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MODAL OPERATORS ON L-ALGEBRAS

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Abstract. The main goal of this paper is to introduce analogously modal operators on L-algebras and study their properties. To begin with, we introduce the notion of modal operators on L-algebras and investigate some important properties of this operator. In order for the kernel of modal operator to be ideal, we investigate what conditions are required. Relations between modal operator and endomorphism of L-algebras are investigated. Also, we define the concept of positive L-algebra and some characterizations of positive L-algebra are established. Finally, we introduce a map k_a and show that k_a is a modal operator and we prove that the set of all k_a on a positive L-algebra makes a dual BCK-algebra.

1. Introduction

L-algebras, which are related to algebraic logic and quantum structures, were introduced by Rump [12]. Many examples shown that L-algebras are very useful. Yang and Rump [14], characterized pseudo-MV-algebras and Bosbach's non-commutative bricks as L-algebras.

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Wu and Yang [16] proved that orthomodular lattices form a special class of L-algebras in different ways. It was shown that every lattice-ordered effect algebra has an underlying L-algebra structure in Wu et al. [15]. Also, other mathematicians studied the relationship between basic algebras and L-algebras. They proved that a basic algebra which satisfies $(z \oplus \neg x) \oplus \neg(y \oplus \neg x) = (z \oplus \neg y) \oplus \neg(x \oplus \neg y)$ can be converted into an L-algebra. Conversely, if an L-algebra with the least element "0" and some conditions such that it is an involutive bounded lattice can be organized into a basic algebra, it must be a lattice-ordered effect algebra. We refer the reader to the following sources for further study in the field of L-algebras [3, 4, 5].

In 1981, modal operators (special cases of closure operators) on Heyting algebras were introduced and studied by Macnab [9]. Since then, properties of modal operators were considered on other algebraic structures such as MV-algebra [7], bounded commutative residuated Rℓ-monoids (simply called Rℓ-monoids) [11], commutative residuated lattices [8] and so on. The essence of modal operator is closure operator, and closure operator is an important part of the theoretical study of partial order sets.

In this paper, we introduce the notion of modal operator on L-algebras and investigate some important properties of this operator. In order for the kernel of modal operator to be ideal, we investigate what conditions are required. Relations between modal operator and endomorphism of L-algebras are investigated. Also, we define the concept of positive L-algebra and some characterizations of positive L-algebra are established. Finally, we introduce a map k_a and show that k_a is a modal operator and we prove that the set of all k_a on a positive L-algebra makes a dual BCK-algebra.

2. Preliminaries

This section lists the known default contents that will be used later.

Definition 2.1. [6] An *L-algebra* is an algebraic structure $(\mathcal{L}; \rightsquigarrow, 1)$ of type $(2, 0)$ satisfying
$(L1)$ $x \rightsquigarrow x = x \rightsquigarrow 1 = 1$ and $1 \rightsquigarrow x = x$,
$(L2)$ $(x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z) = (y \rightsquigarrow x) \rightsquigarrow (y \rightsquigarrow z)$,
$(L3)$ if $x \rightsquigarrow y = y \rightsquigarrow x = 1$, then $x = y$,
for any $x, y, z \in \mathcal{L}$. Condition $(L1)$ states that 1 is a logical unit, while $(L2)$ is related to the quantum Yang-Baxter equation. Note that a logical unit is always unique. In addition, easily it can be seen that the relation

$$x \leq y \text{ if and only if } x \rightsquigarrow y = 1,$$

defines a partial order for any L-algebra $\mathcal{L}$. If $\mathcal{L}$ admits a smallest element 0 with respect to the ordering $\leq$, then it is called a *bounded L-algebra.*

We say that a bounded L-algebra $\mathcal{L}$ has *negation* if the map $x \longmapsto x'$ is bijective, where $x' = x \rightsquigarrow 0$. The inverse map will then be denoted by $x \mapsto x^{\sim}$. If $x^{\sim} = x'$, then L is called an *L-algebra with double negation.*

