (0), (0) \oplus \mathbb{Z}_8, (0) \oplus 2\mathbb{Z}_8, (0) \oplus 4\mathbb{Z}_8\}$ and its centeral vertex is $\mathbb{Z}_3 \oplus (0)$.

Corollary 2.12. *Let R be an Artinian ring and M be a faithful R-module. Then there exists a vertex in $\mathbb{SAG}^*(M)$ that is adjacent to every other vertex if and only if $M = M_1 \oplus M_2$ where M_1 and M_2 are both simple or R is a local ring with a nonzero maximal ideal.*

Proof. First suppose that N is a vertex in $\mathbb{SAG}^*(M)$ that is adjacent to every other vertex. Since M is faithful, $V(\mathbb{SAG}^*(M)) = V(\mathbb{AG}^*(M))$ and we know that any edge in $\mathbb{SAG}^*(M)$ is an edge in $\mathbb{AG}^*(M)$. Thus N is adjacent to every other vertex in $\mathbb{AG}^*(M)$. Now the assertion follows from [3, Corollary 2.12]. Conversely, suppose that $M = M_1 \oplus M_2$, where M_1 and M_2 are both simple or R is a local ring with a nonzero maximal ideal. By [3, Corollary 2.12], there exists a vertex N in $\mathbb{AG}^*(M)$ that is adjacent to every other vertex. Thus $M(N :_R M)(K :_R M) = 0$, for every other vertex K in $\mathbb{AG}^*(M)$. Since M is faithful, $M(N :_R M) \neq 0$. Also $0 \neq (N :_R M) \subseteq (M(N :_R M) :_R M)$. Thus $M(N :_R M)(K :_R M) = 0$ implies that $M(N :_R M)$ is a vertex in $\mathbb{SAG}^*(M)$ that is joined to every other vertex. □

Proposition 2.13. *Let $M = M_1 \oplus M_2$, where $\mathrm{ann}_R(M)$ is a nil ideal of R, M_1 is a simple submodule of M and M_2 is a prime submodule of M. Then there exists a vertex in $\mathbb{AG}(M)$ that is joined to every other vertex.*

Proof. Due to simplicity of M_1 and being prime of M_2, we conclude that $\mathrm{ann}_R(M_1)$ is a maximal ideal of R and $\mathrm{ann}_R(M_2)$ is a prime ideal of R. The following two situations may occur:

(a) $\text{ann}_R(M_1) = \text{ann}_R(M_2)$. This implies that M_1 is a vertex that is joined to all other submodules of M, because for any $0 \neq N \leq M$;

$$N(M_1 :_R M) = N\text{ann}_R(M_2) = N\text{ann}_R(M) = 0.$$

(b) $\text{ann}_R(M_1) \neq \text{ann}_R(M_2)$. If R is local, then $\text{ann}_R(M_2) \subseteq \text{ann}_R(M_1)$ and therefore $\text{ann}_R(M) = \text{ann}_R(M_2) \cap \text{ann}_R(M_1) = \text{ann}_R(M_2)$. Thus for any nonzero submodule N of M we have

$$\begin{aligned} M(N :_R M)(M_1 :_R M) &= M(N :_R M)\text{ann}_R(M_2) \\ &\subseteq M\text{ann}_R(M_2) = M\text{ann}_R(M) = 0. \end{aligned}$$

Hence M_1 is a vertex that is joined to all nonzero submodules of M. Now we suppose that R is not local. By [3, Lemma 2.4], since M_1 is minimal, we have either $M(M_1 :_R M)(M_1 :_R M) = 0$ or $M_1 = Me$, where e is an idempotent element in R. First we assume that $M(M_1 :_R M)(M_1 :_R M) = 0$. If $M(M_1 :_R M) = 0$, then M_1 is joined to M and so it is joined to all nonzero submodules of M. Now if $M(M_1 :_R M) \neq 0$, then since M_1 is minimal, $M(M_1 : M) = M_1$ and hence $M_1(M_1 :_R M) = 0$. Thus $M_1\text{ann}_R(M_2) = 0$ and so $\text{ann}_R(M_2) \subseteq \text{ann}_R(M_1)$. It follows that $\text{ann}_R(M) = \text{ann}_R(M_2)$. Hence for any nonzero submodule N of M;

$$M(N :_R M)(M_1 :_R M) = M(N :_R M)\text{ann}_R(M_2) = 0.$$

This means that M_1 is adjacent to any submodule of M. Now, if the second case occurs, then we will have $M = Me \oplus M(1-e)$ and it can be easily seen that $M(1-e) \cong M_2$. Thus $M(1-e)$ is a prime submodule of M. Now by [3, Lemma 2.4], there exists a vertex in $\mathbb{AG}(M)$ that is joined to all other vertices. □

