Algebraic Structures and Their Applications Vol. 10 No. 2 (2023) pp 173-181.

Research Paper

THE EFFECT OF SINGULARITY ON A TYPE OF SUPPLEMENTED MODULES

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Abstract. Let R be a ring, M a right R-module, and $S = End_R(M)$ the ring of all R-Endomorphisms of M. We say that M is Endomorphism δ-H-supplemented (briefly, E-δ-H-supplemented) provided that for every $\varphi \in S$, there exists a direct summand D of M such that $M = Im\varphi + X$ if and only if $M = D + X$ for every submodule X of M with M/X singular. In this paper, we prove that a non-δ-cosingular module M is E-δ-H-supplemented if and only if M is dual Rickart. We also show that every direct summand of a weak duo E-δ-H-supplemented module inherits the property.

1. Introduction

Throughout this paper, all rings are associative ring with identity, and all modules are unitary right R-modules. Let M and N be R-modules. Then by $S = End_R(M)$, we denote

DOI: 10.22034/as.2023.3044

MSC(2010): Primary 05C50.

Keywords: Dual Rickart module, δ-small-submodule, E-δ-H-supplemented modules, E-H-supplemented module, H-supplemented module.

Received: 19 December 2022, Accepted: 08 April 2023.

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the ring of all endomorphisms of M, and by $N \leq M$, we mean that N is a submodule of M. A submodule N of M is said to be *small* in M if $N + K \neq M$ for any proper submodule K of M, and we denote it by $N \ll M$. As a generalization, Zhou [12] was introduced the concept of δ-small submodules. A submodule N of M is called δ-small in M (denoted by $N \ll_\delta M$) if $M \neq N + K$ for any proper submodule K of M with M/K singular. General properties and some useful characterizations of δ-small submodules of a module was investigated in [12].

The notion of *H-supplemented* modules was introduced by Mohamed and Muller in [5]. A module M was called *H-supplemented* if for every submodule N of M there exists a direct summand D of M such that $M = N + X$ if and only if $M = D + X$ for every submodule X of M. Different definition's style, unusual properties and being a generalization of lifting modules, all led many researchers to study and investigate H-supplemented modules further than what was investigated in [5]. Maybe the first serious effort has been made in [2]. He investigated some general properties of H-supplemented modules, such as homomorphic images and direct summands of these modules. The authors in reference [3], proposed some equivalent conditions for a module to be H-supplemented, which show that this class of modules is closely related to the concept of small submodules. They proved that a module M is H-supplemented if and only if for every submodule N of M there is a direct summand D of M such that $(N + D)/N \ll M/N$ and $(N + D)/D \ll M/D$. In [7], Moniri and etc. investigated, the E-H-supplemented definition of module M, a homological approach of a H-supplemented modules. A module M is called E-H-supplemented, if for every endomorphism φ of M there exists a direct summand D of M such that $Imf + X = M$ if and only if $D + X = M$.

Let M be a module over a ring R. Following [11], M is called *(non)cosingular* if ($\overline{Z}(M) = M$) $\overline{Z}(M) = 0$, where $\overline{Z}(M) = \bigcap\{Kerf \mid f : M \to U\}$, in which U is an arbitrary small right R-module. The author in [8] considered the class of right δ-small R-modules in the definition of $\overline{Z}(M)$, and defined $\overline{Z}_\delta(M)$ to be $\bigcap\{Kerg \mid g : M \to V\}$ where V is a δ-small module (i.e. there exists another module U such that $V \ll_\delta U$). In [8], M is said to be *(non-)δ-cosingular* in case ($\overline{Z}_\delta(M) = M$) $\overline{Z}_\delta(M) = 0$. Hence for module M, we have $\overline{Z}_\delta(M) \subseteq \overline{Z}(M)$. Therefore, every cosingular right R-module is δ-cosingular, and every non-δ-cosingular module is noncosingular.

As a pioneer research on lifting modules, supplemented modules, $\oplus$-supplemented modules, and others concept of singularity. Zhou [12] made a different impression on works that made on supplemented modules and related concepts. The first person who worked on δ-version was Koşan, that introduced δ-lifting modules and δ-supplemented modules and tried to investigate their natural properties [4]. According to [6], a module M is said to be δ-H-supplemented, if for every submodule N of M there is a direct summand D of M such that $M = N + X$ if and only if $M = D + X$, for every submodule X of M with M/X singular.