Proposition 2.2. [14] *Let $\mathcal{L}$ be an L-algebra. Then $x \leq y$ implies $z \rightsquigarrow x \leq z \rightsquigarrow y$, for any $x, y, z \in \mathcal{L}$.*

Proposition 2.3. [14] *For an L-algebra $\mathcal{L}$, the following are equivalent:*
(i) $x \leq y \rightsquigarrow x$,
(ii) if $x \leq z$, *then* $z \rightsquigarrow y \leq x \rightsquigarrow y$,
(iii) $((x \rightsquigarrow y) \rightsquigarrow z) \rightsquigarrow z \leq ((x \rightsquigarrow y) \rightsquigarrow z) \rightsquigarrow ((y \rightsquigarrow x) \rightsquigarrow z)$,
for any $x, y, z \in \mathcal{L}$.

Definition 2.4. [13] An L-algebra $\mathcal{L}$ which satisfies

$$x \rightsquigarrow (y \rightsquigarrow x) = 1, \qquad (K)$$

for any $x, y \in \mathcal{L}$ is called a *KL-algebra.*

A *CKL-algebra* is an L-algebra which satisfies

$$x \rightsquigarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightsquigarrow z), \qquad (C)$$

for any $x, y, z \in \mathcal{L}$ (see [13]).

Clearly, every CKL-algebra is a KL-algebra, since for any $x, y \in \mathcal{L}$, we have

$$x \rightsquigarrow (y \rightsquigarrow x) = y \rightsquigarrow (x \rightsquigarrow x) = y \rightsquigarrow 1 = 1.$$

Proposition 2.5. [1] *Assume $(\mathcal{L}, \rightsquigarrow, 1)$ is a CKL-algebra. Then for any $x, y, z \in \mathcal{L}$, the following properties hold:*
(i) if $x \leq y$, *then* $z \rightsquigarrow x \leq z \rightsquigarrow y$,
(ii) $x \rightsquigarrow (y \rightsquigarrow x) = 1$, *i.e.,* $x \leq y \rightsquigarrow x$,
(iii) $x \leq (x \rightsquigarrow y) \rightsquigarrow y$,
(iv) $x \leq y \rightsquigarrow z$ *if and only if* $y \leq x \rightsquigarrow z$,
(v) if $x \leq y$, *then* $y \rightsquigarrow z \leq x \rightsquigarrow z$,
(vi) $((x \rightsquigarrow y) \rightsquigarrow z) \rightsquigarrow z \leq ((x \rightsquigarrow y) \rightsquigarrow z) \rightsquigarrow ((y \rightsquigarrow x) \rightsquigarrow z)$,
(vii) $z \rightsquigarrow y \leq (y \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow x)$,
(viii) $z \rightsquigarrow y \leq (x \rightsquigarrow z) \rightsquigarrow (x \rightsquigarrow y)$,
If $\mathcal{L}$ has a least element as 0, *then*
(ix) if $x \leq y$, *then* $y' \leq x'$, *where* $x' = x \rightsquigarrow 0$,
(x) $x \leq x''$, *and* $x' = x'''$,
(xi) $x' \leq x \rightsquigarrow y$,

(xii) $((x \rightsquigarrow y) \rightsquigarrow y) \rightsquigarrow y = x \rightsquigarrow y$,
(xiii) If $\mathcal{L}$ *has double negation, then* $x \rightsquigarrow y = y' \rightsquigarrow x'$.

Definition 2.6. [12] A subset $\mathcal{I}$ of an L-algebra $\mathcal{L}$ is called an *ideal of* $\mathcal{L}$ if it satisfies the following conditions for all $x, y \in \mathcal{I}$,
(I_1) $1 \in \mathcal{I}$,
(I_2) if $x \in \mathcal{I}$ and $x \rightsquigarrow y \in \mathcal{I}$, then $y \in \mathcal{I}$,
(I_3) if $x \in \mathcal{I}$, then $(x \rightsquigarrow y) \rightsquigarrow y \in \mathcal{I}$,
(I_4) if $x \in \mathcal{I}$, then $y \rightsquigarrow x \in \mathcal{I}$ and $y \rightsquigarrow (x \rightsquigarrow y) \in \mathcal{I}$.

