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Research Paper

CHARACTERIZATION OF ZERO-DIMENSIONAL RINGS SUCH THAT THE CLIQUE NUMBER OF THEIR ANNIHILATING-IDEAL GRAPHS IS AT MOST FOUR

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Abstract. The rings considered in this article are commutative with identity which are not integral domains. Let R be a ring. An ideal I of R is said to be an annihilating ideal of R if there exists $r \in R\backslash\{0\}$ such that $Ir = (0)$. Let $\mathbb{A}(R)$ denote the set of all annihilating ideals of R and let $\mathbb{A}(R)^* = \mathbb{A}(R)\backslash\{(0)\}$. Recall that the annihilating-ideal graph of R, denoted by $\mathbb{AG}(R)$, is an undirected graph whose vertex set is $\mathbb{A}(R)^*$ and distinct vertices I and J are adjacent in this graph if and only if $IJ = (0)$. The aim of this article is to characterize zero-dimensional rings such that the clique number of their annihilating-ideal graphs is at most four.

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1. Introduction

The rings considered in this article are commutative with identity which admit at least one non-zero annihilating-ideal. The study of associating a graph with a ring and investigating the interplay between the ring-theoretic properties of the ring and the graph-theoretic properties of the graph associated with it began with the research work of Beck in [9]. In [9], Beck was mainly interested in colorings. Let R be a ring. Let $Z(R)$ denote the set of all zero-divisors of R and let us denote $Z(R)\backslash\{0\}$ by $Z(R)^*$. The graphs considered in this article are undiredted and simple. For a graph G, we denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. Recall from [6] that the *zero-divisor graph of* R, denoted by $\Gamma(R)$, is an undirected graph with $V(\Gamma(R)) = Z(R)^*$ and distinct vertices x and y are adjacent in $\Gamma(R)$ if and only if $xy = 0$. During the last two decades, several mathematicians contributed to the area of zero-divisor graphs in commutative rings. For an excellent and interesting survey on zero-divisor graphs in commutative rings, the reader is referred to [4].

Let R be a ring. As in [10], we denote the set of all annihilating ideals of R by $\mathbb{A}(R)$ and we denote $\mathbb{A}(R)\backslash\{(0)\}$ by $\mathbb{A}(R)^*$. Let R be such that $\mathbb{A}(R)^* \neq \emptyset$. The concept of the annihilating-ideal graph of a ring was introduced by Behboodi and Rakeei in [10]. Recall from [10] that the *annihilating-ideal graph of* R, denoted by $\mathbb{AG}(R)$, is an undirected graph with $V(\mathbb{AG}(R)) = \mathbb{A}(R)^*$ and distinct vertices I and J are adjacent in this graph if and only if $IJ = (0)$. Several interesting and inspiring theorems were proved on $\mathbb{AG}(R)$ in [10, 11]. The annihilating-ideal graph of a commutative ring was also considered by several other researchers, for example, refer [1, 2, 3, 14].

Let $G = (V, E)$ be a graph. We say that G *satisfies* (A) if G does not contain $K_{3,3}$ as a subgraph. We say that G *satisfies* (B) if G does not contain K_5 as a subgraph. A complete subgraph of a graph G is called a *clique* of G ([8], Definition 1.2.2). Let $k \in \mathbb{N}$ be such that each clique of G is a clique on at most k vertices. The *clique number* of G, denoted by $\omega(G)$, is defined as the largest positive integer n such that G contains a clique on n vertices ([8], page 185). If G contains a clique on n vertices for all $n \geq 1$, then we define $\omega(G) = \infty$. Observe that a graph G satisfies (B) if and only if $\omega(G) \leq 4$.

Let $G = (V, E)$ be a graph. A *vertex coloring* of G is a map $f : V \to S$, where S is a set of distinct colors. A vertex coloring $f : V \to S$ is said to be *proper*, if adjacent vertices of G receive distinct colors of S; that is, if $u, v \in V$ are adjacent in G, then $f(u) \neq f(v)$ ([8], page 129). Recall from ([8], Definition 7.1.2) that the *chromatic number* of G, denoted by $\chi(G)$, is the minimum number of colors needed for a proper vertex coloring of G. It is well-known that $\omega(G) \leq \chi(G)$.

Let R be a ring. The ring R is said to be *quasi-local* (respectively, *semi-quasi-local*) if R has only one maximal ideal (respectively, has only a finite number of maximal ideals). If

R is quasi-local with $\mathfrak{m}$ as its unique maximal ideal, then we denote it using the notation $(R, \mathfrak{m})$. A Noetherian quasi-local (respectively, semi-quasi-local) ring is referred to as a *local* (respectively, *semi-local*) ring. The Krull dimension of R is simply referred to as the dimension of R. We denote the dimension of R by $dim R$. We denote the set of all maximal ideals of R by $Max(R)$. We denote the cardinality of a set A by $|A|$. This article is a continuation of our work which appeared in [16] regarding the planarity of $\mathbb{AG}(R)$, where R is a zero-dimensional semi-quasi-local ring, which is not quasi-local. Let $n \in \mathbb{N}$ be such that $n \geq 2$. Let R be a zero-dimensional ring with $|Max(R)| \geq n$. It was shown in ([17], Lemma 3.15) that there exist zero-dimensional rings $R_1, R_2, \ldots, R_n$ such that $R \cong R_1 \times R_2 \times \cdots \times R_n$ as rings. Let R be the direct product of n rings $R_1, \ldots, R_n$. It was shown in ([16], Lemma 2.3) that if $\mathbb{AG}(R)$ satisfies (A), then $n \leq 3$. Hence, it follows from ([17], Lemma 3.15) and ([16], Lemma 2.3) that if $\mathbb{AG}(R)$ satisfies (A) for a zero-dimensional ring R, then $|Max(R)| \leq 3$. Thus the assumption that R is semi-quasi-local in the statement of Theorem 5.1 of [16] is superfluous. For a zero-dimensional non-quasi-local ring R, it was shown in ([16], Theorem 5.1) that $\mathbb{AG}(R)$ satisfies (A) if and only if $\mathbb{AG}(R)$ is planar and moreover, such rings R were characterized in ([16], Statement (iii) of Theorem 5.1). Notice that it follows from Kuratowski's Theorem ([13], Theorem 5.9) and $(ii) \Rightarrow (iv)$ of ([16], Theorem 5.1) that if $\mathbb{AG}(R)$ satisfies (A), then $\mathbb{AG}(R)$ satisfies (B). Moreover, in ([16], Example 6.13), an example of a local Artinian ring $(R, \mathfrak{m})$ was provided such that $\mathbb{AG}(R)$ is K_5. Hence, we obtain that $\mathbb{AG}(R)$ satisfies (A) but it does not satisfy (B).

The aim of this article is to characterize zero-dimensional rings R such that $\mathbb{AG}(R)$ satisfies (B) and to determine $\chi(\mathbb{AG}(R))$ in the case when $\mathbb{AG}(R)$ satisfies (B). In Section 2 of this article, we state and prove several supporting results for proving the main theorems which characterize zero-dimensional rings R for which $\mathbb{AG}(R)$ satisfies (B). We observe in Corollary 2.2 that if $\mathbb{AG}(R)$ satisfies (B), then $|Max(R)| \leq 4$. In Theorem 3.2, we characterize zero-dimensional rings R with $|Max(R)| = 4$ such that $\mathbb{AG}(R)$ satisfies (B). In Section 4, we consider zero-dimensional rings R with $|Max(R)| = 3$ and in Theorem 4.16, we are able to characterize such rings R in order that $\mathbb{AG}(R)$ to satisfy (B). In Section 5, we consider the problem of characterizing zero-dimensional rings R with $|Max(R)| = 2$ such that $\mathbb{AG}(R)$ satisfies (B). We are not able to solve this problem completely. However, Proposition 5.8, Lemmas 5.11 to 5.13, and Lemma 5.15 contain the required characterization in certain special cases. In Section 6, we try to characterize zero-dimensional quasi-local rings R such that $\mathbb{AG}(R)$ satisfies (B). Let $(R, \mathfrak{m})$ be a quasi-local zero-dimensional ring. In Propositions 6.2 and 6.3, we provide a characterization of R such that $\omega(\mathbb{AG}(R)) \leq 4$ in some special cases. In Remark 6.5, we provide a characterization of R such that $\omega(\mathbb{AG}(R)) \in \{1, 2\}$ and moreover, we are able to provide a characterization of R such that $\omega(\mathbb{AG}(R)) \in \{3,4\}$ in some special cases. In Remark 6.5, we also mention the problems that remain to be solved.

Let R be a ring. We denote the nilradical of R by $nil(R)$. We say that R is *reduced* if $nil(R) = (0)$. We denote the set of all minimal prime ideals of R by $Min(R)$. We denote the group of units of R by $U(R)$.

2. Some preliminary results

Lemma 2.1. *Let $n \in \mathbb{N}$ be such that $n \geq 2$ and let $R = R_1 \times R_2 \times \cdots \times R_n$, where R_i is a ring for each $i \in \{1, 2, \ldots, n\}$. If $\mathbb{AG}(R)$ satisfies (B), then $n \leq 4$.*

Proof. Assume that $\mathbb{AG}(R)$ satisfies (B). For any $i \in \{1, 2, \ldots, n\}$, let $e_i \in R$ be such that its i-th coordinate is 1, whereas its j-th coordinate is 0 for all $j \in \{1, 2, \ldots, n\} \backslash \{i\}$. It is clear that the subgraph of $\mathbb{AG}(R)$ induced by $\{Re_i \mid i \in \{1, 2, \ldots, n\}\}$ is a clique. This implies that $\omega(\mathbb{AG}(R)) \geq n$. Since $\omega(\mathbb{AG}(R)) \leq 4$, it follows that $n \leq 4$. □

Corollary 2.2. *Let R be a ring such that $dim R = 0$. If $\mathbb{AG}(R)$ satisfies (B), then $|Max(R)| \leq 4$.*

Proof. Assume that $\mathbb{AG}(R)$ satisfies (B). If $|Max(R)| \geq 5$, then it follows from ([17], Lemma 3.15) that there exist zero-dimensional rings $R_1, R_2, \ldots, R_5$ such that $R \cong R_1 \times R_2 \times \cdots \times R_5$ as rings. It follows from the proof of Lemma 2.1 that $\omega(\mathbb{AG}(R_1 \times R_2 \times \cdots \times R_5)) \geq 5$ and so, $\omega(\mathbb{AG}(R)) \geq 5$, which contradicts $\mathbb{AG}(R)$ satisfies (B). Therefore, $|Max(R)| \leq 4$. □

Let $T = \mathbb{Z} \times \mathbb{Z}$. Notice that T is a reduced ring with $|Min(T)| = 2$. Hence, it follows from ([11], Corollary 2.11) that $\omega(\mathbb{AG}(T)) = \chi(\mathbb{AG}(T)) = 2$. Hence, $\mathbb{AG}(T)$ satisfies (B). Notice that $Max(T)$ is infinite. Thus this example illustrates that Corollary 2.2 can fail to hold for a ring of positive dimension.

For a ring R with $\mathbb{A}(R)^* \neq \emptyset$, we know from ([10], Theorem 2.1) that $\mathbb{AG}(R)$ is connected. Thus if $|\mathbb{A}(R)^*| \geq 2$, then it is possible to find distinct $I, J \in \mathbb{A}(R)^*$ such that $IJ = (0)$. We use this remark in the proofs of some of the results of this article.

Lemma 2.3. *Let $R = R_1 \times R_2 \times R_3 \times R_4$, where R_i is a ring for each $i \in \{1, 2, 3, 4\}$. If $\mathbb{AG}(R)$ satisfies (B), then $|\mathbb{A}(R_i)^*| \leq 1$ for each $i \in \{1, 2, 3, 4\}$.*

Proof. Assume that $\mathbb{AG}(R)$ satisfies (B). Suppose that for some $i \in \{1, 2, 3, 4\}$, $|\mathbb{A}(R_i)^*| \geq 2$. Without loss of generality, we can assume that $|\mathbb{A}(R_1)^*| \geq 2$. Hence, there exist distinct $I_{11}, I_{12} \in \mathbb{A}(R_1)^*$ such that $I_{11}I_{12} = (0)$. Let $I_1 = I_{11} \times (0) \times (0) \times (0), I_2 = I_{12} \times (0) \times (0) \times (0), I_3 = (0) \times R_2 \times (0) \times (0), I_4 = (0) \times (0) \times R_3 \times (0)$, and $I_5 = (0) \times (0) \times (0) \times R_4$. It is clear that the subgraph of $\mathbb{AG}(R)$ induced by $\{I_j \mid j \in \{1, 2, \ldots, 5\}\}$ is a clique. This is impossible, since $\omega(\mathbb{AG}(R)) \leq 4$. Therefore, $|\mathbb{A}(R_i)^*| \leq 1$ for each $i \in \{1, 2, 3, 4\}$. $\square$

Recall that a principal ideal ring R is called a *special principal ideal ring* (SPIR) if R has a unique prime ideal. If $\mathfrak{m}$ is the only prime ideal of R, then it follows from ([7], Proposition 1.8) that $\mathfrak{m}$ is necessarily nilpotent. If R is an SPIR with $\mathfrak{m}$ as its only prime ideal, then we denote it by saying that $(R, \mathfrak{m})$ is an SPIR. Let $(R, \mathfrak{m})$ be an SPIR which is not a field. Then $\mathfrak{m} = Rm$ is principal and let $n \geq 2$ be least with the property that $\mathfrak{m}^n = (0)$. Then it follows from the proof of $(iii) \Rightarrow (i)$ of ([7], Proposition 8.8) that $\{\mathfrak{m}^i = Rm^i \mid i \in \{1, \ldots, n-1\}\}$ is the set of all non-zero proper ideals of R.

For a ring R, we know from ([10], Corollary 2.9(a)) that $|\mathbb{A}(R)^*| = 1$ if and only if $(R, Z(R))$ is an SPIR with $(Z(R))^2 = (0)$. One can also refer ([15], Lemma 2.6) for a proof of ([10], Corollary 2.9(a)).

