THE EFFECT OF SINGULARITY ON A TYPE OF SUPPLEMENTED MODULES

ALI REZA MONIRI HAMZEKOLAEE* AND ALI VALINEJAD

ABSTRACT. Let $R$ be a ring, $M$ a right $R$-module, and $S = \text{End}_R(M)$ the ring of all $R$-Endomorphisms of $M$. We say that $M$ is Endomorphism $\delta$-$H$-supplemented (briefly, $E$-$\delta$-$H$-supplemented) provided that for every $\varphi \in S$, there exists a direct summand $D$ of $M$ such that $M = \text{Im}\varphi + X$ if and only if $M = D + X$ for every submodule $X$ of $M$ with $M/X$ singular. In this paper, we prove that a non-$\delta$-cosingular module $M$ is $E$-$\delta$-$H$-supplemented if and only if $M$ is dual Rickart. We also show that every direct summand of a weak duo $E$-$\delta$-$H$-supplemented module inherits the property.

1. Introduction

Throughout this paper, all rings are associative ring with identity, and all modules are unitary right $R$-modules. Let $M$ and $N$ be $R$-modules. Then by $S = \text{End}_R(M)$, we denote
the ring of all endomorphisms of $M$, and by $N \subseteq M$, we mean that $N$ is a submodule of $M$.

A submodule $N$ of $M$ is said to be small in $M$ if $N + K \neq M$ for any proper submodule $K$ of $M$, and we denote it by $N \ll M$. As a generalization, Zhou [12] introduced the concept of $\delta$-small submodules. A submodule $N$ of $M$ is called $\delta$-small in $M$ (denoted by $N \ll_{\delta} M$) if $M \neq N + K$ for any proper submodule $K$ of $M$ with $M/K$ singular. General properties and some useful characterizations of $\delta$-small submodules of a module were investigated in [12].

The notion of $H$-supplemented modules was introduced by Mohamed and Muller in [3]. A module $M$ was called $H$-supplemented if for every submodule $N$ of $M$ there exists a direct summand $D$ of $M$ such that $M = N + X$ if and only if $M = D + X$ for every submodule $X$ of $M$. Different definition’s style, unusual properties and being a generalization of lifting modules, all led many researchers to study and investigate $H$-supplemented modules further than what was investigated in [3]. Maybe the first serious effort has been made in [3]. He investigated some general properties of $H$-supplemented modules, such as homomorphic images and direct summands of these modules. The authors in reference [3], proposed some equivalent conditions for a module to be $H$-supplemented, which show that this class of modules is closely related to the concept of small submodules. They proved that a module $M$ is $H$-supplemented if and only if for every submodule $N$ of $M$ there is a direct summand $D$ of $M$ such that $(N + D)/N \ll M/N$ and $(N + D)/D \ll M/D$. In [3], Moniri and etc. investigated, the $E$-$H$-supplemented definition of module $M$, a homological approach of a $H$-supplemented modules. A module $M$ is called $E$-$H$-supplemented, if for every endomorphism $\varphi$ of $M$ there exists a direct summand $D$ of $M$ such that $Im \varphi + X = M$ if and only if $D + X = M$.

Let $M$ be a module over a ring $R$. Following [11], $M$ is called (non)cosingular if $(\overline{Z}(M) = M)$ $\overline{Z}(M) = 0$, where $\overline{Z}(M) = \bigcap \{Ker f \mid f : M \to U\}$, in which $U$ is an arbitrary small right $R$-module. The author in [8] considered the class of right $\delta$-small $R$-modules in the definition of $\overline{Z}(M)$, and defined $\overline{Z}_\delta(M)$ to be $\bigcap \{Ker g \mid g : M \to V\}$ where $V$ is a $\delta$-small module (i.e. there exists another module $U$ such that $V \ll_{\delta} U$). In [8], $M$ is said to be (non)-$\delta$-cosingular in case $(Z_\delta(M) = M)$ $\overline{Z}_\delta(M) = 0$. Hence for module $M$, we have $\overline{Z}_\delta(M) \subseteq \overline{Z}(M)$. Therefore, every cosingular right $R$-module is $\delta$-cosingular, and every non-$\delta$-cosingular module is noncosingular.