If we consider the ideal of CKL-algebra, the conditions (I_3) and (I_4) can be dropped. In fact, for any $x \in \mathcal{I}$, by (C) and (I_1) we have

$$x \rightsquigarrow ((x \rightsquigarrow y) \rightsquigarrow y) = (x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow y) = 1 \in \mathcal{I},$$

for any $y \in \mathcal{L}$. It follows by (I_2) that $(x \rightsquigarrow y) \rightsquigarrow y \in \mathcal{I}$. Thus (I_3) holds. Furthermore, if $x \in \mathcal{I}$, then for any $y \in \mathcal{L}$, by (K) we have $x \rightsquigarrow (y \rightsquigarrow x) = 1 \in \mathcal{I}$ and by (I_2), $y \rightsquigarrow x \in \mathcal{I}$.

For an L-algebra, a binary relation $\sim$ is a *congruence relation* [12] on $\mathcal{L}$ if it is an equivalence relation such that for any $x, y, z \in \mathcal{L}$,

$$x \sim y \Leftrightarrow \ (z \rightsquigarrow x) \sim (z \rightsquigarrow y) \ \text{ and } \ (x \rightsquigarrow z) \sim (y \rightsquigarrow z).$$

Theorem 2.7. [12] *Let* $(\mathcal{L}, \rightsquigarrow, 1)$ *be an* L*-algebra. Then every ideal* $\mathcal{I}$ *of* $\mathcal{L}$ *defines a congruence relation on* $\mathcal{L}$*, for any* $x, y \in \mathcal{L}$*, where*

$$x \sim y \ \Leftrightarrow \ \ x \rightsquigarrow y, \ y \rightsquigarrow x \in \mathcal{I}.$$

Conversely, every congruence relation $\sim$ *defines an ideal* $\mathcal{I} = \{x \in \mathcal{L} \mid x \sim 1\}$.

Definition 2.8. [12] Let $\mathcal{L}$ and $\mathcal{H}$ be two L-algebras. Then a map $f : \mathcal{L} \to \mathcal{H}$ is called an L-*homomorphism* if for any $x, y \in \mathcal{L}$ we have $f(x \rightsquigarrow_{\mathcal{L}} y) = f(x) \rightsquigarrow_{\mathcal{H}} f(y)$.

If f is an injective, then f is called a *monomorphism* and if f is onto, then f is called an *epimorphism*. In addition, if f is a bijective function, then f is called an *isomorphism*.

Note. From now on, we let $(\mathcal{L}, \rightsquigarrow, 1)$ or $\mathcal{L}$, for short, be an L-algebra.

3. Modal operators on L-algebras

In this section, we introduce the notion of modal operators on L-algebras and investigate some important properties of this operator. In order for the kernel of modal operator to be ideal, we investigate what conditions are required.

Definition 3.1. A mapping $m : \mathcal{L} \to \mathcal{L}$ is called a ***modal operator*** on $\mathcal{L}$ if for each $x, y \in \mathcal{L}$ we have:
(M_1) $x \leq m(x)$,
(M_2) $m(m(x)) = m(x)$,
(M_3) $m(x \rightsquigarrow y) \leq m(x) \rightsquigarrow m(y)$.

The set of all modal operators on $\mathcal{L}$ is denoted by $\mathcal{M}od(\mathcal{L})$.

Remark 3.2. Assume $m \in \mathcal{M}od(\mathcal{L})$. Then
(i) Since for any $x \in \mathcal{L}$, $x \leq m(x)$, we get $1 \leq m(1)$, and so $m(1) = 1$.
(ii) If $x \leq y$, then $x \rightsquigarrow y = 1$, and so $1 = m(1) = m(x \rightsquigarrow y)$. Also, since $m \in \mathcal{M}od(\mathcal{L})$, by (M_3) we have

$$1 = m(1) = m(x \rightsquigarrow y) \leq m(x) \rightsquigarrow m(y),$$

and so $m(x) \rightsquigarrow m(y) = 1$. Hence, $m(x) \leq m(y)$. Therefore, every modal operator on $\mathcal{L}$ is order preserving.