Inspired by [7] and [6], in this manuscript, we are interested in studying δ-H-supplemented modules via homomorphisms. Combining the two concepts E-H-supplemented modules and δ-H-supplemented modules, we call a module M, a E-δ-H-supplemented module if for every endomorphism φ of M there exists a direct summand D of M such that $Im\varphi + X = M$ if and only if $D + X = M$, for all submodules X of M with M/X singular. We introduce some equivalent conditions for this definition impressing the close relation of δ-H-supplemented modules to the concept of δ-small submodules.

2. Singularity and endomorphism H-supplemented modules

In this section, we introduce a new generalization of the class of E-H-supplemented modules and δ-H-supplemented modules, namely Endomorphism δ-H-supplemented modules. We work on factor modules, particularly direct summands of Endomorphism δ-H-supplemented modules.

Definition 2.1. A module M is called *Endomorphism δ-H-supplemented* (*E-δ-H-supplemented*, for short) if for every $f \in S$, there exists a direct summand D of M such that $Imf + X = M$ if and only if $D + X = M$ for every submodule X of M with M/X singular.

Every δ-H-supplemented module is E-δ-H-supplemented. We shall present some conditions showing that the concept of E-δ-H-supplemented modules is closely related to the concept δ-small submodules.

Theorem 2.2. *The following are equivalent for a module M:*

(1) *M is E-H-supplemented;*

(2) *For every $f \in S$, there exists a direct summand D of M with $\frac{Imf+D}{D} \ll_\delta \frac{M}{D}$ and $\frac{Imf+D}{Imf} \ll_\delta \frac{M}{Imf}$;*

(3) *For every $f \in S$, there exist a direct summand D and a submodule N of M with $Imf \subseteq N$ and $D \subseteq N$ such that $\frac{N}{D} \ll_\delta \frac{M}{D}$ and $\frac{N}{Imf} \ll_\delta \frac{M}{Imf}$.*

Proof. (1) $\Rightarrow$ (2) Let $f \in S$. By (1), there exists a direct summand D of M such that $Imf + X = M$ if and only if $D + X = M$ for every submodule X of M with M/X singular. Let $(Imf + D)/Imf + X/Imf = M/Imf$ for a submodule X of M containing Imf such that M/X is singular. It follows that $D + X = M$. Now, (1) implies $Imf + X = M$. Therefore, $X = M$, showing that $(Imf + D)/Imf \ll_\delta M/Imf$. For the second one, suppose that $(Imf + D)/D + Y/D = M/D$ where Y is a submodule of M, which contains D with M/Y singular. Then $Imf + Y = M$ combining with (1) implies $M = Y$, as required.

(2) $\Rightarrow$ (3) Set $N = Imf + D$.

(3) $\Rightarrow$ (1) Let $f \in S$. Then by assumption there is a submodule N and a direct summand D of M such that $N/D \ll_\delta M/D$ and $N/Imf \ll_\delta M/Imf$. Suppose that $Imf + X = M$ for a submodule X of M with M/X singular. Then $M = N + X$. Now, $N/D + (X+D)/D = M/D$. As M/X is singular, we conclude that $M/(X+D)$ is singular. Being N/D a δ-small submodule of M/D implies $M = X + D$. For the converse, let $M = Y + D$ for a submodule Y of M with M/Y singular. Then $M = N + Y$ which implies $N/Imf + (Y + Imf)/Imf = M/Imf$. Note also that $M/(Y + Imf)$ is singular and M/Y. Therefore, $M = Imf + Y$ is desired. □

We present some assumptions, which under two concepts E-H-supplemented modules and E-δ-H-supplemented modules are coincide.

Proposition 2.3. *Let M be a module. In each of the following cases, M is E-H-supplemented if and only if M is E-δ-H-supplemented.*

(1) *M is a singular module.*

(2) *M has no simple projective submodule.*

Proof. (1) This follows from the fact that every homomorphic image of a singular module is singular. In fact, every δ-small submodule of a singular module is a small submodule of that module.