Lemma 2.14. *Let R be an Artinian ring and* $\text{ann}_R(M)$ *be a nil ideal of R. If $\mathbb{SAG}(M)$ is a star graph, Then $M = M_1 \oplus M_2$, where M_1 and M_2 are both simple or R is a local ring with the maximal ideal $P = \text{ann}_R(M)$, that $MP^4 = 0$ or M is a vertex in $\mathbb{SAG}(M)$*

Proof. Suppose that M is not vertex. Since $\mathbb{AG}(M)$ is star, Corollary 2.10 implies that $M = M_1 \oplus M_2$, where M_1 is simple and M_2 is homogeneous semisimple or R is a local ring with the maximal ideal $P = \text{ann}_R(M)$. In the first case we show that M_2 is simple too. If not, then $M_2 = \oplus_{i\in I} S_i$ and $|I| \geq 2$. Therefore $\mathbb{SAG}(M)$ includes the triangle $S_1 - M_1 - S_2 - S_1$ which contradicts being the star of $\mathbb{SAG}(M)$. Now suppose that R is local and $P = \text{ann}_R(M)$. Since R is Artinian, we can consider n to be the smallest positive integer such that $MP^n = 0$ and $MP^{n-1} \neq 0$. If $MP^2 = MP^{n-2}$, then $MP^4 = 0$. Thus we assume that $MP^2 \neq MP^{n-2}$. It is clear that MP^2 and MP^{n-2} are adjacent. But $0 \neq MP^{n-1}$ is the central vertex of the $\mathbb{SAG}(M)$, so $MP^{n-1} = MP^{n-2}$ or $MP^{n-1} = MP^2$. Multiplying the ideal P in the first case we have $MP^{n-1} = 0$, a contradiction. Therefore $MP^{n-1} = MP^2$ and so $MP^3 = 0$. □

Remark 2.15. If M is a faithful R-module and $\mathbb{AG}(M)$ is a complete graph, then $\mathbb{AG}(R)$ is also complete.

Proof. Suppse that I and J are two vertices in $\mathbb{AG}(R)$. Then there exist $I', J' \in V(AG(M))$ such that $II' = JJ' = 0$. Now we have

$$M(MI :_R M)(MI' :_R M) = MI(MI' :_R M) = M(MI' :_R M)I = MI'I = 0.$$

Thus MI is a vertex in $\mathbb{AG}(M)$. Similarly, MJ is a vertex. Due to the completeness of $\mathbb{AG}(M)$ we have

$$0 = M(MI :_R M)(MJ :_R M) = MIJ.$$

Since M is faithful, $IJ = 0$ and hence I and J are adjacent in $\mathbb{AG}(M)$. □

Theorem 2.16. *Let R be an Artinian ring and M be an R-module such that* $\mathrm{ann}_R(M)$ *is a nil ideal of R and M is not a vertex in $\mathbb{SAG}(M)$. If $\mathbb{SAG}(M)$ is a nonempty star graph, then $M = M_1 \oplus M_2$, where M_1 and M_2 both are simple or R is a local ring with the maximal ideal P, where $P \in \mathrm{Ass}(M)$ and one of the following conditions occurs;*

1. *$MP^2 = 0$ and MP is the only minimal submodule of M that $M(N :_R M) = MP$, for any nonzero proper submodule N of M.*
2. *$MP^3 = 0$ and $0 \neq MP^2 = mR$ is the only minimal submodule of M, for some $m \in M$ and $NP(N :_R M) = MP^2$, for any submodule N of $M$with $P^2 \not\subseteq \mathrm{ann}_R(N)$.*
3. *$MP^4 = 0$ and $0 \neq MP^3 = mR$ and $MP = Ma$, for some $m \in M$ and $0 \neq a \in R$, and every nonzero proper submodule of M is a vertex.*