Example 3.3. (i) Clearly, $id_{\mathcal{L}} \in \mathcal{M}od(\mathcal{L})$.
(ii) Let $(\mathcal{L} = \{a, b, c, 1\}, \leq)$ be a chain where $a \leq b \leq c \leq 1$. Define the operation $\rightsquigarrow$ on $\mathcal{L}$ in Table 1:

$\rightsquigarrow$	a	b	c	1
a	1	1	1	1
b	a	1	1	1
c	a	b	1	1
1	a	b	c	1

Table 1

Then $(\mathcal{L}, \rightsquigarrow, 1)$ is an L-algebra. If we consider a map $m : \mathcal{L} \to \mathcal{L}$ as follows:

$$m(1) = 1, \ m(a) = m(b) = b, \ m(c) = c,$$

it is easily to see that $m \in \mathcal{M}od(\mathcal{L})$.
(iii) Let $(\mathcal{L} = \{a, b, c, 1\}, \leq)$ be a poset where $a, c \leq b \leq 1$. Then $(\mathcal{L}, \rightsquigarrow, 1)$ is an L-algebra where the operation $\rightsquigarrow$ is defined in Table 2:

$\rightsquigarrow$	a	b	c	1
a	1	1	a	1
b	a	1	c	1
c	a	1	1	1
1	a	b	c	1

Table 2

If $m : \mathcal{L} \to \mathcal{L}$ is $m(a) = m(b) = m(c) = b$ and $m(1) = 1$, then $m \in \mathcal{M}od(\mathcal{L})$.

Proposition 3.4. *Let $m \in \mathcal{M}od(\mathcal{L})$. Then the following statements hold:*
(i) $m(x) \rightsquigarrow m(y) = m(m(x) \rightsquigarrow m(y))$.
(ii) If $\mathcal{L}$ is a KL-algebra, then $m(m(x) \rightsquigarrow m(y)) = x \rightsquigarrow m(y) = m(x \rightsquigarrow m(y))$.
(iiii) If $\mathcal{L}$ is a bounded CKL-algebra, then $m(x) \leq (m(x) \rightsquigarrow m(0)) \rightsquigarrow m(0)$.
(iv) If $\mathcal{L}$ is bounded, then $m(x') \leq m(x) \rightsquigarrow m(0)$.

Proof. (*i*) Assume $x, y \in \mathcal{L}$. Then by (M_1), $m(x) \rightsquigarrow m(y) \leq m(m(x) \rightsquigarrow m(y))$. Conversely, since $m(x), m(y) \in \mathcal{L}$ and $m \in \mathcal{M}od(\mathcal{L})$, by (M_3) and (M_2) we have

$$m(m(x) \rightsquigarrow m(y)) \leq m(m(x)) \rightsquigarrow m(m(y)) = m(x) \rightsquigarrow m(y).$$

Hence, $m(x) \rightsquigarrow m(y) = m(m(x) \rightsquigarrow m(y))$.
(*ii*) By (M_1), for any $x \in \mathcal{L}$, $x \leq m(x)$. Since $\mathcal{L}$ is a KL-algebra, by Proposition 2.3(ii), $m(x) \rightsquigarrow m(y) \leq x \rightsquigarrow m(y)$. Then by (i), $m(x) \rightsquigarrow m(y) = m(m(x) \rightsquigarrow m(y))$ and so $m(m(x) \rightsquigarrow m(y)) \leq x \rightsquigarrow m(y)$. Conversely, since $m \in \mathcal{M}od(\mathcal{L})$, by (M_1), (M_3) and (M_2), respectively, we have

$$x \rightsquigarrow m(y) \leq m(x \rightsquigarrow m(y)) \leq m(x) \rightsquigarrow m(m(y)) = m(x) \rightsquigarrow m(y) \leq m(m(x) \rightsquigarrow m(y)).$$