(2) Let M be a E-δ-H-supplemented module with simple projective submodule. Suppose that f is an endomorphism of M. Then there is a direct summand D of M such that $(Imf + D)/Imf \ll_\delta M/Imf$ and $(Imf + D)/D \ll_\delta M/D$. Let $(Imf + D)/Imf + T/Imf = M/Imf$ for a submodule T/Imf of M/Imf. Then, by [12, Lemma 1.2], $(Imf + D)/Imf$ contains a semisimple projective direct summand Y/Imf of M/Imf such that $Y/Imf \oplus T/Imf = M/Imf$. So, there is a submodule N' of Y such that $Y = Imf \oplus N'$, since Y/Imf is projective. It follows that N' contains a simple projective submodule. Now, $Y = Imf$, and consequently $T/Imf = M/Imf$ implies that $(Imf + D)/Imf \ll M/Imf$. Applying the same argument, we can prove $(Imf + D)/D \ll M/D$. Therefore, M is H-supplemented. □

Corollary 2.4. *Let R be a ring such that every simple right R-module is singular(consider the ring $\mathbb{Z}$). Then a right R-module M is E-H-supplemented if and only if M is E-δ-H-supplemented. Particularly, an $\mathbb{Z}$-module M is E-H-supplemented if and only if M is E-δ-H-supplemented.*

Proposition 2.5. *Let M be an indecomposable module. Then M is E-δ-H-supplemented if and only if the image of each endomorphism of M is δ-small in M or every endomorphism of M is an epimorphism.*

Proof. Let M be an indecomposable E-δ-H-supplemented module. Consider a nonzero endomorphism f of M. Then there is a direct summand D of M such that $(Imf + D)/Imf \ll_\delta M/Imf$ and $(Imf + D)/D \ll_\delta M/D$. Suppose $D = 0$. Then clearly, $Imf \ll_\delta M$. Otherwise, $D = M$ implies $M/Imf \ll_\delta M/Imf$. Now [12, Lemma 1.2] yields that M/Imf is projective and semisimple (it is sufficient in [12, Lemma 1.2] that we set $M = M/Imf$, $N = M/Imf$ and $X = 0$). It follows now that Imf must be a direct summand of M. Being M indecomposable implies $Imf = 0$, a contradiction. The converse is straightforward to check. □

We next present some examples of E-δ-H-supplemented modules.

Example 2.6. (1) Suppose that M_1 is a H-supplemented module with a unique composition series $M_1 \supset U \supset V \supset 0$ (we may choose the $\mathbb{Z}$-module $M_1 = \mathbb{Z}_8$). Now, let $M = M_1 \oplus M_1/U \oplus U/V \oplus V/0$. Then M is a H-supplemented module by [3, Corollary 4.5(2)] and a δ-H-supplemented module. Hence M is E-δ-H-supplemented.

(2) Every H-supplemented module is E-δ-H-supplemented. The converse does not hold in general. Now let $F = \mathbb{Z}_2$, which is a field, and $S = \prod_{i=1}^{\infty} F_i$ where $F_i = F$ for each i. Let R be the subring of S generated by $\oplus_{i=1}^{\infty} F_i$ and 1_S. It is well-known that R is not a semiperfect ring which yields that R_R is not a H-supplemented module. By [12, Example 4.1], R is a δ-semiperfect ring. Now [4, Theorem 3.3] implies that R_R is δ-lifting and consequently R_R is δ-H-supplemented. Hence R_R is E-δ-H-supplemented.

Note if the image of every endomorphism of M is a direct summand of M, that module M is dual Rickart.

Theorem 2.7. *Let M be a module. Then the following statements are equivalent:*

(1) *M is dual Rickart;*

(2) *M is E-δ-H-supplemented and δ-noncosingular.*

In particular, if M is a non-δ-cosingular E-δ-H-supplemented module, it is dual Rickart.

Proof. (1) ⇒ (2) It is clear by definitions.
(2) ⇒ (1) Let M be $\mathcal{T}$-δ-noncosingular and E-δ-H-supplemented. Suppose that $f \in S$. Now there is a direct summand D of M such that $(Imf + D)/D \ll_\delta M/D$ and $(Imf + D)/Imf \ll_\delta M/Imf$. Consider the R-homomorphism $\lambda\colon M \to M/D$ defined by $\lambda(m) = f(m) + D$. Set $M = D \oplus D'$ for a submodule D' of M. So that there is an isomorphism $h\colon M/D \to D'$ induced by the decomposition $M = D \oplus D'$. Consider the homomorphism $joho\lambda\colon M \to M$ where $j\colon D' \to M$ is the inclusion map. Since $Im\lambda = (Imf + D)/D \ll_\delta M/D$, we can get $joho\lambda(M) = h((Imf+D)/D) \ll_\delta D' \subseteq M$. So $Im(joho\lambda) \ll_\delta M$. Being M, $\mathcal{T}$-δ-noncosingular implies that $joho\lambda = 0$. It follows that $(Imf + D)/D \subseteq Kerh$. Hence $(Imf + D)/D = D/D$.

