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Research Paper

COMMUTATIVE TRUE-FALSE IDEALS IN BCI/BCK-ALGEBRAS

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Abstract. The notion of a (limited) commutative $T\&F$-ideal in BCK-algebras and BCI-algebras is introduced, and their properties are investigated. A relationship between a $T\&F$-ideal and a commutative $T\&F$-ideal in BCK-algebras and BCI-algebras is established, and examples to show that any $T\&F$-ideal may not be commutative are given. Proper conditions for a $T\&F$-ideal to be commutative are provided. Using a commutative ideal of a BCK-algebra and a BCI-algebra, a commutative $T\&F$-ideal is established. The closed $T\&F$-ideal in a BCI-algebra is introduced, and a condition for a closed $T\&F$-ideal to be commutative is discussed. Characterization of a commutative $T\&F$-ideal in a BCI-algebra is considered.

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1. Introduction

Fuzzy sets in mathematics are somewhat similar to sets in which elements have a degree of membership. The fuzzy set is an extension of the classical concept of the set and was introduced by Zadeh in 1965 [23]. The idea of fuzzy set is welcome because it handles uncertainty and vagueness which cantorian set could not address. The interval value fuzzy set [6] is an extension of the fuzzy set in which each element in the universe is assigned a closed sub-interval of a unit interval close to an unknown degree of membership. However in reality, it may not always be true that the degree of non-membership of an element in a fuzzy set is equal to 1 minus the membership degree because there may be some hesitation degree. Therefore, Attanassov [1] introduced a generalization of fuzzy sets and his called intuitionistic fuzzy sets (IFS) which incorporate the degree of hesitation called hesitation margin (and is defined as 1 minus the sum of membership and non-membership degrees respectively). The vague set [5] is an extension of fuzzy sets and regarded as a special case of context-dependent fuzzy set which has the ability to overcome the problems faced when using fuzzy sets by providing us with an interval-based membership which clearly separates the evidence for and against an element. Later on, Jun et al. [11] gave the idea of a cubic set and it was characterized by an interval valued fuzzy set and a fuzzy set, which is a more general tool to capture uncertainty and vagueness, since a fuzzy set deals with single-value membership while an interval valued fuzzy set ranges the membership in the form of intervals. Neutrosophic logic was introduced by Florentin Smarandache [22] in 1995. It is a logic in which each proposition is estimated to have a degree of truth (T), a degree of indeterminacy (I) and a degree of falsity (F). Neutrosophic sets are indeed more general than IFS as there are no constraints between the degree of truth, degree of indeterminacy and degree of falsity. All these degrees can individually vary within $[0, 1]$. In 2018, Mohseni et al. [18] introduced the notion of MBJ-neutrosophic sets which is another generalization of neutrosophic sets. They used the interval valued fuzzy set as the indeterminate membership function because they claimed that an interval-valued fuzzy set can be better-shown hesitancy as an indeterminate membership function and also, an interval valued fuzzy set is a generalization of a fuzzy set. Also, they studied it on some logical algebras; [21, 12, 17]. Later on, Borzooei et al. [3] introduced the concept of true-false sets (TFS) and claimed that a True-False set is a generalization of fuzzy sets, an intuitionistic fuzzy set, vague sets, interval-valued fuzzy sets, cubic sets, neutrosophic sets, an MBJ-neutrosophic set and etc. Mohseni et al. [19] constructed True-False structures to study the basic properties and applied them to groups and BCK/BCI-algebras at the same time. Using this structure, Mohseni et al. [20] discussed ideal theory in BCK/BCI-algebras. As algebraic structures in universal algebra, BCI-algebras and BCK-algebras were introduced by Imai, Iséki and Tanaka in 1966; [9, 10], and they described fragments of the propositional calculus involving implication known as BCK and BCI logics. These BCI-algebras and BCK-algebras are studied in various ways by many researchers; [2, 4]. Meng studied commutative ideals in a BCK-algebra [13] and commutative ideals in a BCI-algebra [14].

The purpose of this paper is to study commutative ideals in a BCK-algebra and a BCI-algebra by applying the True-False structure. We introduce the notion of (limited) commutative $T\&F$-ideals in BCK-algebras and BCI-algebras and investigate their properties. We establish a relationship between a $T\&F$-ideal and a commutative $T\&F$-ideal in BCK-algebras and BCI-algebras and provide examples to show that any $T\&F$-ideal may not be a commutative $T\&F$-ideal. We give proper conditions so that a $T\&F$-ideal can be a commutative $T\&F$-ideal. Using a commutative ideal of a BCK-algebra and a BCI-algebra, we establish a commutative $T\&F$-ideal. We introduce the closed $T\&F$-ideal in a BCI-algebra, and provide a condition for a closed $T\&F$-ideal to be commutative. We consider the characterization of a commutative $T\&F$-ideal in a BCI-algebra.

2. Preliminaries

2.1. **Basic concepts about BCK/BCI-algebras.** A *BCI-algebra* is an algebraic structure $(P;*,\theta)$ which satisfies:

(I) $(\forall d,j,r\in P)\ (((d*j)*(d*r))*(r*j)=\theta)$,

(II) $(\forall d,j\in P)\ ((d*(d*j))*j=\theta)$,

(III) $(\forall d\in P)\ (d*d=\theta)$,

(IV) $(\forall d,j\in P)\ (d*j=\theta,\ j*d=\theta\ \Rightarrow\ d=j)$.

If a BCI-algebra P satisfies the following identity:

(V) $(\forall \iota\in P)\ (\theta*\iota=\theta)$,

then P is called a *BCK-algebra.* An order relation "$\leq$" on a BCK/BCI-algebra P is given as follows:

$$(\forall d,j\in P)(d\leq j\ \Leftrightarrow\ d*j=\theta). \tag{1}$$

Every BCK/BCI-algebra P satisfies:

$$(\forall \iota\in P)(\iota*\theta=\iota), \tag{2}$$

$$(\forall \iota,\zeta,\varrho\in P)(\iota\leq\zeta\ \Rightarrow\ \iota*\varrho\leq\zeta*\varrho,\ \varrho*\zeta\leq\varrho*\iota), \tag{3}$$

$$(\forall \iota,\zeta,\varrho\in P)((\iota*\zeta)*\varrho=(\iota*\varrho)*\zeta). \tag{4}$$

Every BCI-algebra P satisfies:

$$(\forall \iota,\zeta\in P)(\iota*(\iota*(\iota*\zeta))=\iota*\zeta), \tag{5}$$

$$(\forall \iota,\zeta\in P)(\theta*(\iota*\zeta)=(\theta*\iota)*(\theta*\zeta)). \tag{6}$$

A BCK-algebra P is said to be *commutative* (see [15]) if $d*(d*j) = j*(j*d)$ for all $d, j \in P$.