So, $x \rightsquigarrow m(y) \leq m(m(x) \rightsquigarrow m(y))$. Hence, $m(m(x) \rightsquigarrow m(y)) = x \rightsquigarrow m(y)$. Also, clearly, by (M_1), $x \rightsquigarrow m(y) \leq m(x \rightsquigarrow m(y))$. Also, by (M_1), for any $x \in \mathcal{L}$, $x \leq m(x)$. Since $\mathcal{L}$ is a KL-algebra, by Proposition 2.3(ii), $m(x) \rightsquigarrow m(y) \leq x \rightsquigarrow m(y)$. Then by (M_3) and (M_2), $m(x \rightsquigarrow m(y)) \leq m(x) \rightsquigarrow m(m(y)) = m(x) \rightsquigarrow m(y)$, and so $m(x \rightsquigarrow m(y)) \leq x \rightsquigarrow m(y)$. Therefore, $x \rightsquigarrow m(y) = m(x \rightsquigarrow m(y))$.
(*iii*) By assumption, $\mathcal{L}$ is bounded, so $m(0)$ is well-known. Since $m(x) \rightsquigarrow m(0) \leq m(x) \rightsquigarrow m(0)$, by Proposition 2.5(iv) we have $m(x) \leq (m(x) \rightsquigarrow m(0)) \rightsquigarrow m(0)$.
(*iv*) By (M_1), since $\mathcal{L}$ is bounded, we obtain $0 \leq m(0)$, then by Proposition 2.5(i), $x \rightsquigarrow 0 \leq x \rightsquigarrow m(0)$, and so $x' \leq x \rightsquigarrow m(0)$. Thus, we have

$$\begin{aligned} m(x') &\leq m(x \rightsquigarrow m(0)) && \text{by Remark 3.2(ii)} \\ &\leq m(x) \rightsquigarrow m(m(0)) && \text{by } (M_3) \\ &= m(x) \rightsquigarrow m(0). && \text{by } (M_2) \end{aligned}$$

□

Theorem 3.5. *Let $\mathcal{L}$ be a KL-algebra and $m : \mathcal{L} \to \mathcal{L}$ be a map. Then $m \in \mathcal{M}od(\mathcal{L})$ if and only if m satisfies in the following conditions:*
(1) *$m(x \rightsquigarrow y) \leq m(x) \rightsquigarrow m(y)$,*
(2) *$m(x) \rightsquigarrow m(y) = x \rightsquigarrow m(y)$.*

Proof. ($\Rightarrow$) By the definition of a modal operator and Proposition 3.4(i) and (ii), the proof is clear.

($\Leftarrow$) Let $x \in \mathcal{L}$ and (1) and (2) hold. Clearly, (M_3) holds. Since $m(x) \rightsquigarrow m(x) = 1$, by (2) we have $1 = m(x) \rightsquigarrow m(x) = x \rightsquigarrow m(x)$. Thus, $x \leq m(x)$, and so (M_1) holds. Now, we prove $m(x) = m(m(x))$. For this, by (M_1), obviously, $m(x) \leq m(m(x))$. On the other side, by (2), $m(m(x)) \rightsquigarrow m(x) = m(x) \rightsquigarrow m(x) = 1$, and so $m(m(x)) \leq m(x)$. Thus, (M_2) holds. Therefore, $m \in \mathcal{M}od(\mathcal{L})$. □

Corollary 3.6. *If $\mathcal{L}$ is a CKL-algebra and $m : \mathcal{L} \to \mathcal{L}$ is a map, then $m \in \mathcal{M}od(\mathcal{L})$ if and only if for any $x, y \in \mathcal{L}$, $m(x) \rightsquigarrow m(y) = x \rightsquigarrow m(y)$.*

Proof. Since every CKL-algebra is a KL-algebra, it follows from Theorem 3.5.