Proof. By Lemma 2.14, $M = M_1 \oplus M_2$ where both M_1 and M_2 are simple or R is a local ring with the maximal ideal P such that $MP^4 = 0$. Suppose that the second case holds. Note that since R is Artinian, there is a minimal submodule K of M and so $P = \mathrm{ann}_R(K)$. Since K is a prime R-module, $P \in \mathrm{Ass}(M)$. Then one of the following cases occurs:

1. $MP^2 = 0$. Since $(N :_R M) \subseteq P$, for any nonzero proper submodule N of M, we have $MP(N :_R M) \subseteq MP^2 = 0$. Then MP is joined to all other vertices in $\mathbb{SAG}(M)$ and since $\mathbb{SAG}(M)$ is star, MP is the central vertex. Also note that for $0 \neq x \in MP$, $\mathrm{ann}_R(x) = P$. We claim that MP is a minimal submodule of M. Otherwise let $0 \neq N \subsetneq MP$. Now since $\mathbb{SAG}(M)$ is star, M has no other nontrivial submodule than MP and N. For any $x \in MP \setminus N$, we have $MP = xR$ and since N is simple, $N = yR$, where $0 \neq y \in N$. On the other hand since $P = \mathrm{ann}_R(xR) = \mathrm{ann}_R(yR)$, it can be easily seen that $MP = xR \cong yR = N$, a contradiction. Hence MP is minimal. Since M is not a vertex and P is maximal, we conclude that $M(N :_R M) = MP$, for any nonzero proper submodule N of M.

(2) $MP^3 = 0$ and $MP^2 \neq 0$. Then MP^2 is the central vertex in $\mathbb{SAG}(M)$. Since $P \in \mathrm{Ass}(M)$, we have $P = \mathrm{ann}_R(m)$, for some $0 \neq m \in M$. Thus $mR(N :_R M) \subseteq mP = 0$, for any nonzero proper submodule N of M. Therefore $mR = MP^2$. If there exists $0 \neq N \lneq MP^2$, then we have the cycle $MP - N - MP^2 - MP$ that is a contradiction. Thus MP^2 is a minimal submodule of M. If $T \neq MP^2$ is a minimal submodule of M, then $\mathrm{ann}_R(T)$ is a maximal ideal and since R is local, $P = \mathrm{ann}_R(T)$. Therefore we have $MP(T :_R M) = M(T :_R M)P \subseteq TP = 0$, contradicting the fact that $\mathbb{SAG}(M)$ is star. Thus MP^2 is the only minimal submodule of M. Now let N be a submodule of M such that $P^2 \nsubseteq \mathrm{ann}_R(N)$. Then $NP(N :_R M) \subseteq NP^2 \subseteq MP^2$. If $NP(N :_R M) = 0$, then since $\mathbb{SAG}(M)$ is a star graph, we have $NP = N$, $NP = MP^2$ or $N = MP^2$. In any case we conclude that $P^2 \nsubseteq \mathrm{ann}_R(N)$, a contradiction. Therefore $NP(N :_R M) \neq 0$ and so $NP(N :_R M) = MP^2$.

(3) $MP^4 = 0$ and $MP^3 \neq 0$. In this case we show that $\mathbb{AG}(M)$ is also a star graph, i.e, $\mathbb{AG}(M) = \mathbb{SAG}(M)$. First note that for any ideal I of R and any submodule N of M, if $MI - N$ is an edge in $\mathbb{AG}(M)$, then $MI - N$ is also an edge in $\mathbb{SAG}(M)$, because $M(MI :_R M) = MI$. Now suppose that $\mathbb{AG}(M)$ is not star and $N - K$ is an edge in $\mathbb{AG}(M)$ such that $N \neq K$ and $N, K \notin \{MP, MP^2, MP^3\}$. Thus $M(N :_R M)(K :_R M) = 0$ and since $\mathbb{SAG}(M)$ is star, one of the following occurs:

   (a) $M(N :_R M) = N$. Then $N(K :_R M) = 0$ and so $N - K$ is an edge in $\mathbb{SAG}(M)$, a contradiction.