A BCI-algebra P is said to be *commutative* (see [16]) if it satisfies:

$$(\forall d, j \in P)\,(d \leq j \;\Rightarrow\; d = j * (j * d))\,. \tag{7}$$

A subset L of a BCK/BCI-algebra P is called a *subalgebra* of P if $\iota * \zeta \in L$ for all $\iota, \zeta \in L$. A subset L of a BCK/BCI-algebra P is called an *ideal* of P if it satisfies:

$$\theta \in L, \tag{8}$$

$$(\forall \iota, \zeta \in P)(\iota * \zeta \in L,\, \zeta \in L \;\Rightarrow\; \iota \in L). \tag{9}$$

An ideal L of a BCI-algebra P is said to be *closed*; [7], if it satisfies:

$$(\forall d \in P)\,(d \in L \;\Rightarrow\; \theta * d \in L)\,. \tag{10}$$

A subset L of a BCK-algebra P is called a *commutative ideal* of P; [13], if it satisfies (8) and

$$(\forall \iota, \zeta, \varrho \in P)((\iota * \zeta) * \varrho \in L,\, \varrho \in L \;\Rightarrow\; \iota * (\zeta * (\zeta * \iota)) \in L). \tag{11}$$

A subset L of a BCI-algebra P is called a *commutative ideal* of P (see [14]) if it satisfies (8) and

$$(\iota * \zeta) * \varrho \in L,\, \varrho \in L \;\Rightarrow\; \iota * ((\zeta * (\zeta * \iota)) * (\theta * (\theta * (\iota * \zeta)))) \in L, \tag{12}$$

for all $\iota, \zeta, \varrho \in P$.

For more information on BCI-algebra and BCK-algebra, please refer to the books [8] and [15].

2.2. **Basic concepts about True-False structures.** Let W be a universal set. A *True-False structure* (briefly, *$T\&F$-structure*) over W [19] is defined to be a pair $(W, \mathcal{B})$ with $\mathcal{B} := (\varphi_A,\ \widetilde{\varphi}_A,\ \partial_A,\ \widetilde{\partial}_A)$ given by the following function:

$$\begin{aligned} &\mathcal{B} : W \to [0,1] \times int[0,1] \times [0,1] \times int[0,1], \\ &\qquad \iota \mapsto (\varphi_A(\iota), \widetilde{\varphi}_A(\iota), \partial_A(\iota), \widetilde{\partial}_A(\iota)). \end{aligned} \tag{13}$$

A $T\&F$-structure $(W, \mathcal{B})$ over W with $\mathcal{B} := (\varphi_A,\ \widetilde{\varphi}_A,\ \partial_A,\ \widetilde{\partial}_A)$ is said to be *limited*; [19], if $\varphi_A(\iota) + \partial_A(\iota) \leq 1$ and $\sup \widetilde{\varphi}_A(\iota) + \sup \widetilde{\partial}_A(\iota) \leq 1$ for all $\iota \in W$.

Given a (limited) $T\&F$-structure $(W, \mathcal{B})$ over W with $\mathcal{B} := (\varphi_A, \widetilde{\varphi}_A, \partial_A, \widetilde{\partial}_A)$, consider the sets

$$U(\varphi_A;\alpha) := \{\iota \in W \mid \varphi_A(\iota) \geq \alpha\}, \qquad U(\widetilde{\varphi}_A;\widetilde{t}) := \{\iota \in W \mid \widetilde{\varphi}_A(\iota) \succcurlyeq \widetilde{t}\},$$
$$L(\partial_A;\beta) := \{\iota \in W \mid \partial_A(\iota) \leq \beta\}, \qquad L(\widetilde{\partial}_A;\widetilde{s}) := \{\iota \in W \mid \widetilde{\partial}_A(\iota) \preccurlyeq \widetilde{s}\},$$
$$\mathcal{L}_{\mathcal{B}}(\alpha,\widetilde{t},\beta,\widetilde{s}) := U(\varphi_A;\alpha) \cap U(\widetilde{\varphi}_A;\widetilde{t}) \cap L(\partial_A;\beta) \cap L(\widetilde{\partial}_A;\widetilde{s}),$$

where $\alpha, \beta \in [0,1]$ and $\widetilde{t} = [t^-, t^+], \widetilde{s} = [s^-, s^+] \in int([0,1])$. We say that $\mathcal{L}_{\mathcal{B}}(\alpha,\widetilde{t},\beta,\widetilde{s})$ is a $T\&F$-*level set* of $\mathcal{B}$ over W.

Given a $T\&F$-structure $(P, \mathcal{B})$ over a set P with $\mathcal{B} := (\varphi_A, \widetilde{\varphi}_A, \partial_A, \widetilde{\partial}_A)$, we consider the following sets:

$$\Omega_T^{\varphi} := \{(\iota,\zeta) \in P \times P : \varphi_A(\iota) \geq \varphi_A(\zeta),\, \widetilde{\varphi}_A(\iota) \succcurlyeq \widetilde{\varphi}_A(\zeta)\}, \tag{14}$$

$$\Omega_F^{\partial} := \{(\iota,\zeta) \in P \times P : \partial_A(\iota) \leq \partial_A(\zeta),\, \widetilde{\partial}_A(\iota) \preccurlyeq \widetilde{\partial}_A(\zeta)\}, \tag{15}$$

$$\Omega^{\varphi}_{T(\min,\mathrm{rmin})} := \left\{ \tfrac{\iota}{(\zeta,\varrho)} \in \tfrac{P}{P\times P} : \begin{array}{l} \varphi_A(\iota) \geq \min\{\varphi_A(\zeta), \varphi_A(\varrho)\} \\ \widetilde{\varphi}_A(\iota) \succcurlyeq \mathrm{rmin}\{\widetilde{\varphi}_A(\zeta), \widetilde{\varphi}_A(\varrho)\} \end{array} \right\}, \tag{16}$$

and

$$\Omega^{\partial}_{F(\max,\mathrm{rmax})} := \left\{ \tfrac{\iota}{(\zeta,\varrho)} \in \tfrac{P}{P\times P} : \begin{array}{l} \partial_A(\iota) \leq \max\{\partial_A(\zeta), \partial_A(\varrho)\} \\ \widetilde{\partial}_A(\iota) \preccurlyeq \mathrm{rmax}\{\widetilde{\partial}_A(\zeta), \widetilde{\partial}_A(\varrho)\} \end{array} \right\}. \tag{17}$$