As a pioneer research on lifting modules, supplemented modules, $\oplus$-supplemented modules, and others concept of singularity. Zhou [12] made a different impression on works that made on supplemented modules and related concepts. The first person who worked on $\delta$-version was Koşan, that introduced $\delta$-lifting modules and $\delta$-supplemented modules and tried to investigate their natural properties [4]. According to [4], a module $M$ is said to be $\delta$-$H$-supplemented, if for every submodule $N$ of $M$ there is a direct summand $D$ of $M$ such that $M = N + X$ if and only if $M = D + X$, for every submodule $X$ of $M$ with $M/X$ singular.
Inspired by [7] and [6], in this manuscript, we are interested in studying $\delta$-$H$-supplemented modules via homomorphisms. Combining the two concepts $E$-$H$-supplemented modules and $\delta$-$H$-supplemented modules, we call a module $M$, an $E$-$\delta$-$H$-supplemented module if for every endomorphism $\varphi$ of $M$ there exists a direct summand $D$ of $M$ such that $\text{Im}\varphi + X = M$ if and only if $D + X = M$, for all submodules $X$ of $M$ with $M/X$ singular. We introduce some equivalent conditions for this definition impressing the close relation of $\delta$-$H$-supplemented modules to the concept of $\delta$-small submodules.

2. SINGULARITY AND ENDOMORPHISM $H$-SUPPLEMENTED MODULES

In this section, we introduce a new generalization of the class of $E$-$H$-supplemented modules and $\delta$-$H$-supplemented modules, namely Endomorphism $\delta$-$H$-supplemented modules. We work on factor modules, particularly direct summands of Endomorphism $\delta$-$H$-supplemented modules.

**Definition 2.1.** A module $M$ is called *Endomorphism $\delta$-$H$-supplemented* ($E$-$\delta$-$H$-supplemented, for short) if for every $f \in S$, there exists a direct summand $D$ of $M$ such that $\text{Im} f + X = M$ if and only if $D + X = M$ for every submodule $X$ of $M$ with $M/X$ singular.

Every $\delta$-$H$-supplemented module is $E$-$\delta$-$H$-supplemented. We shall present some conditions showing that the concept of $E$-$\delta$-$H$-supplemented modules is closely related to the concept $\delta$-small submodules.

**Theorem 2.2.** The following are equivalent for a module $M$:

1. $M$ is $E$-$H$-supplemented;
2. For every $f \in S$, there exists a direct summand $D$ of $M$ with $\frac{\text{Im} f + D}{\text{Im} f} \ll_\delta \frac{M}{D}$ and $\frac{\text{Im} f + D}{\text{Im} f} \ll_\delta \frac{M}{M_f}$;
3. For every $f \in S$, there exist a direct summand $D$ and a submodule $N$ of $M$ with $\text{Im} f \subseteq N$ and $D \subseteq N$ such that $\frac{N}{D} \ll_\delta \frac{M}{D}$ and $\frac{N}{\text{Im} f} \ll_\delta \frac{M}{\text{Im} f}$.

**Proof.** (1) ⇒ (2) Let $f \in S$. By (1), there exists a direct summand $D$ of $M$ such that $\text{Im} f + X = M$ if and only if $D + X = M$ for every submodule $X$ of $M$ with $M/X$ singular. Let $(\text{Im} f + D)/\text{Im} f + X/\text{Im} f = M/\text{Im} f$ for a submodule $X$ of $M$ containing $\text{Im} f$ such that $M/X$ is singular. It follows that $D + X = M$. Now, (1) implies $\text{Im} f + X = M$. Therefore, $X = M$, showing that $(\text{Im} f + D)/\text{Im} f \ll_\delta M/\text{Im} f$. For the second one, suppose that $(\text{Im} f + D)/D + Y/D = M/D$ where $Y$ is a submodule of $M$, which contains $D$ with $M/Y$ singular. Then $\text{Im} f + Y = M$ combining with (1) implies $M = Y$, as required.

(2) ⇒ (3) Set $N = \text{Im} f + D$. 


(3) ⇒ (1) Let $f \in S$. Then by assumption there is a submodule $N$ and a direct summand $D$ of $M$ such that $N/D \leq M/D$ and $N/\text{Im} f \leq M/\text{Im} f$. Suppose that $\text{Im} f + X = M$ for a submodule $X$ of $M$ with $M/X$ singular. Then $M = N + X$. Now, $N/D + (X + D)/D = M/D$. As $M/X$ is singular, we conclude that $M/(X + D)$ is singular. Being $N/D$ a $\delta$-small submodule of $M/D$ implies $M = X + D$. For the converse, let $M = Y + D$ for a submodule $Y$ of $M$ with $M/Y$ singular. Then $M = N + Y$ which implies $N/\text{Im} f + (Y + \text{Im} f)/\text{Im} f = M/\text{Im} f$. Note also that $M/(Y + \text{Im} f)$ is singular and $M/Y$. Therefore, $M = \text{Im} f + Y$ is desired. □

We present some assumptions, which under two concepts $E$-$H$-supplemented modules and $E$-$\delta$-$H$-supplemented modules are coincide.