   (b) $M(N :_R M) = MP^3$. Then $0 = MP^3(MP :_R M) = M(N :_R M)(MP :_R M) = M(MP :_R M)(N :_R M) = MP(N :_R M)$ and so $MP - N$ is an edge in $\mathbb{SAG}(M)$, a contradiction.

   (c) $M(N :_R M) = K$. Then similarly, $M(K :_R M) = N$. In this case, we conclude that $K \subseteq N$ and $N \subseteq K$ and so $N = K$, a contradiction.

Therefore, $\mathbb{AG}(M)$ is also a star graph and we are done by Case 3 in the proof of Theorem 2.14 in [3].

□

Proposition 2.17. (a) *Let M be a faithful R-module such that it has only one nonzero proper submodule. Then $M \cong R$ as R-modules.*

(b) *Let R be an Artinian ring and M be a finitely generated faithful R-module. Then any nonzero proper submodule of M is a vertex in $\mathbb{SAG}^*(M)$.*

Proof. (a) Suppose that N is the only nonzero proper submodule of M. Clearly $N = xR$, for any $0 \neq x \in N$. Let $y \in M \setminus N$ and we claim that $M = (x + y)R$. If not, then $(x + y)R = 0$ or $(x + y)R = N$. In any case we conclude that $y \in N$, which is a contradiction. Hence $(x+y)R = M$ and one can easily see that $\phi : R \to M$ by $\phi(r) = (x+y)r$ is an R-isomorphism.

(b) Suppose that N is a nonzero proper submodule of M. There exists a maximal submodule K of M containing N. Because of maximality of K, M/K is simple and therefore $(K :_R M)$ is maximal. On the other hand since $\mathrm{ann}_R(M) \subseteq (K :_R M)$, we have $(K :_R M) \in \mathrm{Ass}(M)$. Then there exists $0 \neq m \in M$ such that $(N :_R M) \subseteq (K :_R M) = \mathrm{ann}_R(m)$ and so $mR(N :_R M) = 0$. Thus N is a vertex in $\mathbb{SAG}^*(M)$.

□

Theorem 2.18. *Let M be a faithful R-module that is not a vertex in $\mathbb{SAG}(M)$. Then the following statements hold:*

(a) *$\mathbb{SAG}(M)$ is a graph with only one vertex if and only if M has only one nonzero proper submodule.*

(b) *$\mathbb{SAG}(M)$ is a graph with two vertices if and only if $M = M_1 \oplus M_2$, where M_1 and M_2 are simple or M has exactly two nonzero proper submodules.*

(c) *$\mathbb{SAG}(M)$ is a graph with three vertices if and only if M has exactly three nonzero submodules m_1R, m_2R and m_3R such that*

$$m_3R = m_1R \cap m_2R,$$

$$Z(R) = \mathrm{ann}_R(m_3),$$

$$(m_1R)^2 = (m_2R)^2 = (m_3R)^2 = 0,$$

or

$$\Lambda^* M = \{MZ(R), MZ^2(R), MZ^3(R)\},$$

where $\Lambda^* M$ is the set of nonzero proper submodules of M.

Proof. Since $V(\mathbb{SAG}(M)) = V(\mathbb{AG}(M))$, the proof follows from [3, Corollary 2.16]. □

3. Coloring of $SAG^*(M)$

In a graph G, a *clique* of G is a complete subgraph and the supremum of the sizes of cliques in G, denoted by $cl(G)$, is called the clique number of G. Let $\chi(G)$ denote the *chromatic number* of the graph G, that is, the minimal number of colors needed to color the vertices of G so that no two adjacent vertices have the same color. Clearly $\chi(G) \geq cl(G)$. In this section, we study the coloring of graphs $\mathbb{SAG}^*(M)$ and $\mathbb{SAG}^*(M)$, espicially when they are (complete) bipartite graphs or their chromatic and clique numbers are finite.