It is clear that

$$(\forall \iota,\zeta \in P)\left((\iota,\zeta) \in \Omega_T^{\varphi} \;\Leftrightarrow\; \tfrac{\iota}{(\zeta,\zeta)} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})}\right), \tag{18}$$

$$(\forall \iota,\zeta,\varrho \in P)\left((\iota,\zeta) \in \Omega_T^{\varphi},\, (\zeta,\varrho) \in \Omega_T^{\varphi} \;\Rightarrow\; (\iota,\varrho) \in \Omega_T^{\varphi}\right), \tag{19}$$

$$(\forall \iota,\zeta,\varrho \in P)\left((\iota,\zeta) \in \Omega_F^{\partial},\, (\zeta,\varrho) \in \Omega_F^{\partial} \;\Rightarrow\; (\iota,\varrho) \in \Omega_F^{\partial}\right), \tag{20}$$

$$(\forall \iota,\zeta \in P)\left((\iota,\zeta) \in \Omega_F^{\partial} \;\Leftrightarrow\; \tfrac{\iota}{(\zeta,\zeta)} \in \Omega^{\partial}_{F(\max,\mathrm{rmax})}\right), \tag{21}$$

$$(\forall \iota,\zeta,\varrho \in P)\left(\tfrac{\iota}{(\zeta,\varrho)} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \;\Leftrightarrow\; \tfrac{\iota}{(\varrho,\zeta)} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})}\right), \tag{22}$$

$$(\forall \iota,\zeta,\varrho \in P)\left(\tfrac{\iota}{(\zeta,\varrho)} \in \Omega^{\partial}_{F(\max,\mathrm{rmax})} \;\Leftrightarrow\; \tfrac{\iota}{(\varrho,\zeta)} \in \Omega^{\partial}_{F(\max,\mathrm{rmax})}\right). \tag{23}$$

Proposition 2.1. [20] *Let $(P, \mathcal{B})$ be a $T\&F$-structure over a set P with $\mathcal{B} := (\varphi_A, \widetilde{\varphi}_A, \partial_A, \widetilde{\partial}_A)$. For any $\iota, \iota, \zeta, \varrho \in P$, we have*

$$\begin{gathered} \tfrac{\iota}{(\iota,\zeta)} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})},\, (\zeta,\varrho) \in \Omega_T^{\varphi} \;\Rightarrow\; \tfrac{\iota}{(\iota,\varrho)} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})}, \\ \tfrac{\iota}{(\iota,\zeta)} \in \Omega^{\partial}_{F(\max,\mathrm{rmax})},\, (\zeta,\varrho) \in \Omega_F^{\partial} \;\Rightarrow\; \tfrac{\iota}{(\iota,\varrho)} \in \Omega^{\partial}_{F(\max,\mathrm{rmax})}. \end{gathered} \tag{24}$$

Definition 2.2. [19] A $T\&F$-structure $(P, \mathcal{B})$ over a BCK/BCI-algebra P with $\mathcal{B} := (\varphi_A,\, \widetilde{\varphi}_A,\, \partial_A,\, \widetilde{\partial}_A)$ is called a *$T\&F$-subalgebra* of P if the following assertion is valid.

$$(\forall \iota, \zeta \in P)\left(\tfrac{\iota*\zeta}{(\iota,\zeta)} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})}\right). \tag{25}$$

If a $T\&F$-subalgebra is limited, then we say that it is a *limited $T\&F$-subalgebra*.

Definition 2.3. [20] A $T\&F$-structure $(P, \mathcal{B})$ over a BCK/BCI-algebra P with $\mathcal{B} := (\varphi_A,\, \widetilde{\varphi}_A,\, \partial_A,\, \widetilde{\partial}_A)$ is called a *$T\&F$-ideal* of P if the following assertions are valid.

$$(\forall \iota \in P)\left((\theta, \iota) \in \Omega^{\varphi}_{T} \cap \Omega^{\partial}_{F}\right). \tag{26}$$

$$(\forall \iota, \zeta \in P)\left(\tfrac{\iota}{(\iota*\zeta,\, \zeta)} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})}\right). \tag{27}$$

If a $T\&F$-ideal is limited, then we say that it is a *limited $T\&F$-ideal*.

Proposition 2.4. [20] *Every $T\&F$-ideal $(P, \mathcal{B})$ of a BCK/BCI-algebra P satisfies:*

$$(\forall \iota, \zeta, \varrho \in P)\left(\iota * \zeta \leq \varrho \;\Rightarrow\; \tfrac{\iota}{(\zeta,\varrho)} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})}\right). \tag{28}$$

3. Commutative $T\&F$-ideals of BCK-algebras

Definition 3.1. A $T\&F$-structure $(P, \mathcal{B})$ over a BCK-algebra P with $\mathcal{B} := (\varphi_A,\, \widetilde{\varphi}_A,\, \partial_A,\, \widetilde{\partial}_A)$ is said to be a *commutative $T\&F$-ideal* of P if it satisfies (26) and

$$(\forall \iota, \zeta \in P)\left(\tfrac{\iota*(\zeta*(\zeta*\iota))}{((\iota*\zeta)*\varrho,\, \varrho)} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})}\right). \tag{29}$$

If a commutative $T\&F$-ideal is limited, then we say that it is a *limited commutative $T\&F$-ideal*.

Example 3.2. Consider a BCK-algebra $P := \{\theta, o_1, o_2, o_3\}$ with the binary operation $*$ given in Table 1.

Table 1. Cayley table for the binary operation "$*$"

$*$	θ	o_1	o_2	o_3
θ	θ	θ	θ	θ
o_1	o_1	θ	θ	o_1
o_2	o_2	o_1	θ	o_2
o_3	o_3	o_3	o_3	θ

Let $(P, \mathcal{B})$ be a $T\&F$-structure over P with $\mathcal{B} := (\varphi_A,\, \widetilde{\varphi}_A,\, \partial_A,\, \widetilde{\partial}_A)$ given by Table 2.
It is routine to verify that $(P, \mathcal{B})$ is a commutative $T\&F$-ideal of P which is not limited.