Lemma 2.4. *Let $R = R_1 \times R_2 \times R_3 \times R_4$, where, R_i is a ring for each $i \in \{1, 2, 3, 4\}$. If $\mathbb{AG}(R)$ satisfies (B), then there exists at most one $i \in \{1, 2, 3, 4\}$ such that $\mathbb{A}(R_i)^* \neq \emptyset$.*

Proof. Suppose that $\mathbb{A}(R_i)^* \neq \emptyset$ for at least two values of $i \in \{1, 2, 3, 4\}$. Without loss of generality we can assume that $\mathbb{A}(R_1)^* \neq \emptyset$ and $\mathbb{A}(R_2)^* \neq \emptyset$. Now, it follows from Lemma 2.3 that $|\mathbb{A}(R_1)^*| = |\mathbb{A}(R_2)^*| = 1$. Let $\mathfrak{m}_1$ (respectively, $\mathfrak{m}_2$) be the unique non-zero annihilating ideal of R_1(respectively, R_2). Notice that $(R_i, \mathfrak{m}_i)$ is an SPIR with $\mathfrak{m}_i^2 = (0)$ for each $i \in \{1, 2\}$. Let $I_1 = \mathfrak{m}_1 \times (0) \times (0) \times (0), I_2 = (0) \times \mathfrak{m}_2 \times (0) \times (0), I_3 = \mathfrak{m}_1 \times \mathfrak{m}_2 \times (0) \times (0), I_4 = (0) \times (0) \times R_3 \times (0)$, and $I_5 = (0) \times (0) \times (0) \times R_4$. Observe that the subgraph of $\mathbb{AG}(R)$ induced by $\{I_i \mid i \in \{1, 2, \ldots, 5\}\}$ is a clique and this implies that $\omega(\mathbb{AG}(R)) \geq 5$. This is impossible, since $\omega(\mathbb{AG}(R)) \leq 4$ by assumption. Therefore, there exists at most one $i \in \{1, 2, 3, 4\}$ such that $\mathbb{A}(R_i)^* \neq \emptyset$. $\square$

Lemma 2.5. *Let $(S, \mathfrak{m})$ be an SPIR and $t \geq 2$ be least with the property that $\mathfrak{m}^t = (0)$. Then the following statements hold*

(i) *If $t = 2k$ for some $k \geq 1$, then $\omega(\mathbb{AG}(S)) = \chi(\mathbb{AG}(S)) = k$.*
(ii) *If $t = 2k + 1$ for some $k \geq 1$, then $\omega(\mathbb{AG}(S)) = \chi(\mathbb{AG}(S)) = k + 1$.*

Proof. (i) Notice that the subgraph of $\mathbb{AG}(S)$ induced by $\{\mathfrak{m}^i \mid i \in \{k, k+1, \ldots, 2k-1\}\}$ is a clique on k vertices. Hence, $\omega(\mathbb{AG}(S)) \geq k$. We next verify that $\chi(\mathbb{AG}(S)) \leq k$. Let $\{c_1, \ldots, c_k\}$ be a set of k distinct colors. Let us assign the color c_{i+1} to $\mathfrak{m}^{k+i}$ for each $i \in \{0, \ldots, k-1\}$. Let us assign the color c_j to $\mathfrak{m}^{k-j}$ for each $j \in \{1, \ldots, k-1\}$. It is easy to verify that the above assignment of colors is indeed a proper vertex coloring of $\mathbb{AG}(S)$. Hence, we obtain that $\chi(\mathbb{AG}(S)) \leq k \leq \omega(\mathbb{AG}(S)) \leq \chi(\mathbb{AG}(S))$. Therefore,

$\omega(\mathbb{AG}(S)) = \chi(\mathbb{AG}(S)) = k$.

(ii) Observe that the subgraph of $\mathbb{AG}(S)$ induced by $\{\mathfrak{m}^i \mid i \in \{k, k+1, \ldots, 2k\}\}$ is a clique on $k+1$ vertices. Hence, $\omega(\mathbb{AG}(S)) \geq k+1$. Let $\{c_1, c_2 \ldots, c_{k+1}\}$ be a set of $k+1$ distinct colors. Let us assign the color c_{i+1} to $\mathfrak{m}^{k+i}$ for each $i \in \{0, 1, \ldots, k\}$. Let us assign the color c_j to $\mathfrak{m}^{k-j}$ for each $j \in \{1, \ldots, k-1\}$. It is easy to verify the above assignment of colors is a proper vertex coloring of $\mathbb{AG}(S)$. This proves that $\chi(\mathbb{AG}(S)) \leq k+1 \leq \omega(\mathbb{AG}(S)) \leq \chi(\mathbb{AG}(S))$. Therefore, $\omega(\mathbb{AG}(S)) = \chi(\mathbb{AG}(S)) = k+1$. $\square$

For any ring R, we denote the set of all proper ideals of R by $\mathbb{I}(R)$ and we denote the set $\mathbb{I}(R)\backslash\{(0)\}$ by $\mathbb{I}(R)^*$. Since any proper ideal of an Artinian ring R is an annihilating ideal of R, it follows that $\mathbb{I}(R) = \mathbb{A}(R)$.

Lemma 2.6. *Let D be an integral domain, $(S, \mathfrak{m})$ be a local Artinian ring with $\mathfrak{m} \neq (0)$, and $k \in \mathbb{N}$ be such that $\omega(\mathbb{AG}(S)) = \chi(\mathbb{AG}(S)) = k$. Let $R = D \times S$. Then $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = k+1$.*

Proof. Let $\{I_1, \ldots, I_k\} \subseteq \mathbb{A}(S)^*$ be such that the subgraph of $\mathbb{AG}(S)$ induced by $\{I_1, \ldots, I_k\}$ is a clique. Observe that the subgraph of $\mathbb{AG}(R)$ induced by $\{(0) \times I_1, \ldots, (0) \times I_k, D \times (0)\}$ is a clique. Hence, $\omega(\mathbb{AG}(R)) \geq k+1$. We next verify that $\chi(\mathbb{AG}(R)) \leq k+1$. Let $\{c_1, \ldots, c_{k+1}\}$ be a set of $k+1$ distinct colors. Since $\chi(\mathbb{AG}(S)) = k$, the vertices of $\mathbb{AG}(S)$ can be properly colored using $\{c_1, \ldots, c_k\}$. Let $V_i = \{I \in \mathbb{A}(S)^* \mid I \text{ receives color } c_i\}$ for each $i \in \{1, \ldots, k\}$. Observe that $\mathbb{A}(S)^* = \bigcup_{i=1}^{k} V_i$. Since S is Artinian, $\mathbb{I}(S) = \mathbb{A}(S)$. Let $W_i = \{(0) \times I \mid I \in V_i\}$ for each $i \in \{1, \ldots, k\}$. Let $V = \{A \times I \mid A \in \mathbb{I}(D)^* \cup \{D\}, I \in \mathbb{A}(S)\}$. It is easy to verify that $\mathbb{A}(R)^* = (\bigcup_{i=1}^{k} W_i) \cup V \cup \{(0) \times S\}$. Let us assign the color c_i to all the members of W_i for each $i \in \{1, \ldots, k\}$, assign the color c_{k+1} to all the members of V, and assign the color c_1 to $(0) \times S$. It is clear that the above assignment of colors is a proper vertex coloring of $\mathbb{AG}(R)$. Therefore, $\chi(\mathbb{AG}(R)) \leq k+1 \leq \omega(\mathbb{AG}(R)) \leq \chi(\mathbb{AG}(R))$. This proves that $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = k+1$. $\square$

Lemma 2.7. *Let T be a reduced ring such that $|Min(T)| = n$ for some $n \in \mathbb{N}$ with $n \geq 2$. Let $k \in \mathbb{N}$ and $(S, \mathfrak{m})$ be a local Artinian ring which is not a field such that $\omega(\mathbb{AG}(S)) = \chi(\mathbb{AG}(S)) = k$. Let $R = T \times S$. Then $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = n+k$.*

Proof. As T is a reduced ring with $|Min(T)| = n \geq 2$, we obtain from ([11], Corollary 2.11) that $\omega(\mathbb{AG}(T)) = \chi(\mathbb{AG}(T)) = n$. Let $\{I_i \mid i \in \{1, 2, \ldots, n\}\} \subseteq \mathbb{A}(T)^*$ be such that the subgraph of $\mathbb{AG}(T)$ induced by $\{I_1, I_2, \ldots, I_n\}$ is a clique. By hypothesis, $\omega(\mathbb{AG}(S)) = \chi(\mathbb{AG}(S)) = k$. Let $\{J_1, \ldots, J_k\} \subseteq \mathbb{A}(S)^*$ be such that the subgraph of $\mathbb{AG}(S)$ induced by $\{J_1, \ldots, J_k\}$ is a clique. Notice that the subgraph of $\mathbb{AG}(R)$ induced by

$\{I_i \times (0) \mid i \in \{1, \ldots, n\}\} \cup \{(0) \times J_j \mid j \in \{1, \ldots, k\}\}$ is a clique. Therefore, $\omega(\mathbb{AG}(R)) \geq n+k$. We next verify that $\chi(\mathbb{AG}(R)) \leq n+k$. Let $\{c_1, c_2, \ldots, c_{n+k}\}$ be a set of $n+k$ distinct colors. Since $\chi(\mathbb{AG}(T)) = n$, the vertices of $\mathbb{AG}(T)$ can be properly colored using $\{c_1, c_2 \ldots, c_n\}$. Let $V_i = \{I \in \mathbb{A}(T)^* \mid I$ receives color $c_i\}$ for each $i \in \{1, 2, \ldots, n\}$. Observe that $\mathbb{A}(T)^* = \bigcup_{i=1}^{n} V_i$. As $\chi(\mathbb{AG}(S)) = k$, the vertices of $\mathbb{AG}(S)$ can be properly colored using $\{c_{n+1}, \ldots, c_{n+k}\}$. Let $U_j = \{J \in \mathbb{A}(S)^* \mid J$ receives color $c_{n+j}\}$ for each $j \in \{1, \ldots, k\}$. Since S is Artinian, it follows that $\mathbb{I}(S) = \mathbb{A}(S)$. Notice that $\mathbb{A}(S)^* = \bigcup_{j=1}^{k} U_j$. For each $i \in \{1, 2, \ldots, n\}$, let $W_i = \{I \times J \mid I \in V_i, J \in \mathbb{I}(S) \cup \{S\}\}$. Let $V = \{I \times J \mid I \in (\mathbb{I}(T) \backslash \mathbb{A}(T)) \cup \{T\}, J \in \mathbb{I}(S)\}$. It is easy to verify that $\mathbb{A}(R)^* = (\bigcup_{i=1}^{n} W_i) \cup V \cup (\bigcup_{j=1}^{k} \{(0) \times J \mid J \in U_j\}) \cup \{(0) \times S\}$. We now color the vertices of $\mathbb{AG}(R)$ as follows: Let us assign the color c_i to all the elements of W_i for each $i \in \{1, 2, \ldots, n\}$, assign the color c_1 to all the elements of V, assign the color c_{n+j} to all the elements of $\{(0) \times J | J \in U_j\}$ for each $j \in \{1, \ldots, k\}$, and assign the color c_{n+1} to $(0) \times S$. The above assignment of colors using a set of $n+k$ colors is indeed a proper vertex coloring of $\mathbb{AG}(R)$. This proves that $\chi(\mathbb{AG}(R)) \leq n+k \leq \omega(\mathbb{AG}(R)) \leq \chi(\mathbb{AG}(R))$. Hence, we obtain that $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = n+k$. □

Corollary 2.8. *Let T be a reduced ring and $n \in \mathbb{N} \backslash \{1\}$ be such that $|Min(T)| = n$. Let $(S, \mathfrak{m})$ be an SPIR and $t \geq 2$ be least with the property that $\mathfrak{m}^t = (0)$. Let $R = T \times S$. Then the following statements hold:*

(i) *If $t = 2k$ for some $k \geq 1$, then $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = n+k$.*

(ii) *If $t = 2k+1$ for some $k \geq 1$, then $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = n+k+1$.*

Proof. (i) We know from Lemma 2.5(i) that $\omega(\mathbb{AG}(S)) = \chi(\mathbb{AG}(S)) = k$. It now follows immediately from Lemma 2.7 that $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = n+k$.
(ii) By Lemma 2.5(ii), we get that $\omega(\mathbb{AG}(S)) = \chi(\mathbb{AG}(S)) = k+1$. Hence, we obtain from Lemma 2.7 that $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = n+k+1$. □

3. Characterization of zero-dimensional rings R with $|Max(R)| = 4$ such that $\omega(\mathbb{AG}(R)) \leq 4$

Let R be a zero-dimensional ring such that $|Max(R)| = 4$. It follows from ([17], Lemma 3.15) that there exist zero-dimensional rings R_1, R_2, R_3, R_4 such that $R \cong R_1 \times R_2 \times R_3 \times R_4$ as rings. Since $|Max(R)| = 4$ by assumption, it follows that R_i is quasi-local for each $i \in \{1, 2, 3, 4\}$. The aim of this section is to characterize such rings R in order that $\mathbb{AG}(R)$ to satisfy (B).

Lemma 3.1. *Let $R = R_1 \times R_2 \times R_3 \times R_4$, where R_i is a ring for each $i \in \{1, 2, 3, 4\}$. Suppose that R is not reduced. Then the following statements are equivalent:*

(i) $\mathbb{AG}(R)$ *satisfies* (B).

(ii) R_i *is an integral domain for exactly three values of* $i \in \{1,2,3,4\}$ *and if* $j \in \{1,2,3,4\}$ *is such that* R_j *is not an integral domain, then* R_j *is an SPIR with the square of its unique maximal ideal equals the zero ideal.*

(iii) $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = 4$.

Proof. $(i) \Rightarrow (ii)$ Since R is not reduced, it follows that R_j is not an integral domain for at least one $j \in \{1,2,3,4\}$. As $\mathbb{AG}(R)$ satisfies (B) by assumption, we obtain from Lemma 2.4 that there is exactly one $j \in \{1,2,3,4\}$ such that $\mathbb{A}(R_j)^* \neq \emptyset$ and it follows from Lemma 2.3 that R_j has only one non-zero annihilating ideal. Hence, $(R_j, Z(R_j))$ is an SPIR with $(Z(R_j))^2 = (0)$. It is clear that for a ring T, $\mathbb{A}(T)^* = \emptyset$ if and only if T is an integral domain. From the above arguments, we obtain $(i) \Rightarrow (ii)$.

$(ii) \Rightarrow (iii)$ Without loss of generality, we can assume that R_i is an integral domain for each $i \in \{1,2,3\}$. Notice that R_4 is an SPIR with unique non-zero maximal ideal $\mathfrak{m}_4$ such that $\mathfrak{m}_4^2 = (0)$. Let $T = R_1 \times R_2 \times R_3$. Observe that T is a reduced ring and $|Min(T)| = 3$. Since $R \cong T \times R_4$ as rings, it follows from Corollary 2.8(i) that $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = 4$.

$(iii) \Rightarrow (i)$ This is clear. $\square$

Let $(A, \mathfrak{m})$ be quasi-local. Suppose that A is reduced and zero-dimensional. Notice that $Spec(A) = \{\mathfrak{m}\}$. Since A is reduced, it follows from ([7], Proposition 1.8) that $\mathfrak{m} = (0)$ and so, A is a field.

Theorem 3.2. *Let R be a zero-dimensional ring with $|Max(R)| = 4$. Then the following statements are equivalent:*

(i) $\mathbb{AG}(R)$ *satisfies* (B).

(ii) *Either* $R \cong F_1 \times F_2 \times F_3 \times F_4$ *as rings, where* F_i *is a field for each* $i \in \{1,2,3,4\}$ *or* $R \cong F_1 \times F_2 \times F_3 \times R_4$ *as rings, where* F_i *is a field for each* $i \in \{1,2,3\}$ *and* R_4 *is not a field but* R_4 *is an SPIR with the square of its unique maximal ideal equals the zero ideal.*

(iii) $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = 4$.

Proof. Since R is a zero-dimensional ring with $|Max(R)| = 4$, it follows that $R \cong R_1 \times R_2 \times R_3 \times R_4$ as rings, where R_i is a zero-dimensional quasi-local ring for each $i \in \{1,2,3,4\}$.

$(i) \Rightarrow (ii)$ If R is reduced, then R_i is reduced for each $i \in \{1,2,3,4\}$. Since any zero-dimensional quasi-local reduced ring is a field (see the paragraph which appears just preceding the statement of this theorem), it follows that R_i is a field. With $F_i = R_i$ for each $ii \in \{1,2,3,4\}$, we obtain that F_i is a field and $R \cong F_1 \times F_2 \times F_3 \times F_4$ as rings. Suppose that R is not reduced. Then it follows from $(i) \Rightarrow (ii)$ of Lemma 3.1 that R_i is a field for exactly three values of $i \in \{1, 2, 3, 4\}$. Without loss of generality, we can assume that R_i is a field for each $i \in \{1, 2, 3\}$. Again it follows from $(i) \Rightarrow (ii)$ of Lemma 3.1 that R_4 is an SPIR with the square of its unique maximal ideal equals the zero ideal. With $F_i = R_i$ for each $i \in \{1, 2, 3\}$, we obtain that F_i is a field and $R \cong F_1 \times F_2 \times F_3 \times R_4$ as rings.