**Proposition 2.3.** Let $M$ be a module. In each of the following cases, $M$ is $E$-$H$-supplemented if and only if $M$ is $E$-$\delta$-$H$-supplemented.

1. $M$ is a singular module.
2. $M$ has no simple projective submodule.

**Proof.** (1) This follows from the fact that every homomorphic image of a singular module is singular. In fact, every $\delta$-small submodule of a singular module is a small submodule of that module.

(2) Let $M$ be a $E$-$\delta$-$H$-supplemented module with simple projective submodule. Suppose that $f$ is an endomorphism of $M$. Then there is a direct summand $D$ of $M$ such that $(\text{Im} f + D)/\text{Im} f \leq M/\text{Im} f$ and $(\text{Im} f + D)/D \leq M/D$. Let $(\text{Im} f + D)/\text{Im} f + T/\text{Im} f = M/\text{Im} f$ for a submodule $T/\text{Im} f$ of $M/\text{Im} f$. Then, by [12, Lemma 1.2], $(\text{Im} f + D)/\text{Im} f$ contains a semisimple projective direct summand $Y/\text{Im} f$ of $M/\text{Im} f$ such that $Y/\text{Im} f \oplus T/\text{Im} f = M/\text{Im} f$. So, there is a submodule $N'$ of $Y$ such that $Y = \text{Im} f \oplus N'$, since $Y/\text{Im} f$ is projective. It follows that $N'$ contains a simple projective submodule. Now, $Y = \text{Im} f$, and consequently $T/\text{Im} f = M/\text{Im} f$ implies that $(\text{Im} f + D)/\text{Im} f \leq M/\text{Im} f$. Applying the same argument, we can prove $(\text{Im} f + D)/D \leq M/D$. Therefore, $M$ is $H$-supplemented. □

**Corollary 2.4.** Let $R$ be a ring such that every simple right $R$-module is singular (consider the ring $\mathbb{Z}$). Then a right $R$-module $M$ is $E$-$H$-supplemented if and only if $M$ is $E$-$\delta$-$H$-supplemented. Particularly, an $\mathbb{Z}$-module $M$ is $E$-$H$-supplemented if and only if $M$ is $E$-$\delta$-$H$-supplemented.

**Proposition 2.5.** Let $M$ be an indecomposable module. Then $M$ is $E$-$\delta$-$H$-supplemented if and only if the image of each endomorphism of $M$ is $\delta$-small in $M$ or every endomorphism of $M$ is an epimorphism.
Proof. Let \( M \) be an indecomposable \( E\)-\( \delta \)-\( H \)-supplemented module. Consider a nonzero endomorphism \( f \) of \( M \). Then there is a direct summand \( D \) of \( M \) such that \( (\text{Im} f + D)/\text{Im} f \ll_\delta M/\text{Im} f \) and \( (\text{Im} f + D)/D \ll_\delta M/D \). Suppose \( D = 0 \). Then clearly, \( \text{Im} f \ll_\delta M \). Otherwise, \( D = M \) implies \( M/\text{Im} f \ll_\delta M/\text{Im} f \). Now \[ \text{Lemma 1.2} \] yields that \( M/\text{Im} f \) is projective and semisimple (it is sufficient in \[ \text{Lemma 1.2} \] that we set \( M = M/\text{Im} f \), \( N = M/\text{Im} f \) and \( X = 0 \)). It follows now that \( \text{Im} f \) must be a direct summand of \( M \). Being \( M \) indecomposable implies \( \text{Im} f = 0 \), a contradiction. The converse is straightforward to check. \( \square \)

We next present some examples of \( E\)-\( \delta \)-\( H \)-supplemented modules.

**Example 2.6.** (1) Suppose that \( M_1 \) is a \( H \)-supplemented module with a unique composition series \( M_1 \supset U \supset V \supset 0 \) (we may choose the \( \mathbb{Z} \)-module \( M_1 = \mathbb{Z}_8 \)). Now, let \( M = M_1 \oplus M_1/U \oplus U/V \oplus V/0 \). Then \( M \) is a \( H \)-supplemented module by \[ \text{Corollary 4.5}(2) \] and a \( \delta \)-\( H \)-supplemented module. Hence \( M \) is \( E\)-\( \delta \)-\( H \)-supplemented.