TABLE 2. Tabular representation of $(P, \mathcal{B})$

P	$\varphi_A(\iota)$	$\widetilde{\varphi}_A(\iota)$	$\partial_A(\iota)$	$\widetilde{\partial}_A(\iota)$
θ	0.78	$[0.5, 0.9]$	0.23	$[0.12, 0.25]$
o_1	0.41	$[0.4, 0.7]$	0.32	$[0.53, 0.77]$
o_2	0.41	$[0.4, 0.7]$	0.32	$[0.53, 0.77]$
o_3	0.57	$[0.3, 0.5]$	0.74	$[0.23, 0.35]$

TABLE 3. Cayley table for the binary operation "$*$"

$*$	θ	o_1	o_2	o_3	o_4
θ	θ	θ	θ	θ	θ
o_1	o_1	θ	o_1	θ	o_1
o_2	o_2	o_2	θ	θ	o_2
o_3	o_3	o_2	o_1	θ	o_3
o_4	o_4	o_4	o_4	o_4	θ

Example 3.3. Consider the BCK-algebra $(P = \{\theta, o_1, o_2, o_3, o_4\}, *, \theta)$ in which "$*$" is given by Table 3. Let $(P, \mathcal{B})$ be a $T\&F$-structure over P with $\mathcal{B} := (\varphi_A, \widetilde{\varphi}_A, \partial_A, \widetilde{\partial}_A)$ given by Table 4. By routine calculations, we know that $(P, \mathcal{B})$ is a limited commutative $T\&F$-ideal of P.

TABLE 4. Tabular representation of $(P, \mathcal{B})$

P	$\varphi_A(\iota)$	$\widetilde{\varphi}_A(\iota)$	$\partial_A(\iota)$	$\widetilde{\partial}_A(\iota)$
θ	0.67	$[0.45, 0.68]$	0.28	$[0.16, 0.25]$
o_1	0.33	$[0.28, 0.53]$	0.28	$[0.28, 0.38]$
o_2	0.53	$[0.37, 0.59]$	0.41	$[0.29, 0.41]$
o_3	0.33	$[0.28, 0.53]$	0.41	$[0.29, 0.41]$
o_4	0.48	$[0.45, 0.68]$	0.32	$[0.28, 0.27]$

We establish a relationship between a $T\&F$-ideal and a commutative $T\&F$-ideal in BCK-algebras.

Theorem 3.4. *Every commutative $T\&F$-ideal is a $T\&F$-ideal in a BCK-algebra.*

Proof. Let $(P, \mathcal{B})$ be a commutative $T\&F$-ideal of a BCK-algebra P. Using (V) and (2), we have

$$\frac{\iota}{(\iota*\zeta,\zeta)} = \frac{\iota*(\theta*(\theta*\iota))}{((\iota*\theta)*\zeta,\zeta)} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})},$$

for all $\iota, \zeta \in P$ by (29). Consequently $(P, \mathcal{B})$ is a $T\&F$-ideal of P. □

Corollary 3.5. *In a BCK-algebra, every limited commutative $T\&F$-ideal is a limited $T\&F$-ideal.*

In the next example, we can see that the converse of Theorem 3.4 is not true.

Example 3.6. Consider a BCK-algebra $(P = \{\theta, o_1, o_2, o_3, o_4\}, *, \theta)$ where "$*$" is given by Table 5.

Table 5. Cayley table for the binary operation "$*$"

$*$	θ	o_1	o_2	o_3	o_4
θ	θ	θ	θ	θ	θ
o_1	o_1	θ	o_1	θ	θ
o_2	o_2	o_2	θ	θ	θ
o_3	o_3	o_3	o_3	θ	θ
o_4	o_4	o_4	o_4	o_3	θ

Take a $T\&F$-structure $(P, \mathcal{B})$ over P where $\mathcal{B} := (\varphi_A, \widetilde{\varphi}_A, \partial_A, \widetilde{\partial}_A)$ which is given by Table 6. Then $(P, \mathcal{B})$ is a $T\&F$-ideal of P, which is not a commutative $T\&F$-ideal of P since

Table 6. Tabular representation of $(P, \mathcal{B})$

P	$\varphi_A(\iota)$	$\widetilde{\varphi}_A(\iota)$	$\partial_A(\iota)$	$\widetilde{\partial}_A(\iota)$
θ	0.8	$[0.43, 0.65]$	0.28	$[0.17, 0.28]$
o_1	0.5	$[0.38, 0.57]$	0.28	$[0.27, 0.36]$
o_2	0.6	$[0.43, 0.65]$	0.41	$[0.29, 0.41]$
o_3	0.3	$[0.27, 0.53]$	0.55	$[0.31, 0.67]$
o_4	0.3	$[0.27, 0.53]$	0.55	$[0.31, 0.67]$

$$\frac{o_2*(o_3*(o_3*o_2))}{((o_2*o_3)*\theta,\theta)} = \frac{o_2}{(\theta,\theta)} \notin \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})}.$$

Proposition 3.7. *Every commutative $T\&F$-ideal $(P, \mathcal{B})$ of a BCK-algebra P satisfies:*

$$(\forall \iota, \zeta \in P)\left((\iota * (\zeta * (\zeta * \iota)),\ \iota * \zeta) \in \Omega_T^{\varphi} \cap \Omega_F^{\partial}\right). \tag{30}$$

Proof. If we take $\varrho = \theta$ in (29) and use (2), then

$$\tfrac{\iota*(\zeta*(\zeta*\iota))}{(\iota*\zeta,\,\theta)} = \tfrac{\iota*(\zeta*(\zeta*\iota))}{((\iota*\zeta)*\theta,\,\theta)} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})},$$

for all $\iota, \zeta \in P$. Combining this with (26) induces (30). $\square$

We give proper conditions so that a $T\&F$-ideal can be a commutative $T\&F$-ideal.