Proposition 3.1. *Let M be a faithful R-module. Then $\chi(SAG(M)) = 1$ if and only if M has only one nonzero proper submodule.*

Proof. Suppose that $\chi(SAG(M)) = 1$. By [11, Theorem 2.4], $\mathbb{SAG}(M)$ is connected and can not have more than one vertex. Since M is faithful, according to Theorem 2.18(1), M has only one nonzero proper submodule. □

Remark 3.2. If $\mathbb{AG}^*(M)$ is a bipartite graph, then clearly $\mathbb{SAG}^*(M)$ is a bipartite graph. Also $V(AG^*(M)) \subseteq V(SAG^*(M))$ and if M is faithful or M is not a vertex in $\mathbb{AG}^*(M)$, then $\mathbb{SAG}^*(M)$ is a subgraph of $\mathbb{AG}^*(M)$ and $V(AG^*(M)) = V(SAG^*(M))$. To see this, let N and K be adjacent vertices in $\mathbb{AG}^*(M)$. Then $M(K :_R M) \neq 0$, $M(N :_R M) \neq 0$ and $M(N :_R M)(K :_R M) = 0$. Thus $(K :_R M) \neq 0$, $(N :_R M) \neq 0$ and $K'(N :_R M) = 0$ where $K' = M(K :_R M) \subseteq K$. Also $(K' :_R M) \neq 0$, because

$$0 \neq (K :_R M) \subseteq (M(K :_R M) :_R M) = (K' :_R M).$$

Threfore K' is a vertex in $\mathbb{SAG}^*(M)$ that is joined to N.

Proposition 3.3. *Let M be a faithful R-module. Then,*

(a) *$\mathbb{SAG}^*(M)$ is a bipartite graph if and only if $\mathbb{AG}^*(M)$ is a bipartite graph.*
(b) *If R is a reduced ring, then f $\mathbb{AG}^*(M)$ has an infinite clique number if and only if $\mathbb{SAG}^*(M)$ has an infinite clique number.*

Proof. (a) If $\mathbb{AG}^*(M)$ is a bipartite graph, then by Remark 3.2, $\mathbb{SAG}^*(M)$ is a bipartite graph. Now suppose that $\mathbb{SAG}^*(M)$ is a bipartite graph. If $\mathbb{AG}^*(M)$ is not a bipartite graph, then there are two vertices K and N in one part of the graph $\mathbb{SAG}^*(M)$ such that they are adjacent in the $\mathbb{AG}^*(M)$. By Remark 3.2, $N - K'$ and $N' - K$ are two edges in $\mathbb{SAG}^*(M)$, where $K' = M(K :_R M)$ and $N' = M(N :_R M)$. It follows that $N' - K'$ is also an edge in $\mathbb{SAG}^*(M)$ that contradicts being bipartite graph of $\mathbb{SAG}^*(M)$.

(b) Clearly, if $\mathbb{SAG}^*(M)$ has an infinite clique number, then so is $\mathbb{AG}^*(M)$. Conversely, if $\mathbb{AG}^*(M)$ has an infinite clique, then there exist vertices K and $K_1, K_2, \cdots$ such that K is joined to K_i, for every $i \geq 1$ and also for any $i \neq j$, K_i is joined to K_j in the $\mathbb{AG}^*(M)$. Thus the following hold;

$$M(K :_R M)(K_i :_R M) = 0, \ i \geq 1,$$

$$M(K_i :_R M)(K_j :_R M) = 0, \ i, j \geq 1, \ i \neq j.$$

Set $K_i' = M(K_i :_R M)$ and $K_j' = M(K_j :_R M)$. Similar to part (a) can be shown that K_i' and K_j' are adjacent in $\mathbb{SAG}^*(M)$. Note that $K_i' \neq K_j'$, otherwise;

$$M(K_i :_R M) = M(K_j :_R M),$$

and so

$$M(K_i :_R M)^2 = M(K_i :_R M)(K_j :_R M) = 0.$$

Since M is faithful and R is reduced, we conclude that $(K_i :_R M) = 0$, a contradiction.