$(ii) \Rightarrow (iii)$ This follows from ([11], Corollary 2.11) and $(ii) \Rightarrow (iii)$ of Lemma 3.1.

$(iii) \Rightarrow (i)$ This is clear. □

4. Characterization of zero-dimensional rings R with $|Max(R)| = 3$ such that $\omega(\mathbb{AG}(R)) \leq 4$

The aim of this section is to characterize zero-dimensional rings R with $|Max(R)| = 3$ such that $\omega(\mathbb{AG}(R)) \leq 4$. Notice that it follows from ([17], Lemma 3.15) that there exist zero-dimensional rings R_1, R_2, R_3 such that $R \cong R_1 \times R_2 \times R_3$ as rings. Since $|Max(R)| = 3$ by assumption, it follows that R_i is quasi-local for each $i \in \{1, 2, 3\}$.

We first consider a ring R which is the direct product of three rings and try to determine necessary conditions on R for $\mathbb{AG}(R)$ to satisfy (B).

Lemma 4.1. *Let $R = R_1 \times R_2 \times R_3$, where R_i is a ring for each $i \in \{1, 2, 3\}$. If $\mathbb{AG}(R)$ satisfies (B), then there exists at most one $i \in \{1, 2, 3\}$ such that $|\mathbb{A}(R_i)^*| \geq 2$.*

Proof. Suppose that $|\mathbb{A}(R_i)^*| \geq 2$ for at least two values of $i \in \{1, 2, 3\}$. Without loss of generality, we can assume that $|\mathbb{A}(R_1)^*| \geq 2$ and $|\mathbb{A}(R_2)^*| \geq 2$. Notice that there exist distinct $I_{11}, I_{12} \in \mathbb{A}(R_1)^*$ (respectively, $I_{21}, I_{22} \in \mathbb{A}(R_2)^*$) such that $I_{11}I_{12} = (0)$ (respectively $I_{21}I_{22} = (0)$). It is clear that the subgraph of $\mathbb{AG}(R)$ induced by $\{I_1 = I_{11} \times (0) \times (0), I_2 = I_{12} \times (0) \times (0), I_3 = (0) \times I_{21} \times (0), I_4 = (0) \times I_{22} \times (0), I_5 = (0) \times (0) \times R_3\}$ is a clique and hence, $\omega(\mathbb{AG}(R)) \geq 5$. This contradicts the assumption $\mathbb{AG}(R)$ satisfies (B). Therefore, there exists at most one $i \in \{1, 2, 3\}$ such that $|\mathbb{A}(R_i)^*| \geq 2$. □

Lemma 4.2. *Let $R = R_1 \times R_2 \times R_3$, where R_i is a ring for each $i \in \{1, 2, 3\}$. If $\mathbb{AG}(R)$ satisfies (B), then R_i is reduced for at least one $i \in \{1, 2, 3\}$.*

Proof. Suppose that R_i is not reduced for each $i \in \{1, 2, 3\}$. Then there exists $a_i \in R_i \backslash (0)$ such that $a_i^2 = 0$ for each $i \in \{1, 2, 3\}$. Observe that the subgraph of $\mathbb{AG}(R)$ induced by $\{I_1 = R_1a_1 \times (0) \times (0), I_2 = (0) \times R_2a_2 \times (0), I_3 = R_1a_1 \times R_2a_2 \times (0), I_4 = (0) \times R_2a_2 \times R_3a_3, I_5 = (0) \times (0) \times R_3a_3\}$ is a clique. This implies that $\omega(\mathbb{AG}(R)) \geq 5$. This contradicts the assumption $\mathbb{AG}(R)$ satisfies (B). Therefore, R_i is reduced for at least one $i \in \{1, 2, 3\}$. □

Lemma 4.3. *Let $R = R_1 \times R_2 \times R_3$, where R_i is a ring for each $i \in \{1, 2, 3\}$. Suppose that $\mathbb{A}(R_1)^* \neq \emptyset$. If $AG(R)$ satisfies (B), then $\omega(\mathbb{AG}(R_1)) \leq 2$.*

Proof. Suppose that $\omega(\mathbb{AG}(R_1)) \geq 3$. Then there exist $I_{11}, I_{12}, I_{13} \in \mathbb{A}(R_1)^*$ such that the subgraph of $\mathbb{AG}(R_1)$ induced by $\{I_{11}, I_{12}, I_{13}\}$ is a clique. Notice that the subgraph of $\mathbb{AG}(R)$ induced by $\{I_1 = I_{11} \times (0) \times (0), I_2 = I_{12} \times (0) \times (0), I_3 = I_{13} \times (0) \times (0), I_4 = (0) \times R_2 \times (0), I_5 = (0) \times (0) \times R_3\}$ is a clique. This is impossible, since $\mathbb{AG}(R)$ satisfies (B). Therefore, $\omega(\mathbb{AG}(R_1)) \leq 2$. □

Lemma 4.4. *Let $R = R_1 \times R_2 \times R_3$, where R_i is a ring for each $i \in \{1, 2, 3\}$. Suppose that R_1 is not reduced and $|\mathbb{A}(R_1)^*| \geq 2$. If $\mathbb{AG}(R)$ satisfies (B), then R_2 and R_3 must be integral domains.*

Proof. Since R_1 is not reduced by hypothesis, there exists $a_1 \in R_1 \backslash \{0\}$ such that $a_1^2 = 0$. Let $I_{11} = R_1 a_1$. By hypothesis, $|\mathbb{A}(R_1)^*| \geq 2$. Hence, there exists $I_{12} \in A(R_1)^*, I_{12} \neq I_{11}$ and $I_{11} I_{12} = (0)$. We first verify that R_2 is an integral domain. Suppose that R_2 is not an integral domain. Then there exist $a, b \in R_2 \backslash \{0\}$ such that $ab = 0$. Observe that the subgraph of $\mathbb{AG}(R)$ induced by $\{I_1 = I_{11} \times (0) \times (0), I_2 = I_{12} \times (0) \times (0), I_3 = I_{11} \times R_2 a \times (0), I_4 = (0) \times R_2 b \times (0), I_5 = (0) \times (0) \times R_3\}$ is a clique. This implies that $\omega(\mathbb{AG}(R)) \geq 5$, a contradiction. Therefore, R_2 is an integral domain. Similarly, it can be shown that R_3 is an integral domain. □

We often use the following Lemma 4.5 in the verification of several results of this article.

Lemma 4.5. *Let R be a ring and $a, b \in nil(R)$. If $Ra = Rab$, then $a = 0$.*

Proof. From $Ra = Rab$, it follows that $a = rab$ for some $r \in R$. Hence, $a(1 - rb) = 0$. Since $b \in nil(R)$, we obtain from ([7], Exercise 1, page 10) that $1 - rb \in U(R)$. Hence, from $a(1 - rb) = 0$, we get that $a = 0$. □

Lemma 4.6. *Let R be a ring with $|\mathbb{A}(R)^*| \geq 1$ and $m \in \mathbb{N}$. If $\omega(\mathbb{AG}(R)) \leq m$, then $(nil(R))^{2m} = (0)$.*

Proof. Suppose that $m = 1$. As $\omega(\mathbb{AG}(R)) = 1$ and $\mathbb{AG}(R)$ is connected by ([10], Theorem 2.1), it follows that $|\mathbb{A}(R)^*| = 1$. Hence, $(R, Z(R))$ is an SPIR with $Z(R) \neq (0)$ but $(Z(R))^2 = (0)$. Notice that $Z(R) = nil(R)$ and $(nil(R))^2 = (0)$. Therefore, in proving this lemma, we can assume that $m \geq 2$. Let $a \in nil(R)$. We assert that $a^{2m} = 0$. Suppose that $a^{2m} \neq 0$. Let $n \in \mathbb{N}$ be least with the property that $a^n = 0$. Then $n \geq 2m + 1$. Let $i \in \{1, 2, \ldots, m + 1\}$ and let $I_i = Ra^{n-i}$. It is clear that $I_i \neq (0)$. It follows from Lemma 4.5 that I_i, I_j are distinct ideals for all distinct $i, j \in \{1, 2, \ldots, m+1\}$. Observe that for all distinct $i, j \in \{1, 2, \ldots, m+1\}$, $i + j \leq 2m + 1$. Hence, $2n - (i + j) \geq 2n - (2m + 1)$. As $n \geq 2m + 1$, it follows that $2n - (2m + 1) \geq n$. Therefore, $I_iI_j = Ra^{2n-(i+j)} = (0)$ for all distinct $i, j \in \{1, 2, \ldots, m+1\}$. This shows that the subgraph of $\mathbb{AG}(R)$ induced by $\{I_1, \ldots, I_{m+1}\}$ is a clique. This implies that $\omega(\mathbb{AG}(R)) \geq m+1$. This contradicts $\omega(\mathbb{AG}(R)) \leq m$. Hence, $a^{2m} = 0$ for any $a \in nil(R)$.

Let $a, b_1, \ldots, b_m \in nil(R)$. We claim that $a^m \prod_{i=1}^{m} b_i = 0$. Suppose that $a^m \prod_{i=1}^{m} b_i \neq 0$. Let $I_1 = Ra^m$ and for each $j \in \mathbb{N}$ with $2 \leq j \leq m+1$, let $I_j = R(a^m \prod_{k=1}^{j-1} b_k)$. As $a^m \prod_{i=1}^{m} b_i \neq 0$, it follows from Lemma 4.5 that the non-zero ideals $I_1, \ldots I_{m+1}$ are all distinct. Moreover, from $a^{2m} = 0$, it follows that $I_iI_j = (0)$ for all distinct $i, j \in \{1, \ldots, m+1\}$. Hence, the subgraph of $\mathbb{AG}(R)$ induced by $\{I_1, \ldots, I_{m+1}\}$ is a clique. This implies that $\omega(\mathbb{AG}(R)) \geq m + 1$. This contradicts $\omega(\mathbb{AG}(R)) \leq m$. Thus for any $a, b_1, \ldots, b_m \in nil(R)$, $a^m \prod_{i=1}^{m} b_i = 0$.

Let k be a non-negative integer such that $k < m$. Assume that we have proved, for any $a, b_1, \ldots, b_{m+k} \in nil(R)$, $a^{m-k} \prod_{i=1}^{m+k} b_i = 0$. Suppose that $k + 1 < m$. Let $a, b_1, \ldots, b_{m+k+1} \in nil(R)$. We claim that $a^{m-k-1} \prod_{i=1}^{m+k+1} b_i = 0$. Suppose that $a^{m-k-1} \prod_{i=1}^{m+k+1} b_i \neq 0$. Let $t \in \{1, \ldots, m+1\}$ and let $I_t = R(a^{m-k-1} \prod_{j=1}^{k+t} b_j)$. It is clear that $I_t \neq (0)$. Since $a^{m-k-1} \prod_{j=1}^{m+k+1} b_j \neq 0$, it follows from Lemma 4.5 that $I_i \neq I_j$ for all distinct $i, j \in \{1, \ldots, m+1\}$. Let $t_1, t_2 \in \{1, \ldots, m+1\}$ with $t_1 < t_2$. Observe that $I_{t_1}I_{t_2} = R(a^{m-k}a^{m-k-2}(\prod_{j=1}^{k+t_1} b_j^2)(\prod_{j=k+t_1+1}^{k+t_2} b_j))$. Notice that $m - k - 2 + 2(k + t_1) + t_2 - t_1 = m + k + t_1 + t_2 - 2 > m + k$. As $a^{m-k}c_1c_2 \ldots c_{m+k} = 0$ for any $a, c_1, \ldots, c_{m+k} \in nil(R)$, we obtain that $I_{t_1}I_{t_2} = (0)$ for all distinct $t_1, t_2 \in \{1, \ldots, m+1\}$. This shows that the subgraph of $\mathbb{AG}(R)$ induced by $\{I_1, \ldots, I_{m+1}\}$ is a clique. This contradicts $\omega(\mathbb{AG}(R)) \leq m$. Thus for any $a, b_1, \ldots, b_{m+k+1} \in nil(R)$, $a^{m-k-1} \prod_{i=1}^{m+k+1} b_i = 0$. This shows that for all integers s with $1 \leq s < m$, for any $a, b_1, \ldots, b_{m+s} \in nil(R)$, $a^{m-s} \prod_{i=1}^{m+s} b_i = 0$. Hence, on applying with $s = m - 1$, we obtain that for any $a, b_1, \ldots, b_{2m-1} \in nil(R)$, $a \prod_{i=1}^{2m-1} b_i = 0$. This proves that $(nil(R))^{2m} = (0)$. □

Lemma 4.7. *Let I be a non-zero nilpotent ideal of a ring R and $n \in \mathbb{N}$ be least with the property that $I^n = (0)$. Let $i \in \mathbb{N}$ be such that $i < n$. If an ideal J of R with $J \subseteq I^i$ is such that $I^i = J + I^{i+1}$, then $J = I^i$.*

Proof. From $I^i = J + I^{i+1}$, it follows that $I^i = J + I^iI = J + (J + I^{i+1})I = J + I^{i+2}$. Hence, $I^i = J + I^iI^2 = J + (J + I^{i+2})I^2 = J + I^{i+4}$. Proceeding in this way, we obtain that $I^i = J + I^{i+2^k}$ for all $k \geq 1$. It follows from $I^n = (0)$ that $I^i = J$. □

Lemma 4.8. *Let $m \in \mathbb{N}$ and R be a ring with $|\mathbb{A}(R)^*| \geq 1$. If $\omega(\mathbb{AG}(R)) \leq m$ and $(nil(R))^{2m-1} \neq (0)$, then $nil(R)$ is principal.*

Proof. If $m = 1$, then as is remarked in the proof of Lemma 4.6, we know that $(R, Z(R))$ is an SPIR with $Z(R) \neq (0)$ but $(Z(R))^2 = (0)$. Thus $Z(R) = nil(R)$ is principal. Hence, in proving this lemma, we can assume that $m \geq 2$. For convenience, let us denote $nil(R)$ by $\mathfrak{n}$. We know from Lemma 4.6 that $\mathfrak{n}^{2m} = (0)$. By hypothesis, $\mathfrak{n}^{2m-1} \neq (0)$. Hence, $\mathfrak{n}^m \neq \mathfrak{n}^{m+1}$. Let $x \in \mathfrak{n}^m \backslash \mathfrak{n}^{m+1}$. We claim that $\mathfrak{n}^m = Rx$. Suppose not. Then it follows from Lemma 4.7 that $\mathfrak{n}^m \neq Rx + \mathfrak{n}^{m+1}$. Let $y \in \mathfrak{n}^m \backslash (Rx + \mathfrak{n}^{m+1})$. Let $I_1 = \mathfrak{n}^{m+1}, \ldots, I_{m-1} = \mathfrak{n}^{2m-1}, I_m = Rx$, and $I_{m+1} = Ry$. It is clear from the choice of the elements x, y and from the hypothesis $\mathfrak{n}^{2m-1} \neq (0)$ that I_i, I_j are distinct non-zero ideals for all distinct $i, j \in \{1, \ldots, m+1\}$. It follows from $\mathfrak{n}^{2m} = (0)$ that the subgraph of $\mathbb{AG}(R)$ induced by $\{I_1, \ldots, I_{m+1}\}$ is a clique. This implies that $\omega(\mathbb{AG}(R)) \geq m+1$. This contradicts $\omega(\mathbb{AG}(R)) \leq m$. Thus for any $x \in \mathfrak{n}^m \backslash \mathfrak{n}^{m+1}$, $\mathfrak{n}^m = Rx$.