Lemma 3.8. *Let $(P, \mathcal{B})$ be a $T\&F$-structure over a set P with $\mathcal{B} := (\varphi_A,\ \widetilde{\varphi}_A,\ \partial_A,\ \widetilde{\partial}_A)$. If $(\iota, \zeta) \in \Omega_T^{\varphi} \cap \Omega_F^{\partial}$ and $\tfrac{\zeta}{(\iota,\zeta)} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})}$, then $\tfrac{\iota}{(\iota,\zeta)} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})}$ for all $\iota, \zeta, \iota, \zeta \in P$.*

Theorem 3.9. *If a $T\&F$-ideal $(P, \mathcal{B})$ of a BCK-algebra P satisfies* (30)*, then it is a commutative $T\&F$-ideal of P.*

Proof. Let $(P, \mathcal{B})$ be a $T\&F$-ideal of a BCK-algebra P that satisfies (30). Using (27) induces

$$\tfrac{\iota*\zeta}{((\iota*\zeta)*\varrho,\,\varrho)} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})},$$

for all $\iota, \zeta, \varrho \in P$. Combining this with (30) induces

$$\tfrac{\iota*(\zeta*(\zeta*\iota))}{((\iota*\zeta)*\varrho,\,\varrho)} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})},$$

by Lemma 3.8. Therefore $(P, \mathcal{B})$ is a commutative $T\&F$-ideal of P. $\square$

Lemma 3.10 ([20])**.** *Every $T\&F$-ideal $(P, \mathcal{B})$ of a BCK/BCI-algebra P satisfies:*

$$(\forall \iota, \zeta, \varrho \in P)\left(\iota * \zeta \leq \varrho \ \Rightarrow\ \tfrac{\iota}{(\zeta,\,\varrho)} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})}\right). \tag{31}$$

Theorem 3.11. *Every $T\&F$-ideal is a commutative $T\&F$-ideal in a commutative BCK-algebra.*

Proof. Let P be a commutative BCK-algebra and $(P, \mathcal{B})$ be a $T\&F$-ideal of P. For every elements $\iota, \zeta, \varrho \in P$, we get

$$\begin{aligned}((\iota * (\zeta * (\zeta * \iota))) * ((\iota * \zeta) * \varrho)) * \varrho &= ((\iota * (\zeta * (\zeta * \iota))) * \varrho) * ((\iota * \zeta) * \varrho) \\ &\leq (\iota * (\zeta * (\zeta * \iota))) * (\iota * \zeta) \\ &= (\iota * (\iota * \zeta)) * (\zeta * (\zeta * \iota)) = \theta,\end{aligned}$$

by (I), (III), (4) and the commutativity of P, and so

$$((\iota * (\zeta * (\zeta * \iota))) * ((\iota * \zeta) * \varrho)) * \varrho = \theta,$$

that is, $(\iota * (\zeta * (\zeta * \iota))) * ((\iota * \zeta) * \varrho) \leq \varrho$. Hence

$$\tfrac{\iota*(\zeta*(\zeta*\iota))}{((\iota*\zeta)*\varrho,\, \varrho)} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})},$$

by Lemma 3.10. Thus $(P, \mathcal{B})$ is a commutative $T\&F$-ideal of P. □

Using a commutative ideal of a BCK-algebra, we establish a commutative $T\&F$-ideal.

Theorem 3.12. *Given a commutative ideal L of a BCK-algebra P, let $(P, \mathcal{B})$ be a $T\&F$-structure over P with $\mathcal{B} := (\varphi_A, \widetilde{\varphi}_A, \partial_A, \widetilde{\partial}_A)$ which is given as follows:*

$$\varphi_A : P \to [0,1],\ \iota \mapsto \begin{cases} \alpha & \text{if } \iota \in L, \\ 0 & \text{otherwise,} \end{cases}$$

$$\widetilde{\varphi}_A : P \to int[0,1],\ \iota \mapsto \begin{cases} \widetilde{t} & \text{if } \iota \in L, \\ [0,0] & \text{otherwise,} \end{cases}$$

$$\partial_A : P \to [0,1],\ \iota \mapsto \begin{cases} \beta & \text{if } \iota \in L, \\ 1 & \text{otherwise,} \end{cases}$$

$$\widetilde{\partial}_A : P \to int[0,1],\ \iota \mapsto \begin{cases} \widetilde{s} & \text{if } \iota \in L, \\ [1,1] & \text{otherwise,} \end{cases}$$

where $\alpha \in (0,1]$, $\beta \in [0,1)$, $\widetilde{t} \in int[0,1] \setminus \{[0,0]\}$ and $\widetilde{s} \in int[0,1] \setminus \{[1,1]\}$. Then $(P, \mathcal{B})$ is a commutative $T\&F$-ideal of P.

Proof. It is clear that $(\theta, \iota) \in \Omega^{\varphi}_{T} \cap \Omega^{\partial}_{F}$ for all $\iota \in P$. Let $\iota, \zeta, \varrho \in P$. If $(\iota * \zeta) * \varrho \in L$ and $\varrho \in L$, then $\iota * (\zeta * (\zeta * \iota)) \in L$, since L is a commutative ideal of P. Then $\varphi_A((\iota * \zeta) * \varrho) = \varphi_A(\varrho) = \varphi_A(\iota * (\zeta * (\zeta * \iota))) = \alpha$, $\widetilde{\varphi}_A((\iota * \zeta) * \varrho) = \widetilde{\varphi}_A(\varrho) = \widetilde{\varphi}_A(\iota * (\zeta * (\zeta * \iota))) = \widetilde{t}$, $\partial_A((\iota*\zeta)*\varrho) = \partial_A(\varrho) = \partial_A(\iota*(\zeta*(\zeta*\iota))) = \beta$, and $\widetilde{\partial}_A((\iota*\zeta)*\varrho) = \widetilde{\partial}_A(\varrho) = \widetilde{\partial}_A(\iota*(\zeta*(\zeta*\iota))) = \widetilde{s}$. Hence

$$\tfrac{\iota*(\zeta*(\zeta*\iota))}{((\iota*\zeta)*\varrho,\, \varrho)} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})}.$$

Assume that $(\iota * \zeta) * \varrho \notin L$ or $\varrho \notin L$. Then

$$\varphi_A((\iota * \zeta) * \varrho) = 0 \text{ or } \varphi_A(\varrho) = 0,$$

$$\widetilde{\varphi}_A((\iota * \zeta) * \varrho) = [0,0] \text{ or } \widetilde{\varphi}_A(\varrho) = [0,0],$$

$$\partial_A((\iota * \zeta) * \varrho) = 1 \text{ or } \partial_A(\varrho) = 1,$$

$$\widetilde{\partial}_A((\iota * \zeta) * \varrho) = [1,1] \text{ or } \widetilde{\partial}_A(\varrho) = [1,1].$$

It follows that $\tfrac{\iota*(\zeta*(\zeta*\iota))}{((\iota*\zeta)*\varrho,\, \varrho)} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})}$. Therefore $(P, \mathcal{B})$ is a commutative $T\&F$-ideal of P. □

4. Commutative $T\&F$-ideals of BCI-algebras

Definition 4.1. A $T\&F$-structure $(P, \mathcal{B})$ over a BCI-algebra P with $\mathcal{B} := (\varphi_A,\, \widetilde{\varphi}_A,\, \partial_A,\, \widetilde{\partial}_A)$ is said to be a *commutative $T\&F$-ideal* of P if it satisfies (26) and

$$(\forall \iota, \zeta \in P)\left(\frac{\iota*((\zeta*(\zeta*\iota))*(\theta*(\theta*(\iota*\zeta))))}{((\iota*\zeta)*\varrho,\, \varrho)} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})}\right). \tag{32}$$

If a commutative $T\&F$-ideal is limited, then we say that it is a *limited commutative $T\&F$-ideal*.