(2) Every \( H \)-supplemented module is \( E\)-\( \delta \)-\( H \)-supplemented. The converse does not hold in general. Now let \( F = \mathbb{Z}_2 \), which is a field, and \( S = \prod_{i=1}^{\infty} F_i \) where \( F_i = F \) for each \( i \). Let \( R \) be the subring of \( S \) generated by \( \bigoplus_{i=1}^{\infty} F_i \) and \( 1_S \). It is well-known that \( R \) is not a semiperfect ring which yields that \( R_R \) is not a \( H \)-supplemented module. By \[ \text{Example 4.1} \], \( R \) is a \( \delta \)-semiperfect ring. Now \[ \text{Theorem 3.3} \] implies that \( R_R \) is \( \delta \)-lifting and consequently \( R_R \) is \( \delta \)-\( H \)-supplemented. Hence \( R_R \) is \( E\)-\( \delta \)-\( H \)-supplemented.

Note if the image of every endomorphism of \( M \) is a direct summand of \( M \), that module \( M \) is dual Rickart.

**Theorem 2.7.** Let \( M \) be a module. Then the following statements are equivalent:

(1) \( M \) is dual Rickart;

(2) \( M \) is \( E\)-\( \delta \)-\( H \)-supplemented and \( \delta \)-noncosingular.

In particular, if \( M \) is a non-\( \delta \)-cosingular \( E\)-\( \delta \)-\( H \)-supplemented module, it is dual Rickart.

**Proof.** (1) \( \Rightarrow \) (2) It is clear by definitions.

(2) \( \Rightarrow \) (1) Let \( M \) be \( \mathcal{T}\)-\( \delta \)-noncosingular and \( E\)-\( \delta \)-\( H \)-supplemented. Suppose that \( f \in S \). Now there is a direct summand \( D \) of \( M \) such that \( (\text{Im} f + D)/D \ll_\delta M/D \) and \( (\text{Im} f + D)/\text{Im} f \ll_\delta M/\text{Im} f \). Consider the \( R \)-homomorphism \( \lambda : M \to M/D \) defined by \( \lambda(m) = f(m) + D \). Set \( M = D \oplus D' \) for a submodule \( D' \) of \( M \). So that there is an isomorphism \( h : M/D \to D' \) induced by the decomposition \( M = D \oplus D' \). Consider the homomorphism \( \text{joho}\lambda : M \to M \) where \( j : D' \to M \) is the inclusion map. Since \( \text{Im}\lambda = (\text{Im} f + D)/D \ll_\delta M/D \), we can get \( \text{joho}\lambda(M) = h((\text{Im} f + D)/D) \ll_\delta D' \subseteq M \). So \( \text{Im}(\text{joho}\lambda) \ll_\delta M \). Being \( M \), \( \mathcal{T}\)-\( \delta \)-noncosingular implies that \( \text{joho}\lambda = 0 \). It follows that \( (\text{Im} f + D)/D \subseteq \text{Ker} h \). Hence \( (\text{Im} f + D)/D = D/D \).
Therefore, $\text{Im}f \subseteq D$. Since $D/\text{Im}f \ll_\delta M/\text{Im}f$ and $D/\text{Im}f + (D' + \text{Im}f)/\text{Im}f = M/\text{Im}f$, we conclude that $D' + \text{Im}f = M$. By modularity, $\text{Im}f = D$ is a direct summand of $M$. \(\square\)

**Remark 2.8.** By the last result, every dual Rickart module is $E-\delta$-$H$-supplemented, while the other side may not hold. Let $M$ be a hollow module with at least an endomorphism $f$ which is distinct from zero and $id_M$ (for example the $\mathbb{Z}$-module $\mathbb{Z}_{p^n}$ where $p$ is prime and $n > 1$). Then clearly, $M$ is $E-\delta$-$H$-supplemented, which is not dual Rickart.

The following indicates that the class of $E-\delta$-$H$-supplemented modules properly contains the class of $H$-supplemented modules.