Since $\mathfrak{n}^{2m-1} \neq (0)$, it follows that $\mathfrak{n}^{2m-2}a \neq (0)$ for some $a \in \mathfrak{n}$. It follows from $\mathfrak{n}^{2m} = (0)$ that $a \notin \mathfrak{n}^2$. We assert that $\mathfrak{n} = Ra + \mathfrak{n}^2$. Suppose not. Then there exists $b \in \mathfrak{n} \backslash (Ra + \mathfrak{n}^2)$. It follows from $\mathfrak{n}^{2m-2}a \neq (0)$ that either $\mathfrak{n}^{2m-2}b \neq (0)$ or $\mathfrak{n}^{2m-2}(a+b) \neq (0)$. It is clear that $a + b \in \mathfrak{n} \backslash (Ra + \mathfrak{n}^2)$. Hence, on replacing b by $a + b$ if necessary, we can assume without loss of generality that $\mathfrak{n}^{2m-2}b \neq (0)$. Since $\mathfrak{n}^{2m} = (0)$ but $\mathfrak{n}^{2m-2}a \neq (0)$, it follows that $\mathfrak{n}^{m-1}a \not\subseteq \mathfrak{n}^{m+1}$. Similarly, it follows from $\mathfrak{n}^{2m-2}b \neq (0)$ that $\mathfrak{n}^{m-1}b \not\subseteq \mathfrak{n}^{m+1}$. Let $c \in (\mathfrak{n}^{m-1}a) \backslash \mathfrak{n}^{m+1}$. Then it follows from the previous paragraph that $\mathfrak{n}^m = Rc = \mathfrak{n}^{m-1}a$. Let $d \in (\mathfrak{n}^{m-1}b) \backslash \mathfrak{n}^{m+1}$. Then $Rd = \mathfrak{n}^m = \mathfrak{n}^{m-1}b$. This shows that $\mathfrak{n}^{m-1}a = \mathfrak{n}^{m-1}b$. Hence, we obtain that $\mathfrak{n}^m a = \mathfrak{n}^m b$. Let $x \in \mathfrak{n}^m \backslash \mathfrak{n}^{m+1}$. Then $\mathfrak{n}^m = Rx$. Therefore, it follows that $Rxa = Rxb$. Hence, there exists $r \in R$ such that $xb = rxa$ and so, $x(b - ra) = 0$. Notice that $b - ra \in \mathfrak{n} \backslash \mathfrak{n}^2$. Moreover, $\mathfrak{n}^m(b - ra) = (0)$. Let $I_1 = \mathfrak{n}^m, I_2 = \mathfrak{n}^{m+1}, \ldots, I_m = \mathfrak{n}^{2m-1}$, and $I_{m+1} = R(b - ra)$. From the choice of the elements a, b and from the hypothesis $\mathfrak{n}^{2m-1} \neq (0)$, we obtain that I_i, I_j are distinct non-zero ideals for all distinct $i, j \in \{1, \ldots, m+1\}$. As $\mathfrak{n}^{2m} = (0)$ and $\mathfrak{n}^m(b - ra) = (0)$, it follows that the subgraph of $\mathbb{AG}(R)$ induced by $\{I_1, \ldots, I_{m+1}\}$ is a clique. This contradicts $\omega(\mathbb{AG}(R)) \leq m$. Hence, $\mathfrak{n} = Ra + \mathfrak{n}^2$. Therefore, it follows from Lemma 4.7 that $\mathfrak{n} = Ra$. This proves that $nil(R)$ is principal. $\square$

Lemma 4.9. *Let $m \in \mathbb{N}$ be such that $m \geq 2$ and R be a ring such that $(nil(R))^{2m-1} = (0)$ but $(nil(R))^{2m-2} \neq (0)$. If $\omega(\mathbb{AG}(R)) \leq m$, then $(nil(R))^j$ is principal for each $j \in \mathbb{N}$ such that $m \leq j \leq 2m - 2$.*

Proof. Let $j \in \mathbb{N}$ be such that $m \leq j \leq 2m - 2$. It is convenient to denote $nil(R)$ by $\mathfrak{n}$. It follows from $\mathfrak{n}^{2m-1} = (0)$, whereas $\mathfrak{n}^{2m-2} \neq (0)$ that $\mathfrak{n}^j \neq \mathfrak{n}^{j+1}$. Let $x \in \mathfrak{n}^j \backslash \mathfrak{n}^{j+1}$. We assert that $\mathfrak{n}^j = Rx$. Suppose that $\mathfrak{n}^j \neq Rx$. Then it follows from Lemma 4.7 that there exists $y \in \mathfrak{n}^j \backslash (Rx + \mathfrak{n}^{j+1})$. Let $\mathcal{A} = \{\mathfrak{n}^t \mid t \in \{m-1, m, \ldots, 2m-2\} \backslash \{j\}\}$. It is clear that $\mathcal{A} \cup \{Rx, Ry\}$ is a collection of $m + 1$ distinct and non-zero ideals of R. It follows from

$2j \geq 2m$ and $\mathfrak{n}^{2m-1} = (0)$ that the subgraph of $\mathbb{AG}(R)$ induced by $\mathcal{A} \cup \{Rx, Ry\}$ is a clique. This implies that $\omega(\mathbb{AG}(R)) \geq m+1$. This contradicts $\omega(\mathbb{AG}(R)) \leq m$. Thus if $j \in \mathbb{N}$ with $m \leq j \leq 2m-2$, then for any $x \in \mathfrak{n}^j \backslash \mathfrak{n}^{j+1}$, $\mathfrak{n}^j = Rx$. This proves that $\mathfrak{n}^j$ is principal for each $j \in \{m, \ldots, 2m-2\}$. □

Lemma 4.10. *Let $m \in \mathbb{N}$ be such that $m \geq 2$ and R be a ring such that $(nil(R))^{2m-1} = (0)$. Let $z \in (nil(R))^{m-1}$ be such that $z^2 \neq (0)$. If $\omega(\mathbb{AG}(R)) \leq m$, then $nil(R)$ is principal.*

Proof. It is convenient to denote $nil(R)$ by $\mathfrak{n}$. We are assuming that $z^2 \neq 0$ for some $z \in \mathfrak{n}^{m-1}$. Since $z^2 \in \mathfrak{n}^{2m-2}$, it follows that $\mathfrak{n}^{2m-2} \neq (0)$. Let $x \in \mathfrak{n}^{2m-2}, x \neq 0$. It follows from the proof of Lemma 4.9 that $\mathfrak{n}^{2m-2} = Rx$. Since $z^2 \in \mathfrak{n}^{2m-2} \backslash \{0\}$, we obtain that $\mathfrak{n}^{2m-2} = Rz^2$.

From $\mathfrak{n}^{2m-1} = (0)$, whereas $\mathfrak{n}^{2m-2} \neq (0)$, it follows that $\mathfrak{n}^m \neq \mathfrak{n}^{m+1}$. Let $x \in \mathfrak{n}^m \backslash \mathfrak{n}^{m+1}$. It follows from the proof of Lemma 4.9 that $\mathfrak{n}^m = Rx$.

Since $\mathfrak{n}^{2m-2} \neq (0)$, we obtain that $\mathfrak{n}^{2m-3}a \neq (0)$ for some $a \in \mathfrak{n}$. By hypothesis, $\mathfrak{n}^{2m-1} = (0)$. Hence, $a \notin \mathfrak{n}^2$. We claim that $\mathfrak{n} = Ra + \mathfrak{n}^2$. Suppose not. Then there exists $b \in \mathfrak{n} \backslash (Ra + \mathfrak{n}^2)$. It is clear that either $\mathfrak{n}^{2m-3}b \neq (0)$ or $\mathfrak{n}^{2m-3}(a+b) \neq (0)$. Notice that $a + b \in \mathfrak{n} \backslash (Ra + \mathfrak{n}^2)$. Therefore, on replacing b by $a+b$ if necessary, we can assume without loss of generality that $\mathfrak{n}^{2m-3}b \neq (0)$. It follows from $\mathfrak{n}^{2m-3}a \neq (0), \mathfrak{n}^{2m-3}b \neq (0)$, $\mathfrak{n}^{2m-1} = (0)$ that $\mathfrak{n}^{m-1}a \not\subseteq \mathfrak{n}^{m+1}$ and $\mathfrak{n}^{m-1}b \not\subseteq \mathfrak{n}^{m+1}$. Since for any $x \in \mathfrak{n}^m \backslash \mathfrak{n}^{m+1}$, $\mathfrak{n}^m = Rx$, we get that $\mathfrak{n}^{m-1}a = \mathfrak{n}^{m-1}b$.

We next verify that $\mathfrak{n}^{m-1} = Rz$. Consider the map $f : \mathfrak{n}^{m-1} \to \mathfrak{n}^{2m-2}$ defined by $f(w) = wz$. It is clear that f is a homomorphism of R-modules. Since $z \in \mathfrak{n}^{m-1}$ and $\mathfrak{n}^{2m-2} = Rz^2$, we obtain that f is onto. As $\mathfrak{n}^{2m-1} = (0)$, we get that $\mathfrak{n}^m \subseteq Ker(f)$. It is clear from the definition of f that $(Rz)Ker(f) = (0)$. It follows from $z^2 \neq 0$ that $z \notin Ker(f)$. We claim that $Ker(f) = \mathfrak{n}^m$. Suppose that $Ker(f) \neq \mathfrak{n}^m$. Let $I_i = \mathfrak{n}^{m+i-1}$ for each $i \in \{1, \ldots, m-1\}$, $I_m = Rz$, and $I_{m+1} = Ker(f)$. It is clear from the above discussion that I_i, I_j are distinct non-zero ideals for all distinct $i, j \in \{1, \ldots, m+1\}$. Moreover, it follows from $(Rz)Ker(f) = (0)$ and $\mathfrak{n}^{2m-1} = (0)$ that the subgraph of $\mathbb{AG}(R)$ induced by $\{I_1, I_2, \ldots, I_{m+1}\}$ is a clique. This is impossible, since $\omega(\mathbb{AG}(R)) \leq m$. Therefore, $Ker(f) = \mathfrak{n}^m$. Now, we obtain from the fundamental theorem of homomorphism of modules that $\frac{\mathfrak{n}^{m-1}}{Ker(f) = \mathfrak{n}^m} \cong \mathfrak{n}^{2m-2}$ as R-modules. As $\mathfrak{n}^{2m-2}$ is generated by any non-zero element of it and $z \notin Ker(f)$, it follows that $\mathfrak{n}^{m-1} = Rz + \mathfrak{n}^m$. Hence, we obtain from Lemma 4.7 that $\mathfrak{n}^{m-1} = Rz$.

It follows from $\mathfrak{n}^{m-1}a = \mathfrak{n}^{m-1}b$ and $\mathfrak{n}^{m-1} = Rz$ that $Rza = Rzb$. Hence, $z(b - ra) = 0$ for some $r \in R$. Let $I_i = \mathfrak{n}^{m-1+i-1}$ for each $i \in \{1, \ldots, m\}$, and $I_{m+1} = R(b - ra)$. From $\mathfrak{n}^{2m-2} \neq (0), \mathfrak{n}^{2m-1} = (0)$, it follows that I_i, I_j are distinct non-zero ideals for all distinct $i, j \in \{1, \ldots, m+1\}$ and moreover, the subgraph of $\mathbb{AG}(R)$ induced by $\{I_1, \ldots, I_{m+1}\}$ is a clique. This is impossible. Therefore, $\mathfrak{n} = Ra + \mathfrak{n}^2$ and so, we obtain from Lemma 4.7 that $nil(R) = \mathfrak{n} = Ra$ is principal. □

Lemma 4.11. *Let $m \in \mathbb{N}$ be such that $m \geq 2$ and R be a ring such that $(nil(R))^{2m-1} = (0)$. Suppose that $(nil(R))^{2m-2} \neq (0)$, whereas $z^2 = 0$ for each $z \in (nil(R))^{m-1}$. If $\omega(\mathbb{AG}(R)) \leq m$, then $(nil(R))^i$ can be generated by two elements for each $i \in \{1, \ldots, m-1\}$ and $(nil(R))^{m-1}$ is not principal. Moreover, $m = 2$.*

Proof. It is convenient to denote $nil(R)$ by $\mathfrak{n}$. Let $i \in \{1, \ldots, m-1\}$. By hypothesis, $\mathfrak{n}^{2m-2} \neq (0)$. Hence, there exist elements $a_1, \ldots, a_{2m-2} \in \mathfrak{n}$ such that $\prod_{k=1}^{2m-2} a_k \neq 0$. Consider the map $f : \mathfrak{n}^i \to \mathfrak{n}^{m+i-1}$ defined by $f(x) = x(\prod_{s=i+1}^{m+i-1} a_s)$. It is clear that f is a homomorphism of R-modules. It follows from $\mathfrak{n}^{2m-1} = (0)$, $\prod_{k=1}^{2m-2} a_k \neq 0$ that $\prod_{t=1}^{m+i-1} a_t \in \mathfrak{n}^{m+i-1} \backslash \mathfrak{n}^{m+i}$. Hence, we obtain from the proof of Lemma 4.9 that $\mathfrak{n}^{m+i-1} = R(\prod_{t=1}^{m+i-1} a_t)$ and this implies that f is onto. Observe that $\prod_{s=i+1}^{m+i-1} a_s \in \mathfrak{n}^{m-1}$ and as $z^2 = 0$ for each $z \in \mathfrak{n}^{m-1}$, it follows that $\prod_{s=i+1}^{m+i-1} a_s \in Ker(f)$. We claim that $Ker(f) = R(\prod_{s=i+1}^{m+i-1} a_s)$. Suppose that $Ker(f) \neq R(\prod_{s=i+1}^{m+i-1} a_s)$. From the definition of f, it is clear that $(\prod_{s=i+1}^{m+i-1} a_s)Ker(f) = (0)$. Notice that $\prod_{s=i}^{m+i-1} a_s \in \mathfrak{n}^m \backslash \mathfrak{n}^{m+1}$. Hence, from the proof of Lemma 4.9, we obtain that $\mathfrak{n}^m = R(\prod_{s=i}^{m+i-1} a_s)$. Therefore, $\mathfrak{n}^m Ker(f) = (0)$. As $\prod_{s=i+1}^{m+i-1} a_s \in Ker(f)$ and $\prod_{s=i+1}^{m+i-1} a_s \in \mathfrak{n}^{m-1} \backslash \mathfrak{n}^m$, it follows that $Ker(f) \notin \{\mathfrak{n}^j \mid j \in \{m, \ldots, 2m-2\}\}$. Notice that $\mathcal{A} = \{\mathfrak{n}^j \mid j \in \{m, \ldots, 2m-2\}\} \cup \{Ker(f), R(\prod_{s=i+1}^{m+i-1} a_s)\}$ is a set consisting of $m+1$ distinct and non-zero ideals of R. Since $\mathfrak{n}^{2m-1} = (0)$ and $(\prod_{s=i+1}^{m+i-1} a_s)Ker(f) = (0)$, we obtain that the subgraph of $\mathbb{AG}(R)$ induced by $\mathcal{A} \cup \{Ker(f), R(\prod_{s=i+1}^{m+i-1} a_s)\}$ is a clique. This implies that $\omega(\mathbb{AG}(R)) \geq m+1$ and this is a contradiction. Hence, $Ker(f) = R(\prod_{s=i+1}^{m+i-1} a_s)$. It now follows from the fundamental theorem of homomorphism of modules that $\frac{\mathfrak{n}^i}{Ker(f) = R(\prod_{s=i+1}^{m+i-1} a_s)} \cong \mathfrak{n}^{m+i-1}$ as R-modules. It follows from $\mathfrak{n}^{m+i-1} = R(\prod_{s=1}^{m+i-1} a_s)$ that $\mathfrak{n}^i = R(\prod_{s=1}^{i} a_s) + R(\prod_{s=i+1}^{m+i-1} a_s)$. This proves that $(nil(R))^i$ is two generated for each $i \in \{1, \ldots, m-1\}$. If there exists $i \in \mathbb{N}$ with $i < m-1$, then from $\mathfrak{n}^i = R(\prod_{s=1}^{i} a_s) + R(\prod_{s=i+1}^{m+i-1} a_s)$, we obtain that $\mathfrak{n}^i = R(\prod_{s=1}^{i} a_s) + \mathfrak{n}^{i+1}$. Hence, it follows from Lemma 4.7 that $\mathfrak{n}^i = R(\prod_{s=1}^{i} a_s)$ is principal. By hypothesis, $z^2 = 0$ for each $z \in \mathfrak{n}^{m-1}$. As $\mathfrak{n}^{2m-2} \neq (0)$, it follows that $(nil(R))^{m-1}$ is not principal. We next verify that $m = 2$. Suppose that $m \geq 3$. Then $1 < m-1$. Therefore, $\mathfrak{n}$ is principal and so, $\mathfrak{n}^{m-1}$ is principal. This is a contradiction. Therefore, $m = 2$. $\square$