Example 4.2. Let $P = \{\theta, o_1, o_2, o_3, o_4\}$ be a set with the binary operation "$*$" given by Table 7.

Table 7. Cayley table for the binary operation "$*$"

$*$	θ	o_1	o_2	o_3	o_4
θ	θ	θ	o_4	o_3	o_2
o_1	o_1	θ	o_4	o_3	o_2
o_2	o_2	o_2	θ	o_4	o_3
o_3	o_3	o_3	o_2	θ	o_4
o_4	o_4	o_4	o_3	o_2	θ

Then P is a BCI-algebra (see [8]). Let $(P, \mathcal{B})$ be a $T\&F$-structure over P with $\mathcal{B} := (\varphi_A,\, \widetilde{\varphi}_A,\, \partial_A,\, \widetilde{\partial}_A)$ given by Table 8. It is routine to verify that $(\theta, \iota) \in \Omega^{\varphi}_{T} \cap \Omega^{\partial}_{F}$ and

Table 8. Tabular representation of $(P, \mathcal{B})$

P	$\varphi_A(\iota)$	$\widetilde{\varphi}_A(\iota)$	$\partial_A(\iota)$	$\widetilde{\partial}_A(\iota)$
θ	0.98	$[0.43, 0.78]$	0.18	$[0.17, 0.38]$
o_1	0.75	$[0.38, 0.67]$	0.37	$[0.27, 0.46]$
o_2	0.46	$[0.37, 0.53]$	0.62	$[0.29, 0.63]$
o_3	0.46	$[0.37, 0.53]$	0.62	$[0.29, 0.63]$
o_4	0.46	$[0.37, 0.53]$	0.62	$[0.29, 0.63]$

$$\frac{\iota*((\zeta*(\zeta*\iota))*(\theta*(\theta*(\iota*\zeta))))}{((\iota*\zeta)*\varrho,\, \varrho)} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})}$$

for all $\iota, \zeta, \varrho \in P$. Hence $(P, \mathcal{B})$ is a commutative $T\&F$-ideal of P.

Theorem 4.3. *Given a commutative ideal L of a BCI-algebra P, let $(P, \mathcal{B})$ be a $T\&F$-structure over P with $\mathcal{B} := (\varphi_A,\, \widetilde{\varphi}_A,\, \partial_A,\, \widetilde{\partial}_A)$ which is given in Theorem 3.12. Then $(P, \mathcal{B})$ is a commutative $T\&F$-ideal of P.*

Proof. It is identified in the same way as the proof of Theorem 3.12. □

Theorem 4.4. *Every commutative $T\&F$-ideal is a $T\&F$-ideal in a BCI-algebra.*

Proof. Let $(P,\mathcal{B})$ be a commutative $T\&F$-ideal of a BCI-algebra P. If we take $\zeta=\theta$ in (32), then

$$\frac{\iota}{(\iota*\varrho,\,\varrho)}=\frac{\iota*\theta}{((\iota*\theta)*\varrho,\,\varrho)}=\frac{\iota*((\theta*(\theta*\iota))*(\theta*(\theta*(\iota*\theta))))}{((\iota*\theta)*\varrho,\,\varrho)}\in\Omega^{\varphi}_{T(\min,\mathrm{rmin})}\cap\Omega^{\partial}_{F(\max,\mathrm{rmax})}.$$

for all $\iota,\varrho\in P$ by (III) and (2). Hence $(P,\mathcal{B})$ is a $T\&F$-ideal of P. □

Corollary 4.5. *In a BCI-algebra, every limited commutative $T\&F$-ideal is a limited $T\&F$-ideal.*

The converse of Theorem 4.4 is false as shown in the next example.

Example 4.6. Consider the BCK-algebra P in Example 3.6. Then P is also a BCI-algebra since every BCK-algebra is a BCI-algebra. Let $(P,\mathcal{B})$ be a $T\&F$-structure over P with $\mathcal{B}:=(\varphi_A,\,\widetilde{\varphi}_A,\,\partial_A,\,\widetilde{\partial}_A)$ given by Table 9.

Table 9. Tabular representation of $(P,\mathcal{B})$

P	$\varphi_A(\iota)$	$\widetilde{\varphi}_A(\iota)$	$\partial_A(\iota)$	$\widetilde{\partial}_A(\iota)$
θ	0.82	$[0.46, 0.69]$	0.27	$[0.19, 0.48]$
o_1	0.53	$[0.38, 0.58]$	0.39	$[0.27, 0.57]$
o_2	0.42	$[0.32, 0.49]$	0.56	$[0.33, 0.69]$
o_3	0.42	$[0.32, 0.49]$	0.56	$[0.33, 0.69]$
o_4	0.42	$[0.32, 0.49]$	0.56	$[0.33, 0.69]$

Then $(P,\mathcal{B})$ is a $T\&F$-ideal of P and

$$\frac{o_2*((o_3*(o_3*o_2))*(\theta*(\theta*(o_2*o_3))))}{((o_2*o_3)*\theta),\,\theta}=\frac{o_2}{(\theta,\theta)}\notin\Omega^{\varphi}_{T(\min,\mathrm{rmin})}\cap\Omega^{\partial}_{F(\max,\mathrm{rmax})},$$

which proves that $(P,\mathcal{B})$ is not a commutative $T\&F$-ideal of P.