**Example 2.9.** Every injective module over a right hereditary ring is $E-\delta$-$H$-supplemented by [Theorem 2.29]. Consider the $\mathbb{Z}$-module $M = \mathbb{Q}$. It is well-known that $M$ is not supplemented; hence it is not $H$-supplemented while is a dual Rickart $\mathbb{Z}$-module. Therefore, every non-supplemented injective module over a right hereditary ring is $E-\delta$-$H$-supplemented but not $H$-supplemented.

We shall deal with homomorphic images of $E-\delta$-$H$-supplemented modules.

**Proposition 2.10.** Let $M$ be a $E-\delta$-$H$-supplemented module and $N$ a direct summand of $M$. Suppose that for every direct summand $K$ of $M$, there exists a direct summand $T/N$ of $M/N$ such that $(K + T)/T \ll_\delta M/T$ and $(K + T)/(K + N) \ll_\delta M/(K + N)$. Then $M/N$ is $E-\delta$-$H$-supplemented.

**Proof.** Let $M = N \oplus N'$, for some $N' \leq M$, and $f: M/N \to M/N$ be an endomorphism. Consider the natural epimorphism $\pi: M \to M/N$ defined by $\pi(x) = x + N$ and the isomorphism $h: M/N \to N'$ defined by $h(n' + N) = n'$ induced by the decomposition $M = N \oplus N'$. Therefore, $\text{hofo} \pi: M \to M$ is an endomorphism. Set $\text{Im}f = L/N$. It is easy to check that $\text{Im}(\text{hofo} \pi) = L \cap N'$. Since $M$ is $E-\delta$-$H$-supplemented, there exists a direct summand $K$ of $M$ such that $[(L \cap N') + K]/K \ll_\delta M/K$ and $[(L \cap N') + K]/(L \cap N') \ll_\delta M/(L \cap N')$. By assumption, there is a submodule $T$ of $M$ such that $T/N$ is a direct summand of $M/N$ such that $(K + T)/T \ll_\delta M/T$ and $(K + T)/(K + N) \ll_\delta M/(K + N)$. We shall prove that $\frac{L/N + T/N}{L/N} \ll_\delta \frac{M/N}{L/N}$ and $\frac{L/N + T/N}{T/N} \ll_\delta \frac{M/N}{T/N}$.

To verify the last assertions, we assume $(L + T)/L + X/L = M/L$ for a submodule $X$ of $M$ containing $L$ such that $M/X$ is singular. Then, $T + X = M$. Now, $(K + T)/(K + N) + (K + X)/(K + N) = M/(K + N)$. As $M/X$ is singular, we can say $M/(K + X)$ is singular as a homomorphic image of $M/X$. Being $(K + T)/(K + N)$ a $\delta$-small submodule of $M/(K + N)$, we conclude that $M = K + X$. Hence, $[(L \cap N') + X]/(L \cap N') + X/(L \cap N') = M/(L \cap N')$. Therefore, $M = X$ due to $[(L \cap N') + K]/(L \cap N') \ll_\delta M/(L \cap N')$. We turn to the second...
assertion. Suppose that \((L + T)/T + Y/T = M/T\) where \(Y\) is a submodule of \(M\) containing \(T\) such that \(M/Y\) is singular. Then \(L + Y = M\). As \(L\) contains \(N\), we have \(N + (L \cap N') + Y = M\), which implies \((L \cap N') + Y = M\). It follows that \(\[(L \cap N') + K]/K + (Y + K)/K = M/K\). Since \(\[(L \cap N') + K]/K\) is a \(\delta\)-small submodule of \(M/K\) and \(M/Y\) is a singular module, we conclude that \(M = Y + K\). Now \((K + T)/T + Y/T = M/T\) causes \(M = Y\), as required (note that \((K + T)/T \ll_\delta M/T\). \(\square\)

Recall that a submodule \(N\) of \(M\) is said to be fully invariant (projection invariant) if for every endomorphism (idempotent endomorphism) \(f\) of \(M\), we have \(f(N) \subseteq N\). Let \(M\) be a module with a submodule \(N\). The module \(M\) is a (weak) duo if every (direct summand) submodule of \(M\) is fully invariant.

**Proposition 2.11.** Let \(M\) be a module and \(N\) a projection invariant (fully invariant) direct summand of \(M\). If \(M\) is \(E\)-\(\delta\)-\(H\)-supplemented, then \(M/N\) is \(E\)-\(\delta\)-\(H\)-supplemented.