Lemma 4.12. *Let R be a ring such that $nil(R) \neq (0)$, but $(nil(R))^2 = (0)$. If $\omega(\mathbb{AG}(R)) \leq 2$, then $nil(R)$ is principal.*

Proof. Suppose that $nil(R)$ is not principal. Let $x \in nil(R), x \neq 0$. Now, there exists $y \in nil(R) \backslash Rx$. From $(nil(R))^2 = (0)$, it follows that the subgraph of $\mathbb{AG}(R)$ induced by $\{Rx, Ry, nil(R)\}$ is a clique. This implies that $\omega(\mathbb{AG}(R)) \geq 3$, a contradiction. Therefore, we obtain that $nil(R)$ is principal. $\square$

Lemma 4.13. *Let* $(R, \mathfrak{m})$ *be a local Artinian ring with* $\mathfrak{m}^3 = (0), \mathfrak{m}^2 \neq (0)$*,* $z^2 = 0$ *for each* $z \in \mathfrak{m}$*, and* $\mathfrak{m}$ *is generated by two elements. Then* $\omega(\mathbb{AG}(R)) = \chi((\mathbb{AG}(R)) = 2$*.*

Proof. It is clear that the subgraph of $\mathbb{AG}(R)$ induced by $\{\mathfrak{m}, \mathfrak{m}^2\}$ is a clique. Therefore, $\omega(\mathbb{AG}(R)) \geq 2$. We next show that $\chi(\mathbb{AG}(R)) \leq 2$. We first verify that if I and J are any distinct non-zero proper ideals of R different from $\mathfrak{m}^2$, then $IJ \neq (0)$. By hypothesis, there exist $x, y \in \mathfrak{m}$ such that $\mathfrak{m} = Rx + Ry$. From $z^2 = 0$ for each $z \in \mathfrak{m}$, it follows that $\mathfrak{m}^2 = Rxy$. Now, $\mathfrak{m}^3 = (0)$ and so, $\mathfrak{m}^2 = Rxy$ is of dimension one regarded as a vector space over the field $\frac{R}{\mathfrak{m}}$. Since $\mathfrak{m}^2 \neq (0)$ and $z^2 = 0$ for each $z \in \mathfrak{m}$, it follows that $\mathfrak{m}$ is not principal. As $\mathfrak{m}$ is generated by two elements, it follows that $\frac{\mathfrak{m}}{\mathfrak{m}^2}$ is of dimension two regarded as a vector space over $\frac{R}{\mathfrak{m}}$. Let I be any non-zero proper ideal of R different from $\mathfrak{m}^2$. As $dim_{\frac{R}{\mathfrak{m}}}(\mathfrak{m}^2) = 1$, it follows that $I \not\subseteq \mathfrak{m}^2$. Let $a \in I \backslash \mathfrak{m}^2$. Notice that there exist $b \in \mathfrak{m} \backslash \mathfrak{m}^2$ such that $\{a + \mathfrak{m}^2, b + \mathfrak{m}^2\}$ is a basis of $\frac{\mathfrak{m}}{\mathfrak{m}^2}$ as a vector space over $\frac{R}{\mathfrak{m}}$. Hence, we obtain that $\mathfrak{m} = Ra + Rb$. Therefore, $\mathfrak{m}^2 = Rab \subseteq I\mathfrak{m}$. Similarly, it follows that if J is any non-zero proper ideal of R with $J \neq \mathfrak{m}^2$, then $\mathfrak{m}^2 \subseteq J\mathfrak{m}$. Let I, J be non-zero distinct proper ideals of R such that both are different from $\mathfrak{m}^2$. If $I = \mathfrak{m}$ or $J = \mathfrak{m}$, then it is clear that $\mathfrak{m}^2 \subseteq IJ$. Suppose that I and J are both different from $\mathfrak{m}$. Notice that $\mathfrak{m}^2 \subset I \subset \mathfrak{m}$ and $\mathfrak{m}^2 \subset J \subset \mathfrak{m}$. Thus $dim_{\frac{R}{\mathfrak{m}}}(\frac{I}{\mathfrak{m}^2}) = dim_{\frac{R}{\mathfrak{m}}}(\frac{J}{\mathfrak{m}^2}) = 1$. Hence, there exist $a \in I, b \in J$ such that $I = Ra + \mathfrak{m}^2$ and $J = Rb + \mathfrak{m}^2$. As $\mathfrak{m}^2 \subseteq I\mathfrak{m} \cap J\mathfrak{m}$, we obtain that $I = Ra + I\mathfrak{m}$ and $J = Rb + J\mathfrak{m}$. Therefore, it follows from ([7], Corollary 2.7) that $I = Ra$ and $J = Rb$. From $I \neq J$, it follows that $dim_{\frac{R}{\mathfrak{m}}}(\frac{I+J}{\mathfrak{m}^2}) = 2 = dim_{\frac{R}{\mathfrak{m}}}(\frac{\mathfrak{m}}{\mathfrak{m}^2})$. Therefore, $\mathfrak{m} = I + J = Ra + Rb$. Hence, $\mathfrak{m}^2 = Rab \subseteq IJ$. This proves that if I, J are distinct non-zero proper ideals of R which are both different from $\mathfrak{m}^2$, then $\mathfrak{m}^2 \subseteq IJ$ and so, $IJ \neq (0)$ and indeed, $IJ = \mathfrak{m}^2$. This proves that $\mathbb{AG}(R)$ is a star graph. (It is useful to mention here that the local Artinian ring $(R, \mathfrak{m})$ satisfies (ii) of ([10], Theorem 2.6).) Hence, it follows that $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = 2$. □

For any $n \geq 2$, we denote the ring of integers modulo n by $\mathbb{Z}_n$ and we denote the polynomial ring in one variable X (respectively, in two variables X, Y) over $\mathbb{Z}_n$ by $\mathbb{Z}_n[X]$ (respectively, $\mathbb{Z}_n[X, Y]$). We provide some examples in Example 4.14 to illustrate Lemma 4.13.

Example 4.14.

(i) Let $T = \mathbb{Z}_2[X, Y]$ and $I = TX^2 + TY^2$. Let $R = \frac{T}{I}$. It is easy to verify that R is a local Artinian ring with unique maximal ideal $\mathfrak{m} = \frac{TX+TY}{I}$ and R satisfies the hypotheses of Lemma 4.13. Hence, by Lemma 4.13, we obtain that $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = 2$.

(ii) Let $T = \mathbb{Z}_4[X, Y]$ and $I = TX^2 + TY^2 + T(XY - 2)$. Let $R = \frac{T}{I}$. It is easy to verify that R is a local Artinian ring with unique maximal ideal $\mathfrak{m} = \frac{TX+TY}{I}$ and R satisfies the hypotheses of Lemma 4.13. Therefore, we obtain from Lemma 4.13 that $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = 2$.

(iii) Let $T = \mathbb{Z}_4[X]$ and $I = TX^2$. Let $R = \frac{T}{I}$. It is easy to verify that R is a local Artinian ring with unique maximal ideal $\mathfrak{m} = \frac{T2+TX}{I}$ and R satisfies the hypotheses of Lemma 4.13. Therefore, it follows from Lemma 4.13 that $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = 2$.

Examples (i), (ii), and (iii) given above appeared in the list of local rings of order 16 given in ([12], p.475).

Lemma 4.15. *Let $i \in \{1, 2\}$ and $(R_i, \mathfrak{m}_i)$ be an SPIR with $\mathfrak{m}_i \neq (0)$ but $\mathfrak{m}_i^2 = (0)$ for each $i \in \{1, 2\}$. Let $R = R_1 \times R_2 \times F$, where F is a field. Then $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = 4$.*

Proof. Let $V_1 = \{R_1 \times (0) \times (0), R_1 \times (0) \times F, R_1 \times \mathfrak{m}_2 \times (0), R_1 \times \mathfrak{m}_2 \times F, R_1 \times R_2 \times (0)\}$, $V_2 = \{(0) \times R_2 \times (0), (0) \times R_2 \times F, \mathfrak{m}_1 \times R_2 \times (0), \mathfrak{m}_1 \times R_2 \times F\}$, $V_3 = \{(0) \times (0) \times F, (0) \times \mathfrak{m}_2 \times F, \mathfrak{m}_1 \times (0) \times F, \mathfrak{m}_1 \times \mathfrak{m}_2 \times F\}$, and $V_4 = \{\mathfrak{m}_1 \times (0) \times (0), (0) \times \mathfrak{m}_2 \times (0), \mathfrak{m}_1 \times \mathfrak{m}_2 \times (0)\}$. Observe that the subgraph of $\mathbb{AG}(R)$ induced by $V_4 \cup \{\mathfrak{m}_1 \times \mathfrak{m}_2 \times F\}$ is a clique. Hence, $\omega(\mathbb{AG}(R)) \geq 4$. We next verify that $\chi(\mathbb{AG}(R)) \leq 4$. Let $\{c_1, c_2, c_3, c_4\}$ be a set of four distinct colors. Notice that $\mathbb{A}(R)^* = \bigcup_{i=1}^4 V_i$ and $V_i \cap V_j = \emptyset$ for all distinct $i, j \in \{1, 2, 3, 4\}$. Moreover, observe that no two distinct members of V_i are adjacent in $\mathbb{AG}(R)$ for all $i \in \{1, 2, 3\}$. Let us assign the color c_i to all the members of V_i for each $i \in \{1, 2, 3\}$. Let us assign the color c_1 to $\mathfrak{m}_1 \times (0) \times (0)$, color c_2 to $(0) \times \mathfrak{m}_2 \times (0)$, and the color c_4 to $\mathfrak{m}_1 \times \mathfrak{m}_2 \times (0)$. It is clear that the above assignment of colors is indeed a proper vertex coloring of $\mathbb{AG}(R)$. This proves that $\chi(\mathbb{AG}(R)) \leq 4 \leq \omega(\mathbb{AG}(R)) \leq \chi(\mathbb{AG}(R))$. Therefore, we obtain that $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = 4$. □

The following theorem characterizes zero-dimensional rings R with $|Max(R)| = 3$ such that $\mathbb{AG}(R)$ satisfies (B).

Theorem 4.16. *Let R be a zero-dimensional ring with $|Max(R)| = 3$. Then the following statements are equivalent:*

(i) *$\mathbb{AG}(R)$ satisfies (B).*

(ii) *$R \cong R_1 \times R_2 \times R_3$ as rings, where $(R_i, \mathfrak{m}_i)$ is a zero-dimensional quasi-local ring for each $i \in \{1, 2, 3\}$ satisfying exactly one of the following:*

  (a) *R_i is a field for each $i \in \{1, 2, 3\}$.*

  (b) *Exactly two among R_1, R_2, R_3 are fields and if R_i is not a field, then either $(R_i, \mathfrak{m}_i)$ is an SPIR with $\mathfrak{m}_i^4 = (0)$ or $(R_i, \mathfrak{m}_i)$ is a local Artinian ring with $\mathfrak{m}_i^2 \neq (0), \mathfrak{m}_i^3 = (0), z^2 = 0$ for each $z \in \mathfrak{m}_i$, and moreover, $\mathfrak{m}_i$ is generated by two elements and is not principal.*

  (c) *Exactly one among R_1, R_2, R_3 is a field and if R_i and R_j are not fields, then $(R_i, \mathfrak{m}_i)$ (respectively, $(R_j, \mathfrak{m}_j)$) is an SPIR with $\mathfrak{m}_i^2 = (0)$ (respectively, $\mathfrak{m}_j^2 = (0)$).*

(iii) *If* (a) *of* (ii) *holds, then* $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = 3$. *If* (b) *of* (ii) *holds, then* $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) \in \{3, 4\}$. *If* (c) *of* (ii) *holds, then* $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = 4$.

Proof. $(i) \Rightarrow (ii)$ Since R is a zero-dimensional ring with $|Max(R)| = 3$, it follows that there exist zero-dimensional quasi-local rings R_1, R_2, R_3 such that $R \cong R_1 \times R_2 \times R_3$ as rings. Let $\mathfrak{m}_i$ denote the unique maximal ideal of R_i for each $i \in \{1, 2, 3\}$. Since $\mathbb{AG}(R)$ satisfies (B), it follows from Lemma 4.2 that R_i is reduced for at least one $i \in \{1, 2, 3\}$. We can assume without loss of generality that R_3 is reduced. Hence, R_3 is a field. If R_1, R_2 are also reduced, then we obtain that they are also fields. Therefore, (a) holds.

If exactly one between R_1 and R_2 is not reduced, then we can assume without loss of generality that R_1 is not reduced. Now R_2 and R_3 are fields. Notice that $\mathbb{A}(R_1)^* \neq \emptyset$. Since $\mathbb{AG}(R)$ satisfies (B), we obtain from Lemma 4.3 that $\omega(\mathbb{AG}(R_1)) \leq 2$. As $nil(R_1) = \mathfrak{m}_1$, we obtain from Lemma 4.6 that $\mathfrak{m}_1^4 = (0)$. If $\mathfrak{m}_1^3 \neq (0)$, then we obtain from Lemma 4.8 that $\mathfrak{m}_1$ is principal. In such a case, it follows from the proof of $(iii) \Rightarrow (i)$ of ([7], Proposition 8.8) that $\{\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3\}$ is the set of all non-zero proper ideals of R_1. Hence, we obtain that$(R_1, \mathfrak{m}_1)$ is an SPIR. If $\mathfrak{m}_1^3 = (0)$ and $z^2 \neq 0$ for some $z \in \mathfrak{m}_1$, then it follows from Lemma 4.10 that $\mathfrak{m}_1$ is principal. Therefore, $(R_1, \mathfrak{m}_1)$ is an SPIR with $\{\mathfrak{m}_1, \mathfrak{m}_1^2\}$ is the set of all non-zero proper ideals of R_1. If $\mathfrak{m}_1^3 = (0), \mathfrak{m}_1^2 \neq (0)$ but $z^2 = 0$ for each $z \in \mathfrak{m}_1$, then we obtain from Lemma 4.11 that $\mathfrak{m}_1$ is not principal and there exist $a, b \in \mathfrak{m}_1$ such that $\mathfrak{m}_1 = R_1a + R_1b$. If $\mathfrak{m}_1^2 = (0)$, then Lemma 4.12 implies that $\mathfrak{m}_1$ is principal and hence, $(R_1, \mathfrak{m}_1)$ is an SPIR with $\mathfrak{m}_1$ as its only non-zero proper ideal. Thus (b) holds.