□

Lemma 3.4. *Let R be a reduced ring and M be a faithful R-module. Then $\mathbb{AG}^*(M)$ is a (complete) bipartite graph with two nonempty parts if and only if $\mathbb{AG}(R)$ is a (complete) bipartite graph with two nonempty parts.*

Proof. Suppose that $\mathbb{AG}^*(M)$ is a (complete) bipartite graph with two nonempty parts A and B. Then one can easily see that $\mathbb{AG}(R)$ is a (complete) bipartite graph with parts $A' = \{I \leq R \mid MI \in A\}$ and $B' = \{I \leq R \mid MI \in B\}$. Conversely, if $\mathbb{AG}(R)$ is a (complete) bipartite graph with two parts A and B, then it is easy to see that $\mathbb{AG}^*(M)$ is a (complete) bipartite graph with two parts $A' = \{N \leq M \mid (N :_R M) \in A\}$ and $B' = \{N \leq M \mid (N :_R M) \in B\}$ □

Theorem 3.5. *For any faithful R-module M, the following statements are equivalent:*

(a) $\chi(SAG^*(M)) = 2$.

(b) *$SAG^*(M)$ is a bipartite graph with two nonempty parts.*

(c) *R is a reduced ring with exactly two minimal prime ideals or $\mathbb{SAG}^*(M)$ is a star graph with more than one vertex.*

Proof. $(a) \Leftrightarrow (b)$ is trivial.

$(b) \Rightarrow (c)$. Suppose that $\mathbb{SAG}^*(M)$ is a bipartite graph with two nonempty parts. Then $\mathbb{AG}^*(M)$ is the same by Propsition 3.3(a). Therefore by [3, Theorem 3.3], R is a reduced ring with exactly two minimal prime ideals or $\mathbb{AG}^*(M)$ is a star graph with more than one vertex. If $\mathbb{AG}^*(M)$ is a star graph with more than one vertex, then so is $\mathbb{SAG}^*(M)$. To see this, assume that N is a centeral vertex in the $\mathbb{AG}^*(M)$ and $N \neq K$ is an arbitrary vertex in $\mathbb{AG}^*(M)$ that is not joined to N in $\mathbb{SAG}^*(M)$. Then by the proof of Remark 3.2, there is a vertex $0 \neq N' \lneq N$ such that $K - N'$ is an edge in $\mathbb{SAG}^*(M)$. This implies that $K - N'$ is also an edge in $\mathbb{AG}^*(M)$ which contradicts $\mathbb{AG}^*(M)$ being a star.

$(c) \Rightarrow (b)$. If $\mathbb{SAG}^*(M)$ is a star graph with more than one vertex, then it is clearly a (complete) bipartite graph. Now assume that R is a reduced ring with two minimal prime ideals. Then by [7, Theorem 2.3], $\mathbb{AG}(R)$ is a complete bipartite graph with two nonempty parts and so is $\mathbb{AG}^*(M)$ by Lemma 3.4. It follows that $\mathbb{SAG}^*(M)$ is a bipartite graph. □

Corollary 3.6. *Let R be an Artinian ring and M be a faithful R-module. Then the following are equivalent:*

(a) $\chi(\mathbb{SAG}^*(M)) = 2$.

(b) $\mathbb{SAG}^*(M)$ *is a bipartite graph with two nonempty parts.*

(c) $M = M_1 \oplus M_2$ *where M_1 and M_2 are homogeneous semisimple modules or* $\mathbb{SAG}^*(M)$ *is a star graph with more than one vertex.*