**Proof.** Let \(D\) and \(D'\) be submodules of \(M\) such that \(M = D \oplus D'\). By assumption, we have \(N = (D \cap N) \oplus (D' \cap N)\). Then \((D + N) \cap (D' + N) = [D \oplus (D' \cap N)] \cap [(D \cap N) \oplus D'] = (D \cap N) \oplus (D' \cap N) = N\). So \(M/N = [(D + N)/N] \oplus [(D' + N)/N]\). So that for an arbitrary direct summand \(D\) of \(M\), there exists \((D + N)/N\) that is a direct summand of \(M/N\) and \((D + D + N)/(D + N) \ll_\delta M/N\). The result follows from Proposition 2.10. \(\square\)

**Corollary 2.12.** Let \(M\) be a \(E\)-\(\delta\)-\(H\)-supplemented weak duo module. Then every direct summand of \(M\) is \(E\)-\(\delta\)-\(H\)-supplemented.

As a direct consequence of the last proposition, we can say every direct summand of a duo (distributive) \(E\)-\(\delta\)-\(H\)-supplemented module inherits the property.

**Example 2.13.** ([10], Example 3.9) Let \(I\) and \(J\) be two ideals of a commutative local ring \(R\) with maximal ideal \(m\) such that \(I \subset J \subset m\) (e.g., \(R\) is a discrete valuation ring with maximal ideal \(m\), \(I = m^3\) and \(J = m^2\)). Every direct summand of \(M\) is \(H\)-supplemented by [10, Proposition 2.1]. Hence every direct summand of \(M\) is \(E\)-\(\delta\)-\(H\)-supplemented.

**Theorem 2.14.** Let \(M = M_1 \oplus M_2\) be a distributive module. Then \(M\) is \(E\)-\(\delta\)-\(H\)-supplemented module if and only if \(M_1\) and \(M_2\) are \(E\)-\(\delta\)-\(H\)-supplemented.

**Proof.** Let \(M_1\) and \(M_2\) be \(E\)-\(\delta\)-\(H\)-supplemented and \(f \in End_R(M)\). Let \(f(M_i)\) be a submodule of \(M_i\) for \(i = 1, 2\). Then, there is a direct summand \(D_i\) of \(M_i\) for \(i = 1, 2\), such that \((Im f_i + D_i)/Im f_i \ll_\delta M_i/Im f_i\) and \((Im f_i + D_i)/D_i \ll_\delta M_i/D_i\). We shall prove that \((Im f + D)/Im f \ll_\delta M/Im f\) and \((Im f + D)/D \ll_\delta M/D\) where \(D = D_1 \oplus D_2\) which is a direct
summand of $M$. Suppose that $(Imf + D)/Imf + X/Imf = M/Imf$ for a submodule $X$ of $M$ containing $Imf$ with $M/X$ singular. Then $D + X = M$. It follows that $D_1 + (X \cap M_1) = M_1$. Now $(Imf_1 + D_1)/Imf_1 + (X \cap M_1)/Imf_1 = M_1/Imf_1$ and $M_1/(X \cap M_1) \cong X + M_1/X \leq M/X$ is a singular module. Therefore, $X \cap M_1 = M_1$, which implies that $M_1$ is in $X$. Now consider the equality $D + X = M$. Then $D_2 + (X \cap M_2) = M_2$. As $(Imf_2 + D_2)/(X \cap M_2)/Imf_2 = M_2/Imf_2$ and $(Imf_2 + D_2)/Imf_2 \ll_\delta M_2/Imf_2$ and also $M_2/X \cap M_2 \cong (X + M_2)/X \leq M/X$ is singular, we conclude that $X \cap M_2 = M_2$. So $M_2$ is in $X$, which implies that $X = M$. For the other $\delta$-small case, let $(Imf + D)/D + T/D = M/D$ where $T/D \leq M/D$ and $M/T$ is singular. Now $Imf + T = M$ and hence $Imf_1 + (T \cap M_1) = M_1$. Being $(Imf_1 + D_1)/D_1$ a $\delta$-small submodule of $M_1/D_1$ combining with the fact that $M_1/(T \cap M_1)$ is singular and the last equality implies that $T \cap M_1 = M_1$ and therefore $M_1 \subseteq T$. By a same process, $T$ will contain $M_2$. Hence $T = M$ as required. It follows now that $M$ is $E-\delta$-$H$-supplemented. The converse follows from Corollary 2.12. 

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References


**Ali Reza Moniri Hamzekolae**
Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
a.monirih@umz.ac.ir

**Ali Valinejad**
Department of Computer Sciences, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
valinejad@umz.ac.ir