Therefore, $Imf \subseteq D$. Since $D/Imf \ll_\delta M/Imf$ and $D/Imf + (D' + Imf)/Imf = M/Imf$, we conclude that $D' + Imf = M$. By modularity, $Imf = D$ is a direct summand of M. □

Remark 2.8. By the last result, every dual Rickart module is E-δ-H-supplemented, while the other side may not hold. Let M be a hollow module with at least an endomorphism f which is distinct from zero and id_M (for example the $\mathbb{Z}$-module $\mathbb{Z}_{p^n}$ where p is prime and $n > 1$). Then clearly, M is E-δ-H-supplemented, which is not dual Rickart.

The following indicates that the class of E-δ-H-supplemented modules properly contains the class of H-supplemented modules.

Example 2.9. Every injective module over a right hereditary ring is E-δ-H-supplemented by [1, Theorem 2.29]. Consider the $\mathbb{Z}$-module $M = \mathbb{Q}$. It is well-known that M is not supplemented; hence it is not H-supplemented while is a dual Rickart $\mathbb{Z}$-module. Therefore, every non-supplemented injective module over a right hereditary ring is E-δ-H-supplemented but not H-supplemented.

We shall deal with homomorphic images of E-δ-H-supplemented modules.

Proposition 2.10. *Let M be a E-δ-H-supplemented module and N a direct summand of M. Suppose that for every direct summand K of M, there exists a direct summand T/N of M/N such that $(K+T)/T \ll_\delta M/T$ and $(K+T)/(K+N) \ll_\delta M/(K+N)$. Then M/N is E-δ-H-supplemented.*

Proof. Let $M = N \oplus N'$, for some $N' \leq M$, and $f\colon M/N \to M/N$ be an endomorphism. Consider the natural epimorphism $\pi\colon M \to M/N$ defined by $\pi(x) = x + N$ and the isomorphism $h\colon M/N \to N'$ defined by $h(n' + N) = n'$ induced by the decomposition $M = N \oplus N'$. Therefore, $hof o\pi\colon M \to M$ is an endomorphism. Set $Imf = L/N$. It is easy to check that $Im(hof o\pi) = L \cap N'$. Since M is E-δ-H-supplemented, there exists a direct summand K of M such that $[(L\cap N') + K]/K \ll_\delta M/K$ and $[(L \cap N') + K]/(L \cap N') \ll_\delta M/(L \cap N')$. By assumption, there is a submodule T of M such that T/N is a direct summand of M/N such that $(K+T)/T \ll_\delta M/T$ and $(K+T)/(K+N) \ll_\delta M/(K+N)$. We shall prove that $\frac{L/N+T/N}{L/N} \ll_\delta \frac{M/N}{L/N}$ and $\frac{L/N+T/N}{T/N} \ll_\delta \frac{M/N}{T/N}$.

To verify the last assertions, we assume $(L+T)/L + X/L = M/L$ for a submodule X of M containing L such that M/X is singular. Then, $T + X = M$. Now, $(K+T)/(K+N) + (K+X)/(K+N) = M/(K+N)$. As M/X is singular, we can say $M/(K+X)$ is singular as a homomorphic image of M/X. Being $(K+T)/(K+N)$ a δ-small submodule of $M/(K+N)$, we conclude that $M = K + X$. Hence, $[(L\cap N') + X]/(L\cap N') + X/(L\cap N') = M/(L\cap N')$. Therefore, $M = X$ due to $[(L \cap N') + K]/(L \cap N') \ll_\delta M/(L \cap N')$. We turn to the second assertion. Suppose that $(L+T)/T+Y/T=M/T$ where Y is a submodule of M containing T such that M/Y is singular. Then $L+Y=M$. As L contains N, we have $N+(L\cap N')+Y=M$, which implies $(L\cap N')+Y=M$. It follows that $[(L\cap N')+K]/K+(Y+K)/K=M/K$. Since $[(L\cap N')+K]/K$ is a δ-small submodule of M/K and M/Y is a singular module, we conclude that $M=Y+K$. Now $(K+T)/T+Y/T=M/T$ causes $M=Y$, as required (note that $(K+T)/T \ll_\delta M/T$). □

Recall that a submodule N of M is said to be *fully invariant (projection invariant)* if for every endomorphism (idempotent endomorphism) f of M, we have $f(N)\subseteq N$. Let M be a module with a submodule N. The module M is a *(weak) duo* if every (direct summand) submodule of M is fully invariant.