Conversely, by Theorem 3.5, it is enough to prove (M_3). For this, let $x, y \in \mathcal{L}$. Then

$$\begin{aligned}
m(x \rightsquigarrow y) \rightsquigarrow (m(x) \rightsquigarrow m(y)) &= m(x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow m(y)) && \text{by assumption}\\
&= x \rightsquigarrow (m(x \rightsquigarrow y) \rightsquigarrow m(y)) && \text{by } (C)\\
&= x \rightsquigarrow ((x \rightsquigarrow y) \rightsquigarrow m(y)) && \text{by assumption}\\
&= (x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow m(y)) && \text{by } (C)\\
&= 1. && \text{by } (M_1) \text{ and Proposition 2.5(i)}
\end{aligned}$$

Hence, $m(x \rightsquigarrow y) \rightsquigarrow (m(x) \rightsquigarrow m(y)) = 1$ and so $m(x \rightsquigarrow y) \leq m(x) \rightsquigarrow m(y)$. Therefore, (M_3) holds and so $m \in \mathcal{M}od(\mathcal{L})$. □

Note. For any $m_1, m_2 \in \mathcal{M}od(\mathcal{L})$, $m_1 \leq m_2$ if and only if for any $x \in \mathcal{L}$, $m_1(x) \leq m_2(x)$.

Remark 3.7. Consider $m_1, m_2 \in \mathcal{M}od(\mathcal{L})$. Then the condition (M_1) and (M_3) hold for composition of m_1 and m_2. Because if $x \in \mathcal{L}$, clearly, $m_2(x) \in \mathcal{L}$, then $x \leq m_2(x) \leq m_1(m_2(x))$. Also, since $m_2 \in \mathcal{M}od(\mathcal{L})$, we have $m_2(x \rightsquigarrow y) \leq m_2(x) \rightsquigarrow m_2(y)$. By Remark 3.2(ii),

$$m_1(m_2(x \rightsquigarrow y)) \leq m_1(m_2(x) \rightsquigarrow m_2(y)) \leq m_1(m_2(x)) \rightsquigarrow m_1(m_2(y)).$$

In the next example we show that composition of two modal operators do not meet the condition (M_2).

Example 3.8. Consider $(\mathcal{L}, \rightsquigarrow, 1)$ and $m_1 : \mathcal{L} \to \mathcal{L}$ as L-algebra and modal operator as in Example 3.3(ii). Define $m_2 : \mathcal{L} \to \mathcal{L}$ as follows:

$$m_2(1) = 1, \ m_2(a) = a, \ m_2(b) = m_2(c) = c.$$

Obviously, $m_2 \in \mathcal{M}od(\mathcal{L})$, but $m_1m_2 \notin \mathcal{M}od(\mathcal{L})$, because

$$m_1(m_2(m_1(m_2(a)))) = m_1(m_2(m_1(a))) = m_1(m_2(b)) = m_1(c) = c \neq b = m_1(a) = m_1(m_2(a)).$$

Thus, (M_3) does not hold.

Theorem 3.9. *Consider $m_1, m_2 \in \mathcal{M}od(\mathcal{L})$. Then $m_1m_2 \in \mathcal{M}od(\mathcal{L})$ if and only if $m_1m_2 = m_2m_1$.*

Proof. $(\Rightarrow)$ Assume $m_1m_2 \in \mathcal{M}od(\mathcal{L})$. Since $m_1, m_2 \in \mathcal{M}od(\mathcal{L})$, by Remark 3.2(ii), they are order preserving, so m_1m_2 is order preserving, too. For proving $m_1m_2 = m_2m_1$, we show that $m_1m_2 \leq m_2m_1$. For this, since $m_1m_2 \in \mathcal{M}od(\mathcal{L})$, by (M_1), for $x \in \mathcal{L}$, we have $x \leq m_1(m_2(x))$. In addition, by using Remark 3.2(ii) twice we get $m_1(x) \leq m_1(m_1(m_2(x)))$, and so $m_2(m_1(x)) \leq m_2(m_1(m_1(m_2(x))))$. Since $m_1 \in \mathcal{M}od(\mathcal{L})$, by (M_2) we have $m_2(m_1(x)) \leq m_2(m_1(m_2(x)))$. By (M_1) for m_1 we get $m_2(m_1(x)) \leq m_2(m_1(m_2(x))) \leq m_1(m_2(m_1(m_2(x))))$. Since $m_1m_2 \in \mathcal{M}od(\mathcal{L})$, by (M_2), $m_2(m_1(x)) \leq m_1(m_2(x))$. The proof of other side is similar. Hence, $m_1m_2 = m_2m_1$.
$(\Leftarrow)$ Suppose $m_1, m_2 \in \mathcal{M}od(\mathcal{L})$ such that $m_1m_2 = m_2m_1$. By (M_1), for $x \in \mathcal{L}$, $x \leq m_2(x)$, and so $x \leq m_2(x) \leq m_1(m_2(x))$. Thus (M_1) holds. Also, by (M_3) we have $m_2(x \rightsquigarrow y) \leq m_2(x) \rightsquigarrow m_2(y)$. Since $m_1 \in \mathcal{M}od(\mathcal{L})$, by Remark 3.2(ii), we have $m_1(m_2(x \rightsquigarrow y)) \leq m_1(m_2(x) \rightsquigarrow m_2(y))$, and so $m_1(m_2(x \rightsquigarrow y)) \leq m_1(m_2(x)) \rightsquigarrow m_1(m_2(y))$. Thus (M_3) holds. For proving (M_2), by assumption and (M_2) we have