Proof. $(a) \Leftrightarrow (b)$ follows from Theorem 3.5.
$(b) \Rightarrow (c)$. Suppose that $\mathbb{SAG}^*(M)$ is a bipartite graph with two nonempty parts. By Proposition 3.3(a), $\mathbb{AG}^*(M)$ is a bipartite graph and hence by Lemma 3.4, $\mathbb{AG}(R)$ is also a bipartite graph. If R is reduced, then since R is Artinian and commutative, by Wedderburn-Artin Theorem, $R \cong F_1 \times F_2 \times \cdots \times F_n$, where each F_i is a field (see [12, Theorem 3.5]). If $n \geq 3$, then $F_1 - F_2 - F_3 - F_1$ is a triangle in $\mathbb{AG}(R)$, a contradiction. Thus $R \cong F_1 \times F_2$. This implies that there are only two nonisomorphic simple (right) R-modules, up to isomorphism. Therefore M is semisimple and we can write $M = (\underset{I}{\oplus} S) \oplus (\underset{J}{\oplus} T)$, where $|I| \geq 1$, $|J| \geq 1$, S, T are simple and $S \ncong T$. (Note that $\text{ann}_R(S) = F_1 \times (0)$ and $\text{ann}_R(T) = (0) \times F_2$). Now if R is not reduced, then by Theorem 3.5, $\mathbb{SAG}^*(M)$ is a star graph with more than one vertex and the proof is complete.
$(c) \Rightarrow (b)$. Suppose that $M = (\underset{I}{\oplus} S) \oplus (\underset{J}{\oplus} T)$, where $|I| \geq 1$, $|J| \geq 1$ and S, T are simple with $S \ncong T$. Then one can check that $\mathbb{SAG}^*(M)$ is a bipartite graph with two parts A and B, where

$$A = \{0 \neq N \lneq M \mid N \cong (\underset{I}{\oplus} S) \oplus (\underset{J_1}{\oplus} T),\ J_1 \subsetneq J \text{ and } |J_1| \geq 0\},$$

$$B = \{0 \neq K \lneq M \mid K \cong (\underset{I_1}{\oplus} S) \oplus (\underset{J}{\oplus} T),\ I_1 \subsetneq I \text{ and } |I_1| \geq 0\}.$$

□

Corollary 3.7. *Let R be a reduced ring and M be a faithful R-module. The following statements are equivqlent:*

(a) $\chi(\mathbb{SAG}^*(M)) = 2$.

(b) $\mathbb{SAG}^*(M)$ *is a bipartite graph with two nonempty parts.*

(c) *R has only two minimal ideals.*

Proof. Follows from [7, Theorem 2.5], Proposition 3.3 and Lemma 3.4. □

An R-module M is called *semiprime* if, for any $r \in R$ and any submodule N of M, $Nr^2 = 0$ implies that $Nr = 0$.

Lemma 3.8. *Let M be a semiprime R-module such that the clique number of $\mathbb{SAG}^*(M)$ is not infinite. Then the set of all submodules of the form $\mathrm{ann}_M(I)$, where I is an ideal of R, satisfies the ACC condition.*

Proof. Assuming the contrary, there is a strictly ascending chain

$$\mathrm{ann}_M(I_1) \subsetneq \mathrm{ann}_M(I_2) \subsetneq ...,$$

in M. Since for any $i \geq 1$, $\mathrm{ann}_M(I_{i+1})I_i \neq 0$, there exists $r_i \in I_i$ such that $\mathrm{ann}_M(I_{i+1})r_i \neq 0$. We set $J_i = \mathrm{ann}_M(I_{i+1})r_i$ for $i = 1, 2, 3, ...,$ and we show that for any $i < j$, $J_i \neq J_j$. Otherwise $\mathrm{ann}_M(I_{i+1})r_i = \mathrm{ann}_M(I_{j+1})r_j$, where $i < j$. Then

$$0 = \mathrm{ann}_M(I_{i+1})r_i r_j = \mathrm{ann}_M(I_{j+1})r_j^2.$$

Since M is semiprime, $\mathrm{ann}_M(I_{j+1})r_j = 0$, a contradiction. Now for any $i < j$;

$$J_j(J_i :_R M) = \mathrm{ann}_M(I_{j+1})r_j(\mathrm{ann}_M(I_{i+1})r_i :_R M) \subseteq \mathrm{ann}_M(I_{i+1})r_i r_j = 0.$$

Therefore for any $i < j$, J_i and J_j are joined in $\mathbb{SAG}^*(M)$ and hence $\mathbb{SAG}^*(M)$ has an infinite clique number which contradicts the hypothesis. □

Lemma 3.9. *Let $P_1 = \mathrm{ann}_M(r_1)$ and $P_2 = \mathrm{ann}_M(r_2)$ be two distinct prime submodules of R-module M. Then Mr_1 is joined to Mr_2 in $\mathbb{SAG}(M)$.*