Suppose that exactly one among R_1, R_2, R_3 is a field. It is already assumed that R_3 is a field. Since $\mathbb{AG}(R)$ satisfies (B), it follows from Lemma 4.4 that $|\mathbb{A}(R_1)^*| = |\mathbb{A}(R_2)^*| = 1$. Hence, we obtain that $(R_1, \mathfrak{m}_1)$ (respectively $(R_2, \mathfrak{m}_2)$) is an SPIR with $\mathfrak{m}_1^2 = (0)$ (respectively, $\mathfrak{m}_2^2 = (0)$). Thus in this case (c) holds.

$(ii) \Rightarrow (iii)$ Let $T = R_1 \times R_2 \times R_3$. Since $R \cong T$ as rings, it is enough to show that (iii) holds for $\mathbb{AG}(T)$.

Suppose that (a) of (ii) holds. Then T is a reduced ring with $|Min(T)| = 3$. Hence, we obtain from ([11], Corollary 2.11) that $\omega(\mathbb{AG}(T)) = \chi(\mathbb{AG}(T)) = 3$.

Suppose that (b) of (ii) holds. We can assume without loss of generality that R_2 and R_3 are fields. We first assume that $(R_1, \mathfrak{m}_1)$ is an SPIR with $\mathfrak{m}_1^4 = (0)$. Let $2 \leq t \leq 4$ be the least integer such that $\mathfrak{m}_1^t = (0)$. Observe that $R_2 \times R_3$ is a reduced ring and has exactly two minimal prime ideals. If $t = 4$, then it follows from Corollary 2.8(i) that $\omega(\mathbb{AG}(T)) = \chi(\mathbb{AG}(T)) = 4$. If $t = 3$, then it follows from Corollary 2.8(ii) that $\omega(\mathbb{AG}(T)) = \chi(\mathbb{AG}(T)) = 4$. If $t = 2$, then we obtain from Corollary 2.8(i) that $\omega(\mathbb{AG}(T)) = \chi(\mathbb{AG}(T)) = 3$.

Suppose that R_2 and R_3 are fields and $(R_1, \mathfrak{m}_1)$ is a local Artinian ring with $\mathfrak{m}_1^3 = (0), \mathfrak{m}_1^2 \neq (0)$ but $z^2 = 0$ for each $z \in \mathfrak{m}_1$ and $\mathfrak{m}_1$ is generated by two elements. We know from

Lemma 4.13 that $\omega(\mathbb{AG}(R_1)) = \chi(\mathbb{AG}(R_1)) = 2$. It now follows from Lemma 2.7 that $\omega(\mathbb{AG}(T)) = \chi(\mathbb{AG}(T)) = 4$.

Suppose that (c) of (ii) holds. We can assume without loss of generality that R_3 is a field and $(R_i, \mathfrak{m}_i)$ is an SPIR with $\mathfrak{m}_i^2 = (0)$ for each $i \in \{1, 2\}$. On applying Lemma 4.15, we obtain that $\omega(\mathbb{AG}(T)) = \chi(\mathbb{AG}(T)) = 4$.

$(iii) \Rightarrow (i)$ This is clear. □

5. Characterization of zero-dimensional rings R with $|Max(R)| = 2$ such that $\mathbb{AG}(R)$ satisfies (B)

In this section, we try to determine all zero-dimensional rings R with $|Max(R)| = 2$ such that $\mathbb{AG}(R)$ satisfies (B). It follows from ([17], Lemma 3.15) that there exist zero-dimensional rings R_1, R_2 such that $R \cong R_1 \times R_2$ as rings. Since $|Max(R)| = 2$, it follows that R_i is quasi-local for each $i \in \{1, 2\}$. We state and prove several results that are needed for proving the main result of this section.

Lemma 5.1. *Let $R = R_1 \times R_2$, where R_1 and R_2 are rings. Suppose that $\mathbb{A}(R_i)^* \neq \emptyset$ for some $i \in \{1, 2\}$. If $\mathbb{AG}(R)$ satisfies (B), then $\omega(\mathbb{AG}(R_i)) \leq 3$.*

Proof. We can assume without loss of generality that $\mathbb{A}(R_1)^* \neq \emptyset$. Suppose that $\omega(\mathbb{AG}(R_1)) \geq 4$. Then there exist distinct non-zero annihilating ideals $I_{11}, I_{12}, I_{13}, I_{14}$ of R_1 such that the subgraph of $\mathbb{AG}(R_1)$ induced by $\{I_{1i} \mid i \in \{1, 2, 3, 4\}\}$ is a clique. Observe that the subgraph of $\mathbb{AG}(R)$ induced by $\{I_{11} \times (0), I_{12} \times (0), I_{13} \times (0), I_{14} \times (0), (0) \times R_2\}$ is a clique. This implies that $\omega(\mathbb{AG}(R)) \geq 5$ and this contradicts $\mathbb{AG}(R)$ satisfies (B). Therefore, $\omega(\mathbb{AG}(R_1)) \leq 3$. □

Lemma 5.2. *Let R be a ring such that $(nil(R))^2 = (0)$ but $nil(R) \neq (0)$. If $\omega(\mathbb{AG}(R)) \leq 3$, then $nil(R)$ is principal.*

Proof. As $(nil(R))^2 = (0)$ and $\omega(\mathbb{AG}(R)) \leq 3$ by assumption, it follows that $nil(R)$ cannot contain more than three non-zero ideals of R. Therefore, we obtain that $nil(R)$ is finitely generated. Suppose that $nil(R)$ is not principal. Then there exist $x, y \in nil(R)$ such that $x \notin Ry$ and $y \notin Rx$. Notice that in such a case, $R(x + y) \notin \{Rx, Ry\}$. Observe that $Rx, Ry, R(x + y), nil(R)$ are distinct non-zero ideals contained in $nil(R)$. This is impossible. Therefore, we obtain that $nil(R)$ is principal. □

Lemma 5.3. *Let $R = R_1 \times R_2$, where R_1 and R_2 are rings. If $\mathbb{AG}(R)$ satisfies (B), then the following hold:*

(i) *Either $(nil(R_1))^2 = (0)$ or $(nil(R_2))^2 = (0)$.*

(ii) $(nil(R_i))^3 = (0)$ *for each* $i \in \{1, 2\}$.

Proof. (i) Suppose that $(nil(R_1))^2 \neq (0)$ and $(nil(R_2))^2 \neq (0)$. As $\mathbb{AG}(R)$ satisfies (B) by assumption, it follows from Lemmas 5.1 and 4.6 that $(nil(R_1))^6 = (0)$ and $(nil(R_2))^6 = (0)$. Let $m \geq 3$ be least with the property that $(nil(R_1))^m = (0)$ and $n \geq 3$ be least with the property that $(nil(R_2))^n = (0)$. By the choice of m, n, it follows that the subgraph of $\mathbb{AG}(R)$ induced by $\{(nil(R_1))^{m-2} \times (0), (nil(R_1))^{m-1} \times (0), (0) \times (nil(R_2))^{n-2}, (0) \times (nil(R_2))^{n-1}, (nil(R_1))^{m-1} \times (nil(R_2))^{n-1}\}$ is a clique. This implies that $\omega(\mathbb{AG}(R)) \geq 5$. This contradicts $\mathbb{AG}(R)$ satisfies (B). Therefore, either $(nil(R_1))^2 = (0)$ or $(nil(R_2))^2 = (0)$.
(ii) As $\mathbb{AG}(R)$ satisfies (B) by assumption, it follows from (i) that either $(nil(R_1))^2 = (0)$ or $(nil(R_2))^2 = (0)$. Without loss of generality, we can assume that $(nil(R_2))^2 = (0)$. Hence, $(nil(R_2))^3 = (0)$. Suppose that $(nil(R_1))^3 \neq (0)$. We know from Lemmas 5.1 and 4.6 that $(nil(R_1))^6 = (0)$. Let $m \geq 4$ be least with the property that $(nil(R_1))^m = (0)$. Notice that the subgraph of $\mathbb{AG}(R)$ induced by $\{(nil(R_1))^{m-2} \times (0), (nil(R_1))^{m-1} \times (0), (0) \times nil(R_2), (nil(R_1))^{m-2} \times nil(R_2), (nil(R_1))^{m-1} \times nil(R_2)\}$ is a clique on five vertices. This contradicts $\mathbb{AG}(R)$ satisfies (B). Therefore, $(nil(R_1))^3 = (0)$. □

Remark 5.4. Let R be a zero-dimensional ring with $|Max(R)| = 2$. Then there exist zero-dimensional quasi-local rings $(R_1, \mathfrak{m}_1)$ and $(R_2, \mathfrak{m}_2)$ such that $R \cong R_1 \times R_2$ as rings. If R is reduced, then both R_1 and R_2 are fields and in such a case, $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = 2$ and so, $\mathbb{AG}(R)$ satisfies (B). Hence, in characterizing zero-dimensional rings R with $|Max(R)| = 2$ such that $\mathbb{AG}(R)$ satisfies (B), we assume that R is not reduced.

Lemma 5.5. *Let* $R = R_1 \times R_2$, *where* R_1, R_2 *are rings. Let* $\{J_1, J_2, J_3\} \subseteq \mathbb{A}(R_2)^*$ *be such that* $J_1^2 = (0)$ *and the subgraph of* $\mathbb{AG}(R_2)$ *induced by* $\{J_1, J_2, J_3\}$ *is a clique. If* $\mathbb{AG}(R)$ *satisfies* (B), *then* R_1 *is reduced.*

Proof. Suppose that R_1 is not reduced. Then there exists $x \in R_1$ such that $x \neq 0$, but $x^2 = (0)$. Notice that the subgraph of $\mathbb{AG}(R)$ induced by $\{R_1x \times (0), R_1x \times J_1, (0) \times J_1, (0) \times J_2, (0) \times J_3\}$ is a clique on five vertices. This implies that $\omega(\mathbb{AG}(R)) \geq 5$. This contradicts $\mathbb{AG}(R)$ satisfies (B). Therefore, R_1 is reduced. □

Lemma 5.6. *Let* $R = R_1 \times R_2$, *where* $(R_i, \mathfrak{m}_i)$ *is a zero-dimensional quasi-local ring for each* $i \in \{1, 2\}$. *Suppose that both* R_1 *and* R_2 *are not reduced. If* $\mathbb{AG}(R)$ *satisfies* (B), *then* $(R_i, \mathfrak{m}_i)$ *is an SPIR for each* $i \in \{1, 2\}$.

Proof. Notice that $nil(R_i) = \mathfrak{m}_i \neq (0)$ for each $i \in \{1, 2\}$. Assume that $\mathbb{AG}(R)$ satisfies (B). It follows from Lemma 5.1 that $\omega(\mathbb{AG}(R_i)) \leq 3$ for each $i \in \{1, 2\}$. We know from Lemma 5.3(i)

that either $\mathfrak{m}_1^2 = (0)$ or $\mathfrak{m}_2^2 = (0)$. If $\mathfrak{m}_i^2 = (0)$ for some $i \in \{1, 2\}$, then we obtain from Lemma 5.2 that $\mathfrak{m}_i$ is principal. Hence, $(R_i, \mathfrak{m}_i)$ is an SPIR with $\mathfrak{m}_i \neq (0)$ but $\mathfrak{m}_i^2 = (0)$. Without loss of generality, we can assume that $(R_1, \mathfrak{m}_1)$ is an SPIR with $\mathfrak{m}_1 \neq (0)$ but $\mathfrak{m}_1^2 = (0)$. We know from Lemma 5.3(ii) that $\mathfrak{m}_2^3 = (0)$. Suppose that $(R_2, \mathfrak{m}_2)$ is not an SPIR. If $\mathfrak{m}_2$ is principal, then it follows from the proof of $(iii) \Rightarrow (i)$ of ([7], Proposition 8.8) that $(R_2, \mathfrak{m}_2)$ is an SPIR. This contradicts our assumption. Hence, $\mathfrak{m}_2$ is not principal. Therefore, we obtain from Lemma 5.2 that $\mathfrak{m}_2^2 \neq (0)$. We claim that $\omega(\mathbb{AG}(R_2)) = 2$. Since the subgraph of $\mathbb{AG}(R_2)$ induced by $\{\mathfrak{m}_2, \mathfrak{m}_2^2\}$ is a clique, it follows that $\omega(\mathbb{AG}(R_2)) \geq 2$. Thus if $\omega(\mathbb{AG}(R_2)) \neq 2$, then $\omega(\mathbb{AG}(R_2)) = 3$. Let $\mathcal{A} = \{J_1, J_2, J_3\} \subseteq \mathbb{A}(R_2)^*$ be such that the subgraph of $\mathbb{AG}(R_2)$ induced by $\mathcal{A}$ is a clique. Since $\mathfrak{m}_2^3 = (0)$ and $\omega(\mathbb{AG}(R_2)) = 3$ by assumption, it follows that $\mathfrak{m}_2^2 \in \mathcal{A}$. Without loss of generality, we can assume that $J_1 = \mathfrak{m}_2^2$. In such a case, as R_1 is not reduced, we obtain from Lemma 5.5 that $\mathbb{AG}(R)$ does not satisfy (B). This contradicts $\mathbb{AG}(R)$ satisfies (B). Therefore, $\omega(\mathbb{AG}(R_2)) = 2$. Now, as $\mathfrak{m}_2$ is not principal, it follows from Lemma 4.10 that $z^2 = 0$ for each $z \in \mathfrak{m}_2$. Let $z \in \mathfrak{m}_2 \backslash \mathfrak{m}_2^2$. Notice that the subgraph of $\mathbb{AG}(R)$ induced by $\{\mathfrak{m}_1 \times (0), \mathfrak{m}_1 \times R_2 z, \mathfrak{m}_1 \times \mathfrak{m}_2^2, (0) \times R_2 z, (0) \times \mathfrak{m}_2^2\}$ is a clique on five vertices. This contradicts $\mathbb{AG}(R)$ satisfies (B). Therefore, $\mathfrak{m}_2$ is principal and so, we obtain that $(R_2, \mathfrak{m}_2)$ is an SPIR. □

We use Lemma 5.7 in the proof of Proposition 5.8.