Proposition 2.11. *Let M be a module and N a projection invariant (fully invariant) direct summand of M. If M is E-δ-H-supplemented, then M/N is E-δ-H-supplemented.*

Proof. Let D and D' be submodules of M such that $M=D\oplus D'$. By assumption, we have $N=(D\cap N)\oplus(D'\cap N)$. Then $(D+N)\cap(D'+N)=[D\oplus(D'\cap N)]\cap[(D\cap N)\oplus D']=(D\cap N)\oplus(D'\cap N)=N$. So $M/N=[(D+N)/N]\oplus[(D'+N)/N]$. So that for an arbitrary direct summand D of M, there exists $(D+N)/N$ that is a direct summand of M/N and $(D+D+N)/(D+N)\ll_\delta M/N$. The result follows from Proposition 2.10. □

Corollary 2.12. *Let M be a E-δ-H-supplemented weak duo module. Then every direct summand of M is E-δ-H-supplemented.*

As a direct consequence of the last proposition, we can say every direct summand of a duo (distributive) E-δ-H-supplemented module inherits the property.

Example 2.13. ([10, Example 3.9]) *Let I and J be two ideals of a commutative local ring R with maximal ideal m such that $I\subset J\subseteq m$ (e.g., R is a discrete valuation ring with maximal ideal m, $I=m^3$ and $J=m^2$). Every direct summand of M is H-supplemented by* [10, Proposition 2.1]. *Hence every direct summand of M is E-δ-H-supplemented.*

Theorem 2.14. *Let $M=M_1\oplus M_2$ be a distributive module. Then M is E-δ-H-supplemented module if and only if M_1 and M_2 are E-δ-H-supplemented.*

Proof. Let M_1 and M_2 be E-δ-H-supplemented and $f\in End_R(M)$. Let $f(M_i)$ be a submodule of M_i for $i=1,2$. Then, there is a direct summand D_i of M_i for $i=1,2$, such that $(Imf_i+D_i)/Imf_i \ll_\delta M_i/Imf_i$ and $(Imf_i+D_i)/D_i \ll_\delta M_i/D_i$. We shall prove that $(Imf+D)/Imf \ll_\delta M/Imf$ and $(Imf+D)/D \ll_\delta M/D$ where $D=D_1\oplus D_2$ which is a direct summand of M. Suppose that $(Imf+D)/Imf + X/Imf = M/Imf$ for a submodule X of M containing Imf with M/X singular. Then $D+X = M$. It follows that $D_1 + (X \cap M_1) = M_1$. Now $(Imf_1+D_1)/Imf_1+(X\cap M_1)/Imf_1 = M_1/Imf_1$ and $M_1/(X\cap M_1) \cong X+M_1/X \leq M/X$ is a singular module. Therefore, $X\cap M_1 = M_1$, which implies that M_1 is in X. Now consider the equality $D+X = M$. Then $D_2+(X\cap M_2) = M_2$. As $(Imf_2+D_2)+(X\cap M_2)/Imf_2 = M_2/Imf_2$ and $(Imf_2+D_2)/Imf_2 \ll_\delta M_2/Imf_2$ and also $M_2/X\cap M_2 \cong (X+M_2)/X \leq M/X$ is singular, we conclude that $X \cap M_2 = M_2$. So M_2 is in X, which implies that $X = M$. For the other δ-small case, let $(Imf+D)/D+T/D = M/D$ where $T/D \leq M/D$ and M/T is singular. Now $Imf+T = M$ and hence $Imf_1+(T\cap M_1) = M_1$. Being $(Imf_1+D_1)/D_1$ a δ-small submodule of M_1/D_1 combining with the fact that $M_1/(T\cap M_1)$ is singular and the last equality implies that $T \cap M_1 = M_1$ and therefore $M_1 \subseteq T$. By a same process, T will contain M_2. Hence $T = M$ as required. It follows now that M is E-δ-H-supplemented. The converse follows from Corollary 2.12. □

3. Acknowledgments

The authors wish to sincerely thank the referees for several useful comments.

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