Proof. We claim that $Mr_1r_2 = 0$. Otherwise, $\mathrm{ann}_M(r_1)r_1 = 0 \subseteq \mathrm{ann}_M(r_2)$ implies that $\mathrm{ann}_M(r_1) \subseteq \mathrm{ann}_M(r_2)$, because $Mr_1r_2 \neq 0$ and $\mathrm{ann}_M(r_2)$ is a prime submodule of M. Similarly we have $\mathrm{ann}_M(r_2) \subseteq \mathrm{ann}_M(r_1)$, contradicting the hypothesis. Therefore $Mr_1r_2 = 0$ and so $Mr_1(Mr_2 :_R M) \subseteq Mr_1r_2 = 0$, as desired. □

Theorem 3.10. *For a semiprime module M, the following statements are equivqlent;*

(a) $\chi(\mathbb{SAG}^*(M))$ *is finite.*
(b) $cl(\mathbb{SAG}^*(M))$ *is finite.*
(c) $\mathbb{SAG}^*(M)$ *dose not have an infinite clique number.*
(d) *There are prime submodules $P_1, P_2, \ldots, P_k$ in M such that $\bigcap_{i=1}^{k}(P_i :_R M) = (0)$.*

Proof. $(a) \Rightarrow (b)$ and $(b) \Rightarrow (c)$ are clear.
$(c) \Rightarrow (d)$. Suppose that $\mathbb{SAG}^*(M)$ dose not have an infinite clique number. By lemma 3.8, M satisfies the ACC condition on the submodules of the form $\mathrm{ann}_M(I)$, where I is an ideal of R. Thus the set $\{\mathrm{ann}_M(x) \mid Mx \neq 0\}$ has a maximal element. It is easy to check that the

maximal elements of this set are prime submodules of M. By lemma 3.9, the set of distinct maximal elements of the above set is finite. We name these elements $\text{ann}_M(x_1), \ldots, \text{ann}_M(x_k)$. Now we claim that $\cap_{i=1}^{k}(\text{ann}_M(x_i) :_R M) = 0$. Let $0 \neq x \in \cap_{i=1}^{k}(\text{ann}_M(x_i) :_R M)$, then for any i, $Mx \subseteq \text{ann}_M(x_i)$. On the other hand there is $1 \leq j \leq k$ such that $\text{ann}_M(x) \subseteq \text{ann}_M(x_j)$. Thus $Mx_j x = 0$ and so $Mx_j \subseteq \text{ann}_M(x)$. Then $Mx_j \subseteq \text{ann}_M(x_j)$ and hence $Mx_j^2 = 0$. Since M is a semiprime module, we conclude that $Mx_j = 0$, a contradiction.

$(d) \Rightarrow (a)$. Suppose that there are prime submodules $P_1, P_2, \ldots, P_k$ in M such that $\bigcap_{i=1}^{k}(P_i :_R M) = (0)$. For $N \in V(SAG^*(M))$, we define

$$f(N) = \min\{n \in \mathbb{N} \mid (N :_R M) \nsubseteq (P_n :_R M)\}.$$

Now we claim that $\chi(\mathbb{SAG}^*(M)) \leq k$. Let N and K be adjacent in $\mathbb{SAG}^*(M)$. Then $N(K :_R M) = 0$ or $K(N :_R M) = 0$. Anyway $M(N :_R M)(K :_R M) = 0$ and so

$$(N :_R M)(K :_R M) \subseteq \text{ann}_R(M) \subseteq (P_n :_R M).$$

Since $(P_n :_R M)$ is a prime ideal of R, $(N :_R M) \subseteq (P_n :_R M)$ or $(K :_R M) \subseteq (P_n :_R M)$ which is a contradiction in any case. Thus every two adjacent vertices have different colors. □

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Reza Beyranvand

Department of Mathematics, Lorestan university,

Khoramabad, Iran.

beyranvand.r@lu.ac.ir; beyranvand.r94@gmail.com

Parvin Karimi Beiranvand

Department of Mathematics, Islamic Azad University,

Khorramabad branch, Khoramabad, Iran.

karimiparvin87@gmail.com