Proposition 4.7. *Any commutative $T\&F$-ideal $(P,\mathcal{B})$ of a BCI-algebra P satisfies:*

$$(\forall\iota,\zeta\in P)\left((\iota*((\zeta*(\zeta*\iota))*(\theta*(\theta*(\iota*\zeta)))),\,\iota*\zeta)\in\Omega^{\varphi}_{T}\cap\Omega^{\partial}_{F}\right).\tag{33}$$

Proof. If we take $\varrho = \theta$ in (32) and use (2), then

$$\frac{\iota*((\zeta*(\zeta*\iota))*(\theta*(\theta*(\iota*\zeta))))}{(\iota*\zeta, \theta)} = \frac{\iota*((\zeta*(\zeta*\iota))*(\theta*(\theta*(\iota*\zeta))))}{((\iota*\zeta)*\theta, \theta)} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})},$$

for all $\iota, \zeta \in P$. Combining this with (26) induces (33). □

We provide conditions for a $T\&F$-ideal to be commutative.

Theorem 4.8. *A $T\&F$-ideal $(P, \mathcal{B})$ of a BCI-algebra P is commutative if and only if $(P, \mathcal{B})$ satisfies the condition* (33)*.*

Proof. Proposition 4.7 shows that the necessity is valid. Let $(P, \mathcal{B})$ be a $T\&F$-ideal of a BCI-algebra P that satisfies the condition (33). Then $\frac{\iota*\zeta}{((\iota*\zeta)*\varrho, \varrho)} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})}$ for all $\iota, \zeta, \varrho \in P$. It follows from (33) and Lemma 3.8 that

$$\frac{\iota*((\zeta*(\zeta*\iota))*(\theta*(\theta*(\iota*\zeta)))}{((\iota*\zeta)*\varrho, \varrho)} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})},$$

for all $\iota, \zeta, \varrho \in P$. Therefore $(P, \mathcal{B})$ is a commutative $T\&F$-ideal of P. □

Definition 4.9. A $T\&F$-ideal $(P, \mathcal{B})$ of a BCI-algebra P is said to be *closed* if

$$(\forall \iota \in P)\left((\theta * \iota,\ \iota) \in \Omega^{\varphi}_{T} \cap \Omega^{\partial}_{F}\right). \tag{34}$$

Example 4.10. Let $(P, \mathcal{B})$ be a $T\&F$-structure over a BCI-algebra P with $\mathcal{B} := (\varphi_A, \widetilde{\varphi}_A, \partial_A, \widetilde{\partial}_A)$ which is given as follows:

$$\varphi_A : P \to [0,1],\ \iota \mapsto \begin{cases} \alpha & \text{if } \iota \in P_+, \\ 0 & \text{otherwise,} \end{cases}$$

$$\widetilde{\varphi}_A : P \to int[0,1],\ \iota \mapsto \begin{cases} \widetilde{t} & \text{if } \iota \in P_+, \\ [0,0] & \text{otherwise,} \end{cases}$$

$$\partial_A : P \to [0,1],\ \iota \mapsto \begin{cases} \beta & \text{if } \iota \in P_+, \\ 1 & \text{otherwise,} \end{cases}$$

$$\widetilde{\partial}_A : P \to int[0,1],\ \iota \mapsto \begin{cases} \widetilde{s} & \text{if } \iota \in P_+, \\ [1,1] & \text{otherwise,} \end{cases}$$

where $P_+ := \{\iota \in P \mid \theta \leq \iota\}$, $\alpha \in (0,1]$, $\beta \in [0,1)$, $\widetilde{t} \in int[0,1] \backslash \{[0,0]\}$ and $\widetilde{s} \in int[0,1] \backslash \{[1,1]\}$. It is routine to show that $(P, \mathcal{B})$ is a closed $T\&F$-ideal of P.

Theorem 4.11. *Let $(P, \mathcal{B})$ be a closed $T\&F$-ideal of a BCI-algebra P. Then $(P, \mathcal{B})$ is a commutative $T\&F$-ideal of P if and only if the following condition is valid.*

$$(\forall \iota, \zeta \in P)\left((\iota * (\zeta * (\zeta * \iota)),\ \iota * \zeta) \in \Omega^{\varphi}_{T} \cap \Omega^{\partial}_{F}\right). \tag{35}$$

Proof. Assume that $(P,\mathcal{B})$ is a commutative $T\&F$-ideal of P. For every $\iota,\zeta,\varrho\in P$, we have

$$\begin{aligned}
&(\iota*(\zeta*(\zeta*\iota)))*(\iota*((\zeta*(\zeta*\iota))*(\theta*(\theta*(\iota*\zeta)))))\\
&\leq((\zeta*(\zeta*\iota))*(\theta*(\theta*(\iota*\zeta))))*(\zeta*(\zeta*\iota))\\
&=((\zeta*(\zeta*\iota))*(\zeta*(\zeta*\iota)))*(\theta*(\theta*(\iota*\zeta)))\\
&=\theta*(\theta*(\theta*(\iota*\zeta)))=\theta*(\iota*\zeta),
\end{aligned}$$

by (I), (4), (III) and (5). Using Proposition 2.4, we have

$$\tfrac{\iota*(\zeta*(\zeta*\iota))}{(\iota*((\zeta*(\zeta*\iota))*(\theta*(\theta*(\iota*\zeta))),\,\theta*(\iota*\zeta))}\in\Omega^{\varphi}_{T(\min,\mathrm{rmin})}\cap\Omega^{\partial}_{F(\max,\mathrm{rmax})}.$$

It follows from Propositions 2.1 and 4.7 that

$$\tfrac{\iota*(\zeta*(\zeta*\iota))}{(\iota*\zeta,\,\theta*(\iota*\zeta))}\in\Omega^{\varphi}_{T(\min,\mathrm{rmin})}\cap\Omega^{\partial}_{F(\max,\mathrm{rmax})}.$$

Since $(P,\mathcal{B})$ is closed, we conclude that $(\iota*(\zeta*(\zeta*\iota)),\,\iota*\zeta)\in\Omega^{\varphi}_{T}\cap\Omega^{\partial}_{F}$.