Lemma 5.7. *Let $(R_i, \mathfrak{m}_i)$ be an SPIR with $\mathfrak{m}_i \neq (0)$ for each $i \in \{1, 2\}$. Let $n \geq 2$ be least with the property that $\mathfrak{m}_1^n = (0)$ and $m \geq 2$ be least with the property that $\mathfrak{m}_2^m = (0)$. Let $R = R_1 \times R_2$. Then the following statements hold:*

(i) $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = \frac{n}{2} + \frac{m}{2} + \frac{nm}{4}$ *if both n and m are even.*
(ii) $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = \frac{n}{2} + \frac{m+1}{2} + \frac{n(m-1)}{4}$ *if n is even and m is odd.*
(iii) $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = \frac{n+1}{2} + \frac{m+1}{2} + \frac{(n-1)(m-1)}{4}$ *if both n and m are odd.*

Proof. (i) Suppose that $n = 2k$ and $m = 2t$ for some $k, t \in \mathbb{N}$. We know from Lemma 2.5(i) that $\omega(\mathbb{AG}(R_1)) = \chi(\mathbb{AG}(R_1)) = k$ and $\omega(\mathbb{AG}(R_2)) = \chi(\mathbb{AG}(R_2)) = t$. Moreover, it is clear that the subgraph of $\mathbb{AG}(R_1)$ induced by $\{\mathfrak{m}_1^{k+i} \mid i \in \{0, \ldots, k-1\}\}$ is a clique on k vertices and the subgraph of $\mathbb{AG}(R_2)$ induced by $\{\mathfrak{m}_2^{t+j} \mid j \in \{0, \ldots, t-1\}\}$ is a clique on t vertices. It is convenient to denote $\mathfrak{m}_1^{k+i}$ by I_i for each $i \in \{0, \ldots, k-1\}$ and $\mathfrak{m}_2^{t+j}$ by J_j for each $j \in \{0, \ldots, t-1\}$. Observe that $I_i^2 = (0)$ for each $i \in \{0, \ldots, k-1\}$ and $J_j^2 = (0)$ for each $j \in \{0, \ldots, t-1\}$. Notice that the subgraph of $\mathbb{AG}(R)$ induced by $\{I_i \times (0) \mid i \in \{0, \ldots, k-1\}\} \cup \{(0) \times J_j \mid j \in \{0, \ldots, t-1\}\} \cup \{I_i \times J_j \mid i \in \{0, \ldots, k-1\}, j \in \{0, \ldots, t-1\}\}$ is a clique on $k + t + kt$ vertices. Therefore, $\omega(\mathbb{AG}(R)) \geq k + t + kt$. We next verify that $\chi(\mathbb{AG}(R)) \leq k+t+kt$. Let $\{c_1, \ldots, c_k, c_{k+1}, \ldots, c_{k+t}\} \cup \{c_{rs} \mid r \in \{1, \ldots, k\}, s \in \{1, \ldots, t\}\}$ be a set of $k + t + kt$ distinct colors. Since $\omega(\mathbb{AG}(R_1)) = \chi(\mathbb{AG}(R_1)) = k$, the vertices of $\mathbb{AG}(R_1)$

can be properly colored using $\{c_1, \ldots, c_k\}$. Similarly, since $\omega(\mathbb{AG}(R_2)) = \chi(\mathbb{AG}(R_2)) = t$, the vertices of $\mathbb{AG}(R_2)$ can be properly colored using $\{c_{k+1}, \ldots, c_{k+t}\}$. Let $V_r = \{I \in \mathbb{A}(R_1)^* \mid I \text{ receives color } c_r\}$ for each $r \in \{1, \ldots, k\}$ and let $W_s = \{J \in \mathbb{A}(R_2)^* \mid J \text{ receives color } c_{k+s}\}$ for each $s \in \{1, \ldots, t\}$. Notice that $V_r \times \{(0)\} = \{I \times (0) \mid I \in V_r\}$ for each $r \in \{1, \ldots, k\}$, $\{(0)\} \times W_s = \{(0) \times J \mid J \in W_s\}$ for each $s \in \{1, \ldots, t\}$, and $V_r \times W_s = \{I \times J \mid I \in V_r, J \in W_s\}$ for each $r \in \{1, \ldots, k\}$ and $s \in \{1, \ldots, t\}$, $\{R_1\} \times \mathbb{I}(R_2) == \{R_1 \times B \mid B \in \mathbb{I}(R_2)\}$, and $\mathbb{I}(R_1) \times \{R_2\} = \{A \times R_2 \mid A \in \mathbb{I}(R_1)\}$ are subsets of $\mathbb{A}(R)^*$ and it is easy to verify that $\mathbb{A}(R)^* = (\bigcup_{r=1}^{k} V_r \times \{(0)\}) \cup (\bigcup_{s=1}^{t} \{(0)\} \times W_s) \cup (\bigcup_{r \in \{1,\ldots,k\}, s \in \{1,\ldots,t\}} V_r \times W_s) \cup (\{R_1\} \times \mathbb{I}(R_2)) \cup (\mathbb{I}(R_1) \times \{R_2\})$. Let us assign the color c_r to all the elements of $V_r \times \{(0)\}$ for each $r \in \{1, \ldots, k\}$, assign the color c_{k+s} to all the elements of $\{(0)\} \times W_s$ for each $s \in \{1, \ldots, t\}$, assign the color c_{rs} to all the elements of $V_r \times W_s$ for each $r \in \{1, \ldots, k\}$ and $s \in \{1, \ldots, t\}$, assign the color c_1 to all the elements of $\{R_1\} \times \mathbb{I}(R_2)$, and assign the color c_{k+1} to all the elements of $\mathbb{I}(R_1) \times \{R_2\}$.. It is not hard to verify that the above assignment of colors is indeed a proper vertex coloring of $\mathbb{AG}(R)$. Since this proper coloring uses $k+t+kt$ distinct colors, we obtain that $\chi(\mathbb{AG}(R)) \leq k+t+kt$. Hence, $\chi(\mathbb{AG}(R)) \leq k+t+kt \leq \omega(\mathbb{AG}(R)) \leq \chi(\mathbb{AG}(R))$. Therefore, $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = k+t+kt = \frac{n}{2} + \frac{m}{2} + \frac{mn}{4}$.

(ii) Suppose that $n = 2k$ and $m = 2t+1$ for some $k, t \in \mathbb{N}$. Let $I_i = \mathfrak{m}_1^{k+i}$ for each $i \in \{0, \ldots, k-1\}$ be as in the proof of (i). We know from Lemma 2.5(ii) that $\omega(\mathbb{AG}(R_2)) = \chi(\mathbb{AG}(R_2)) = t+1$. Moreover, it is clear that the subgraph of $\mathbb{AG}(R_2)$ induced by $\{\mathfrak{m}_2^{t+j} \mid j \in \{0, 1, \ldots, t\}\}$ is a clique on $t+1$ vertices. For convenience, let us denote $\mathfrak{m}_2^{t+j}$ by J_j for each $j \in \{0, 1, \ldots, t\}$. Notice that $J_0^2 \neq (0)$, whereas $J_j^2 = (0)$ for each $j \in \{1, \ldots, t\}$. Observe that the subgraph of $\mathbb{AG}(R)$ induced by $\{I_i \times (0) \mid i \in \{0, \ldots, k-1\}\} \cup \{(0) \times J_j \mid j \in \{0, 1, \ldots, t\}\} \cup \{I_i \times J_j \mid i \in \{0, \ldots, k-1\}, j \in \{1, \ldots, t\}\}$ is a clique on $k+t+1+kt$ vertices. Hence, $\omega(\mathbb{AG}(R)) \geq k+t+1+kt$. We next verify that $\chi(\mathbb{AG}(R)) \leq k+t+1+kt$. Let $\{c_1, \ldots, c_k, c_{k+1}, \ldots, c_{k+t+1}\} \cup \{c_{rs} \mid r \in \{1, \ldots, k\}, s \in \{2, \ldots, t+1\}\}$ be a set of $k+t+1+kt$ distinct colors. Since $\omega(\mathbb{AG}(R_1)) = \chi(\mathbb{AG}(R_1)) = k$, the vertices of $\mathbb{AG}(R_1)$ can be properly colored using $\{c_1, \ldots, c_k\}$. Let us assign the color c_{k+1} to J_0, the color c_{k+s+1} to both J_s and $\mathfrak{m}_2^{t-s}$ for each $s \in \{1, \ldots, t-1\}$, and the color c_{k+t+1} to J_t. This is a proper vertex coloring of $\mathbb{AG}(R_2)$. Let $V_r, V_r \times \{(0)\}$ be as in the proof of (i) for each $r \in \{1, \ldots, k\}$. Let $U_s = \{J \in \mathbb{A}(R_2)^* | J \text{ receives color } c_{k+s}\}$ and let $\{(0)\} \times U_s = \{(0) \times J \mid J \in U_s\}$ for each $s \in \{1, \ldots, t+1\}$. Let $V_r \times U_s = \{I \times J \mid I \in V_r, J \in U_s\}$ for each $r \in \{1, \ldots, k\}$ and $s \in \{2, \ldots, t+1\}$. Let $\{R_1\} \times \mathbb{I}(R_2) = \{R_1 \times J \mid J \in \mathbb{I}(R_2)\}$ be as in the proof of (i) and let $(\mathbb{I}(R_1) \times \{R_2\}) \cup (\mathbb{A}(R_1)^* \times \{J_0\}) = \{I \times R_2 \mid I \in \mathbb{I}(R_1)\} \cup \{A \times J_0 \mid A \in \mathbb{A}(R_1)^*\}$. It is easy to verify that $\mathbb{A}(R)^* = (\bigcup_{r=1}^{k} V_r \times \{(0)\}) \cup (\bigcup_{s=1}^{t+1} \{(0)\} \times U_s) \cup (\bigcup_{r \in \{1,\ldots,k\}, s \in \{2,\ldots,t+1\}} V_r \times U_s) \cup (\{R_1\} \times \mathbb{I}(R_2)) \cup (\mathbb{I}(R_1) \times \{R_2\}) \cup (\mathbb{A}(R_1)^* \times \{J_0\})$. Let us assign the color c_r to all the elements of $V_r \times \{(0)\}$ for each $r \in \{1, \ldots, k\}$, assign the color c_{k+s} to all the elements of $\{(0)\} \times U_s$ for each $s \in \{1, \ldots, t+1\}$, assign the color c_{rs} to all the elements of $V_r \times U_s$ for each $r \in \{1, \ldots, k\}$ and $s \in \{2, \ldots, t+1\}$, assign the color c_1 to all the elements of $\{R_1\} \times \mathbb{I}(R_2)$, and assign the color c_{k+1} to all the elements of $(\mathbb{I}(R_1) \times \{R_2\}) \cup (\mathbb{A}(R_1)^* \times \{J_0\})$. It is not hard to verify that the above assignment of colors is indeed a proper vertex coloring of $\mathbb{AG}(R)$. As this proper coloring uses $k + t + 1 + kt$ colors, it follows that $\chi(\mathbb{AG}(R)) \leq k + t + 1 + kt$. Hence, $\chi(\mathbb{AG}(R)) \leq k + t + 1 + kt \leq \omega(\mathbb{AG}(R)) \leq \chi(\mathbb{AG}(R))$. Therefore, $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = k + t + 1 + kt = \frac{n}{2} + \frac{m+1}{2} + \frac{n(m-1)}{4}$.

(iii) Suppose that $n = 2k + 1$ and $m = 2t + 1$ for some $k, t \in \mathbb{N}$. Using Lemma 2.5, it can be as shown as in the proof of (ii) that $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = \frac{n+1}{2} + \frac{m+1}{2} + \frac{(n-1)(m-1)}{4}$. □

Proposition 5.8. *Let $R = R_1 \times R_2$, where $(R_i, \mathfrak{m}_i)$ is a zero-dimensional quasi-local ring for each $i \in \{1, 2\}$. Suppose that both R_1 and R_2 are not reduced. Then the following statements are equivalent:*

(i) *$\mathbb{AG}(R)$ satisfies (B).*

(ii) *$(R_i, \mathfrak{m}_i)$ is an SPIR for each $i \in \{1, 2\}$ with either $\mathfrak{m}_1^2 = \mathfrak{m}_2^2 = (0)$ or there exists exactly one $i \in \{1, 2\}$ such that $\mathfrak{m}_i^2 = (0)$ and if $i \in \{1, 2\}$ is such that $\mathfrak{m}_i^2 \neq (0)$, then $\mathfrak{m}_i^3 = (0)$.*

Proof. $(i) \Rightarrow (ii)$ Observe that $nil(R_i) = \mathfrak{m}_i$ for each $i \in \{1, 2\}$. The statement (ii) follows immediately from Lemmas 5.3 and 5.6.

$(ii) \Rightarrow (i)$ By assumption, $(R_i, \mathfrak{m}_i)$ is an SPIR for each $i \in \{1, 2\}$. If $\mathfrak{m}_i^2 = (0)$ for each $i \in \{1, 2\}$, then it follows from Lemma 5.7(i) that $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = 3$. If there exists exactly one $i \in \{1, 2\}$ such that $\mathfrak{m}_i^2 = (0)$, then without loss of generality, we can assume that $i = 1$. In such a case, by assumption $\mathfrak{m}_2^3 = (0)$. Now it follows from Lemma 5.7(ii) that $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = 4$. Hence, $\mathbb{AG}(R)$ satisfies (B). □

Remark 5.9. Let $R = R_1 \times R_2$, where $(R_i, \mathfrak{m}_i)$ is a zero-dimensional quasi-local ring for each $i \in \{1, 2\}$ such that exactly one between R_1 and R_2 is reduced. Without loss of generality, we can assume that R_2 is reduced. In such a case, R_2 is a field. Observe that $nil(R_1) = \mathfrak{m}_1 \neq (0)$. If $\mathbb{AG}(R)$ satisfies (B), then it follows from Lemmas 5.1 and 4.6 that $\mathfrak{m}_1^6 = (0)$. Hence, in characterizing zero-dimensional rings R with $|Max(R)| = 2$ with $R \cong R_1 \times R_2$ as rings, where $(R_1, \mathfrak{m}_1)$ is a zero-dimensional non-reduced ring and R_2 is a field such that $\mathbb{AG}(R)$ satisfies (B), for convenience, after a change of notation, we can assume that $R = S \times F$, where $(S, \mathfrak{m})$ is quasi-local with $\mathfrak{m} \neq (0)$ but $\mathfrak{m}^6 = (0)$ and F is a field.

Corollary 5.10. *Let $(S, \mathfrak{m})$ be an SPIR with $\mathfrak{m} \neq (0)$ but $\mathfrak{m}^6 = (0)$. Let $R = S \times F$, where F is a field. Then $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) \in \{2, 3, 4\}$.*

Proof. Let $t \geq 2$ be the least positive integer with the property that $\mathfrak{m}^t = (0)$. Then $t \in \{2,3,4,5,6\}$. If $t = 2$, then it follows from Lemmas 2.5(i) and 2.6 that $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = 2$. If $t = 3$, then it follows from Lemmas 2.5(ii) and 2.6 that $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = 3$. If $t = 4$, then it follows from Lemmas 2.5(i) and 2.6 that $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = 3$. If $t = 5$, then it follows from Lemmas 2.5(ii) and 2.6 that $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = 4$. If $t = 6$, then it follows from Lemmas 2.5(i) and 2.6 that $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = 4$. □

Lemma 5.11. *Let $R = S \times F$, where $(S, \mathfrak{m})$ is quasi-local with $\mathfrak{m}^6 = (0)$ and F is a field. Suppose that $\mathfrak{m}^5 \neq (0)$. Then $\mathbb{AG}(R)$ satisfies (B) if and only if $(S, \mathfrak{m})$ is an SPIR.*

Proof. Observe that $nil(S) = \mathfrak{m}$. Assume that $\mathbb{AG}(R)$ satisfies (B). Then it follows from Lemmas 5.1 and 4.8 that $\mathfrak{m}$ is principal. Hence, we obtain from the proof of $(iii) \Rightarrow (i)$ of ([7], Proposition 8.8) that $\{\mathfrak{m}^i \mid i \in \{1,2,3,4,5\}\}$ is the set of all non-zero proper ideals of S. Therefore, $(S, \mathfrak{m})$ is an SPIR.