Conversely, suppose that $(P,\mathcal{B})$ satisfies the condition (35). Since

$$\begin{aligned}
&(\iota*((\zeta*(\zeta*\iota))*(\theta*(\theta*(\iota*\zeta)))))*(\iota*(\zeta*(\zeta*\iota)))\\
&\leq(\zeta*(\zeta*\iota))*((\zeta*(\zeta*\iota))*(\theta*(\theta*(\iota*\zeta)))),\\
&\leq\theta*(\theta*(\iota*\zeta)),
\end{aligned}$$

for all $\iota,\zeta\in P$ by (I) and (II), we have

$$\tfrac{\iota*((\zeta*(\zeta*\iota))*(\theta*(\theta*(\iota*\zeta))))}{(\iota*(\zeta*(\zeta*\iota)),\,\theta*(\theta*(\iota*\zeta))}\in\Omega^{\varphi}_{T(\min,\mathrm{rmin})}\cap\Omega^{\partial}_{F(\max,\mathrm{rmax})},$$

by Proposition 2.4. It follows from Proposition 2.1 and (35) that

$$\tfrac{\iota*((\zeta*(\zeta*\iota))*(\theta*(\theta*(\iota*\zeta))))}{(\iota*\zeta,\,\theta*(\theta*(\iota*\zeta))}\in\Omega^{\varphi}_{T(\min,\mathrm{rmin})}\cap\Omega^{\partial}_{F(\max,\mathrm{rmax})}.$$

Combining this with (34) induces

$$(\iota*((\zeta*(\zeta*\iota))*(\theta*(\theta*(\iota*\zeta)))),\,\iota*\zeta)\in\Omega^{\varphi}_{T}\cap\Omega^{\partial}_{F}.$$

Therefore we know that $(P,\mathcal{B})$ is a commutative $T\&F$-ideal of P by Theorem 4.8. □

Lemma 4.12 ([16]). *A BCI-algebra P is commutative if and only if it satisfies:*

$$(\forall\iota,\zeta\in P)\,(\iota*(\iota*\zeta)=\zeta*(\zeta*(\iota*(\iota*\zeta))))\,. \tag{36}$$

Theorem 4.13. *Every closed $T\&F$-ideal of a commutative BCI-algebra is a commutative $T\&F$-ideal.*

Proof. Let $(P, \mathcal{B})$ be a closed $T\&F$-ideal of a commutative BCI-algebra P. Then

$$\begin{aligned}(\iota * (\zeta * (\zeta * \iota))) * (\iota * \zeta) &= (\iota * (\iota * \zeta)) * (\zeta * (\zeta * \iota)) \\ &= (\zeta * (\zeta * (\iota * (\iota * \zeta)))) * (\zeta * (\zeta * \iota)) \\ &= (\zeta * (\zeta * (\zeta * \iota))) * (\zeta * (\iota * (\iota * \zeta))) \\ &= (\zeta * \iota) * (\zeta * (\iota * (\iota * \zeta))) \\ &\leq (\iota * (\iota * \zeta)) * \iota \\ &= \theta * (\iota * \zeta),\end{aligned}$$

for all $\iota, \zeta \in P$ by (I), (III), (4), (5) and Lemma 4.12. It follows from Proposition 2.4 that

$$\tfrac{\iota*(\zeta*(\zeta*\iota))}{(\iota*\zeta,\, \theta*(\iota*\zeta))} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})}.$$

Combining this with (34) induces

$$(\iota * (\zeta * (\zeta * \iota)),\ \iota * \zeta) \in \Omega^{\varphi}_{T} \cap \Omega^{\partial}_{F},$$

for all $\iota, \zeta \in P$. Hence $(P, \mathcal{B})$ is a commutative $T\&F$-ideal of P by Theorem 4.11. □

The converse of Theorem 4.13 is false as shown in the next example.

Example 4.14. Consider the commutative BCI-algebra P in Example 4.2. In that example, it is routine to verify that $(\theta, \iota) \in \Omega^{\varphi}_{T} \cap \Omega^{\partial}_{F}$ and

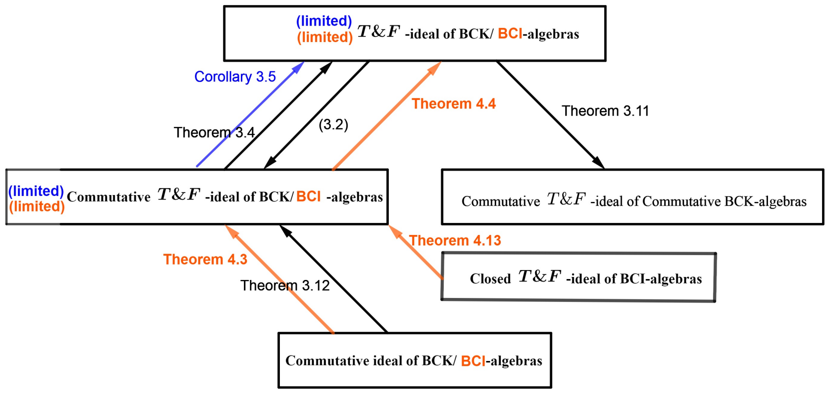
$$\tfrac{\iota*((\zeta*(\zeta*\iota))*(\theta*(\theta*(\iota*\zeta))))}{((\iota*\zeta)*\varrho,\, \varrho)} \in \Omega^{\varphi}_{T(\min,\mathrm{rmin})} \cap \Omega^{\partial}_{F(\max,\mathrm{rmax})},$$

for all $\iota, \zeta, \varrho \in P$. Hence $(P, \mathcal{B})$ is a commutative $T\&F$-ideal of P, but isn't a closed $T\&F$-ideal. Because, if take $\iota = o_2$ in Definition 4.9, then $(\theta * \iota,\ \iota) \notin \Omega^{\varphi}_{T} \cap \Omega^{\partial}_{F}$.

5. Conclusion

Mohseni et al. [19] have constructed True-False structures based on an interval value fuzzy set and a fuzzy set. They have studied the basic properties and have applied it to groups and BCK/BCI-algebras at the same time. Using this structure, Mohseni et al. [20] have discussed ideal theory in BCK/BCI-algebras. Meng have studied a commutative ideal in a BCK-algebra and a BCI-algebra which are algebraic structures in universal algebra. In this paper, we have applied the True-False structure to a commutative ideal in a BCK-algebra and a BCI-algebra. We have introduced the concept of a (limited) commutative $T\&F$-ideal in BCK-algebras and BCI-algebras, and have investigated several properties. We have considered a relationship between a $T\&F$-ideal and a commutative $T\&F$-ideal in BCK-algebras and BCI-algebras, and have provided examples to show that any $T\&F$-ideal may not be a commutative $T\&F$-ideal.

FIGURE 1. The relationship between $T\&F$-ideals



We have given proper conditions so that a $T\&F$-ideal can be a commutative $T\&F$-ideal. We have establish a commutative $T\&F$-ideal by using a commutative ideal of a BCK-algebra and a BCI-algebra. We have introduced the closed $T\&F$-ideal in a BCI-algebra, and have provided a condition for a closed $T\&F$-ideal to be commutative. We have discussed characterization of a commutative $T\&F$-ideal in a BCI-algebra.

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