Conversely, assume that $(S, \mathfrak{m})$ is an SPIR with $\mathfrak{m}^6 = (0)$, but $\mathfrak{m}^5 \neq (0)$. Then we obtain from Corollary 5.10 that $\mathbb{AG}(R)$ satisfies (B). Indeed, it follows from the proof of Corollary 5.10 that $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = 4$. □

Lemma 5.12. *Let $R = S \times F$, where $(S, \mathfrak{m})$ is quasi-local with $\mathfrak{m}^5 = (0)$ but $\mathfrak{m}^4 \neq (0)$ and F is a field. If $\mathbb{AG}(R)$ satisfies (B), then $z^2 \neq 0$ for some $z \in \mathfrak{m}^2$.*

Proof. Notice that $nil(S) = \mathfrak{m}$. As $\mathbb{AG}(R)$ satisfies (B), we obtain from Lemma 5.1 that $\omega(\mathbb{AG}(S)) \leq 3$. Observe that the subgraph of $\mathbb{AG}(S)$ induced by $\{\mathfrak{m}^2, \mathfrak{m}^3, \mathfrak{m}^4\}$ is a clique on three vertices. Hence, it follows that $\omega(\mathbb{AG}(S)) = 3$. Therefore, we obtain from Lemma 4.11 that $z^2 \neq 0$ for some $z \in \mathfrak{m}^2$. □

Lemma 5.13. *Let $R = S \times F$, where $(S, \mathfrak{m})$ is quasi-local with $\mathfrak{m}^5 = (0)$ but $\mathfrak{m}^4 \neq (0)$ and F is a field. Then $\mathbb{AG}(R)$ satisfies (B) if and only if $(S, \mathfrak{m})$ is an SPIR.*

Proof. Notice that $nil(S) = \mathfrak{m}$. Assume that $\mathbb{AG}(R)$ satisfies (B). Then it follows from Lemmas 5.1, 5.12, and 4.10 that $\mathfrak{m}$ is principal. Hence, we obtain from the proof of $(iii) \Rightarrow (i)$ of ([7], Proposition 8.8) that $\{\mathfrak{m}^i \mid i \in \{1,2,3,4\}\}$ is the set of all non-zero proper ideals of S. Therefore, $(S, \mathfrak{m})$ is an SPIR.

Conversely, assume that $(S, \mathfrak{m})$ is an SPIR with $\mathfrak{m}^5 = (0)$ but $\mathfrak{m}^4 \neq (0)$. Then we obtain from the proof of Corollary 5.10 that $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = 4$. Therefore, $\mathbb{AG}(R)$ satisfies (B). □

Example 5.14. Let $T = \mathbb{Z}_2[X, Y]$ and $I = TX^2 + TY^2$. Let $S = \frac{T}{I}$. Then S is a local Artinian ring with $\mathfrak{m} = \frac{TX+TY}{I}$ as its unique maximal ideal such that $\mathfrak{m}^3 = (0 + I)$ but $\mathfrak{m}^2 \neq (0 + I)$. Let $R = S \times F$, where F is a field. Then $\mathbb{AG}(R)$ satisfies (B) but $(S, \mathfrak{m})$ is not an SPIR.

Proof. It is clear that $(S, \mathfrak{m})$ is a local Artinian ring with $\mathfrak{m}^3 = (0 + I)$ but $\mathfrak{m}^2 \neq (0 + I)$. We know from Example 4.14(i) that $\omega(\mathbb{AG}(S)) = \chi(\mathbb{AG}(S)) = 2$. Hence, we obtain from Lemma 2.6 that $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = 3$. Therefore, we get that $\mathbb{AG}(R)$ satisfies (B). As $\mathfrak{m}$ is not a principal ideal of S, it follows that $(S, \mathfrak{m})$ is not an SPIR. □

Lemma 5.15. *Let $R = S \times F$, where $(S, \mathfrak{m})$ is quasi-local with $\mathfrak{m} \neq (0)$ but $\mathfrak{m}^2 = (0)$ and F is a field. Then the following statements are equivalent:*

(i) *$\mathbb{AG}(R)$ satisfies (B).*

(ii) *$(S, \mathfrak{m})$ is an SPIR.*

Proof. Notice that $nil(S) = \mathfrak{m}$.

$(i) \Rightarrow (ii)$ It follows from Lemmas 5.1 and 5.2 that $\mathfrak{m}$ is principal. From $\mathfrak{m}^2 = (0)$, we obtain that $\mathfrak{m}$ is the only non-zero proper ideal of R. Therefore, $(S, \mathfrak{m})$ is an SPIR.

$(ii) \Rightarrow (i)$ It follows from the proof of Corollary 5.10 that $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = 2$. Therefore, $\mathbb{AG}(R)$ satisfies (B). □

6. Characterization of zero-dimensional quasi-local rings R such that $\mathbb{AG}(R)$ satisfies (B)

In this section, we try to characterize zero-dimensional quasi-local rings R such that $\mathbb{AG}(R)$ satisfies (B). We are not able to solve the problem of characterizing such rings. However, we present some partial results regarding this problem. As mentioned in the introduction, we consider rings which admit at least one non-zero annihilating ideal.

Lemma 6.1. *Let $(R, \mathfrak{m})$ be a zero-dimensional quasi-local ring. If $\mathbb{AG}(R)$ satisfies (B), then $\mathfrak{m}^8 = (0)$.*

Proof. Observe that $nil(R) = \mathfrak{m}$. Hence, if $\mathbb{AG}(R)$ satisfies (B), then we obtain from Lemma 4.6 that $\mathfrak{m}^8 = (0)$. □

Proposition 6.2. *Let $(R, \mathfrak{m})$ be a zero-dimensional quasi-local ring such that $\mathfrak{m}^7 \neq (0)$. Then the following statements are equivalent:*

(i) *$\mathbb{AG}(R)$ satisfies (B).*

(ii) *$(R, \mathfrak{m})$ is an SPIR with $\mathfrak{m}^8 = (0)$.*

Proof. $(i) \Rightarrow (ii)$ Notice that $nil(R) = \mathfrak{m}$. It follows from Lemma 6.1 that $\mathfrak{m}^8 = (0)$. Since $\mathfrak{m}^7 \neq (0)$ by hypothesis, we obtain from Lemma 4.8 that $\mathfrak{m}$ is principal. Hence, it follows from the proof of $(iii) \Rightarrow (i)$ of ([7], Proposition 8.8) that $\{\mathfrak{m}^i \mid i \in \{1, 2, \ldots, 7\}\}$ is the set of all non-zero proper ideals of R. Therefore, $(R, \mathfrak{m})$ is an SPIR.

$(ii) \Rightarrow (i)$ As $(R, \mathfrak{m})$ is an SPIR with $\mathfrak{m}^8 = (0)$, whereas $\mathfrak{m}^7 \neq (0)$, we obtain from Lemma 2.5(i) that $\omega(\mathbb{AG}(R)) = 4$. Therefore, $\mathbb{AG}(R)$ satisfies (B). □

Proposition 6.3. *Let $(R, \mathfrak{m})$ be a zero-dimensional quasi-local ring with $\mathfrak{m}^7 = (0)$. If $z^2 \neq (0)$ for some $z \in \mathfrak{m}^3$, then the following statements are equivalent:*

(i) $\mathbb{AG}(R)$ *satisfies* (B).

(ii) $(R, \mathfrak{m})$ *is an SPIR.*

Proof. $(i) \Rightarrow (ii)$ Observe that $nil(R) = \mathfrak{m}$. As $z^2 \neq 0$ for some $z \in \mathfrak{m}^3$, it follows from Lemma 4.10 that $\mathfrak{m}$ is principal. It follows from the proof of $(iii) \Rightarrow (i)$ of ([7], Proposition 8.8) that $\{\mathfrak{m}^i \mid i \in \{1, 2, \ldots, 6\}\}$ is the set of all non-zero proper ideals of R. Therefore, $(R, \mathfrak{m})$ is an SPIR.

$(ii) \Rightarrow (i)$ $(R, \mathfrak{m})$ is an SPIR and by hypothesis, $\mathfrak{m}^7 = (0)$, whereas $\mathfrak{m}^6 \neq (0)$. Hence, we obtain from Lemma 2.5(ii) that $\omega(\mathbb{AG}(R)) = 4$. Therefore, $\mathbb{AG}(R)$ satisfies (B). □

We provide Example 6.4 to illustrate that the hypotheses of Propositions 6.2 and 6.3 cannot be omitted.

Example 6.4. Let $T = \mathbb{Z}_4[X, Y, Z]$ be the polynomial ring in three variables X, Y, Z over $\mathbb{Z}_4$ and I be the ideal of T generated by $\{X^2 - 2, Y^2 - 2, Z^2, XY, YZ - 2, ZX, 2X, 2Y, 2Z\}$. Let $R = \frac{T}{I}$. The ring R appeared in [5] and it was shown there that $\omega(\Gamma_0(R)) = 5 < \chi(\Gamma_0(R)) = 6$, where $\Gamma_0(R)$ is the Beck's zero-divisor graph of R. It was observed in [5] that R is local with $\mathfrak{m} = \frac{TX+TY+TZ}{I}$ as its unique maximal ideal, $\mathfrak{m}^3 = (0 + I)$, and $|R| = 32$. The ring R was also considered in [11] and it was shown in ([11], Proposition 2.1) that $\omega(\mathbb{AG}(R)) = \chi(\mathbb{AG}(R)) = 4$. Hence, $\mathbb{AG}(R)$ satisfies (B) but $\mathfrak{m}$ is not principal and so, $(R, \mathfrak{m})$ is not an SPIR.

Remark 6.5. Let $(R, \mathfrak{m})$ be a zero-dimensional quasi-local ring with $\mathfrak{m} \neq (0)$.

Since $\mathbb{AG}(R)$ is connected by ([10], Theorem 2.1), it is clear that $\omega(\mathbb{AG}(R)) = 1$ if and only if $\mathbb{AG}(R)$ is a graph on a single vertex. This happens if and only if $(R, \mathfrak{m})$ is an SPIR with $\mathfrak{m}^2 = (0)$.

Suppose that $\omega(\mathbb{AG}(R)) = 2$.

As $nil(R) = \mathfrak{m}$, it follows from Lemma 4.6 that $\mathfrak{m}^4 = (0)$. If $\mathfrak{m}^3 \neq (0)$, then we obtain from Lemma 4.8 that $\mathfrak{m}$ is principal and hence, $(R, \mathfrak{m})$ is an SPIR. In such a case, we obtain from Lemma 2.5(i) that $\chi(\mathbb{AG}(R)) = 2$.

Suppose that $\mathfrak{m}^3 = (0)$. We claim that $\mathfrak{m}^2 \neq (0)$. Suppose that $\mathfrak{m}^2 = (0)$. Since we are assuming that $\omega(\mathbb{AG}(R)) = 2$, it follows that $\mathfrak{m}$ cannot be principal. Hence, there exist $a, b \in \mathfrak{m}$ such that $\{a, b\}$ is linearly independent over $\frac{R}{\mathfrak{m}}$. Observe that the subgraph of $\mathbb{AG}(R)$ induced by $\{Ra, Rb, R(a+b), Ra+Rb\}$ is a clique on four vertices. This is impossible. Therefore, $\mathfrak{m}^2 \neq (0)$. If $z^2 \neq 0$ for some $z \in \mathfrak{m}$, then it follows from Lemma 4.10 that $\mathfrak{m}$ is principal. Hence, $(R, \mathfrak{m})$ is an SPIR and moreover, we obtain from Lemma 2.5(ii) that $\chi(\mathbb{AG}(R)) = 2$. If $z^2 = 0$ for each $z \in \mathfrak{m}$, then from Lemma 4.11, we get that $\mathfrak{m}$ is generated by two elements and is not principal. In this case, it is shown in Lemma 4.13 that $\chi(\mathbb{AG}(R)) = 2$.

Suppose that $\omega(\mathbb{AG}(R)) = 3$.

It follows from Lemma 4.6 that $\mathfrak{m}^6 = (0)$. If $\mathfrak{m}^5 \neq (0)$, then we obtain from Lemma 4.8 that $\mathfrak{m}$ is principal. Hence, $(R, \mathfrak{m})$ is an SPIR and in this case, we know from Lemma 2.5(i) that $\chi(\mathbb{AG}(R)) = 3$.

Suppose that $\mathfrak{m}^5 = (0)$ but $\mathfrak{m}^4 \neq (0)$. As $\omega(\mathbb{AG}(R)) = 3$ by assumption, we obtain from Lemma 4.11 that $z^2 \neq 0$ for some $z \in \mathfrak{m}^2$. In such a case, it follows from Lemma 4.10 that $\mathfrak{m}$ is principal. Hence, $(R, \mathfrak{m})$ is an SPIR and we obtain from Lemma 2.5(ii) that $\chi(\mathbb{AG}(R)) = 3$.

Since $\omega(\mathbb{AG}(R)) = 3$ by assumption, it follows as argued above that $\mathfrak{m}^2 \neq (0)$. We are not able to determine rings R with $\omega(\mathbb{AG}(R)) = 3$ such that either $\mathfrak{m}^3 = (0)$ or $\mathfrak{m}^4 = (0)$ but $\mathfrak{m}^3 \neq (0)$.

Suppose that $\omega(\mathbb{AG}(R)) = 4$.

Then we know from Lemma 4.6 that $\mathfrak{m}^8 = (0)$. If $\mathfrak{m}^7 \neq (0)$, then it follows from Lemma 4.8 that $\mathfrak{m}$ is principal. Hence, $(R, \mathfrak{m})$ is an SPIR and we get from Lemma 2.5(i) that $\chi(\mathbb{AG}(R)) = 4$.

Suppose that $\mathfrak{m}^7 = (0)$, whereas $\mathfrak{m}^6 \neq (0)$. Since we are assuming that $\omega(\mathbb{AG}(R)) = 4$, we obtain from Lemma 4.11 that $z^2 \neq 0$ for some $z \in \mathfrak{m}^3$. Therefore, it follows from Lemma 4.10 that $\mathfrak{m}$ is principal. Hence, $(R, \mathfrak{m})$ is an SPIR and we obtain from Lemma 2.5(ii) that $\chi(\mathbb{AG}(R)) = 4$.

Suppose that $\mathfrak{m}^2 = (0)$. As we are assuming that $\omega(\mathbb{AG}(R)) = 4$, it is clear that $\mathfrak{m}$ is not principal. Hence, $dim_{\frac{R}{\mathfrak{m}}}(\mathfrak{m}) \geq 2$. We claim that $dim_{\frac{R}{\mathfrak{m}}}(\mathfrak{m}) = 2$. Otherwise, there exist $a, b, c \in \mathfrak{m}$ such that $\{a, b, c\}$ is linearly independent over $\frac{R}{\mathfrak{m}}$. Observe that the subgraph of $\mathbb{AG}(R)$ induced by $\{Ra, Rb, Rc, R(a+b), Ra+Rb\}$ is a clique. This is impossible, since $\omega(\mathbb{AG}(R)) = 4$. Therefore, $dim_{\frac{R}{\mathfrak{m}}}(\mathfrak{m}) = 2$. Hence, there exist $a, b \in \mathfrak{m}$ such that $\mathfrak{m} = Ra + Rb$. We assert that $|\frac{R}{\mathfrak{m}}| = 2$. Suppose that $|\frac{R}{\mathfrak{m}}| > 2$. Then there exists $r \in R$ such that $r, r-1 \notin \mathfrak{m}$. Notice that the subgraph of $\mathbb{AG}(R)$ induced by $\{Ra, Rb, R(a+b), R(a+rb), Ra+Rb\}$ is a clique. This contradicts $\omega(\mathbb{AG}(R)) = 4$. Therefore, $|\frac{R}{\mathfrak{m}}| = 2$. Observe that $|\mathfrak{m}| = 4$ and $|R| = 8$. Let $T_1 = \mathbb{Z}_2[X, Y]$ be the polynomial ring in two variables X, Y over $\mathbb{Z}_2$ and $T_2 = \mathbb{Z}_4[X]$ be the polynomial ring in one variable X over $\mathbb{Z}_4$. Let $\mathfrak{m}_1 = T_1X + T_1Y$ and $\mathfrak{m}_2 = T_2 2 + T_2X$. It is not hard to show that either $R \cong \frac{T_1}{\mathfrak{m}_1^2}$ or $R \cong \frac{T_2}{\mathfrak{m}_2^2}$ as rings.

Let $i \in \{2, 3, 4, 5\}$. We are not able to characterize rings R such that $\omega(\mathbb{AG}(R)) = 4$ satisfying the condition that $\mathfrak{m}^{i+1} = (0)$ but $\mathfrak{m}^i \neq (0)$.

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