$$m_1(m_2(m_1(m_2(x))) = m_1(m_1(m_2(m_2(x))) = m_1(m_2(x)).$$

Therefore, $m_1m_2 \in \mathcal{M}od(\mathcal{L})$. □

Example 3.10. According to Example 3.8, we have

$$m_1(m_2(a)) = m_1(a) = b \neq c = m_2(b) = m_2(m_1(a)).$$

Thus, the condition $m_1m_2 = m_2m_1$ in Theorem 3.9 is necessary.

Definition 3.11. An L-algebra $(\mathcal{L}, \rightsquigarrow, 1)$ is called a *positive L-algebra* if for any $x, y, z \in \mathcal{L}$ we have

$$x \rightsquigarrow (y \rightsquigarrow z) = (x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z). \qquad (P)$$

Example 3.12. Let $(\mathcal{L} = \{a, b, c, 1\}, \leq)$ be a poset where $a \leq b, c \leq 1$. Then $(\mathcal{L}, \rightsquigarrow, 1)$ is an L-algebra where the operation $\rightsquigarrow$ is defined in Table 3:

$\rightsquigarrow$	a	b	c	1
a	1	1	1	1
b	c	1	c	1
c	b	b	1	1
1	a	b	c	1

Table 3

Obviously, the condition (P) holds, and so $(\mathcal{L}, \rightsquigarrow, 1)$ is a positive L-algebra.

In the following example we show that this is not true that every L-algebra is not a positive L-algebra.

Example 3.13. Let $(\mathcal{L} = \{a, b, c, 1\}, \leq)$ be a poset where $a, c \leq b \leq 1$. Then $(\mathcal{L}, \rightsquigarrow, 1)$ is an L-algebra where the operation $\rightsquigarrow$ is defined in Table 4:

$\rightsquigarrow$	a	b	c	1
a	1	1	a	1
b	a	1	c	1
c	a	1	1	1
1	a	b	c	1

Table 4

Then $(\mathcal{L}, \rightsquigarrow, 1)$ is not a positive L-algebra, because

$$c \rightsquigarrow (a \rightsquigarrow c) = c \rightsquigarrow a = a \neq 1 = a \rightsquigarrow 1 = (c \rightsquigarrow a) \rightsquigarrow (c \rightsquigarrow c).$$

Theorem 3.14. *Every positive L-algebra is a CKL-algebra.*

Proof. Consider L is a positive L-algebra. Then for any $x, y, z \in \mathcal{L}$, we have

$$\begin{aligned} x \rightsquigarrow (y \rightsquigarrow z) &= (x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z) && \text{by (P)} \\ &= (y \rightsquigarrow x) \rightsquigarrow (y \rightsquigarrow z) && \text{by (L2)} \\ &= y \rightsquigarrow (x \rightsquigarrow z). && \text{by (P)} \end{aligned}$$

Hence, L is a CKL-algebra. □

Corollary 3.15. *Every positive L-algebra is a KL-algebra.*

In the following example we show that every CKL-algebra is not a positive L-algebra.

Example 3.16. Let $(\mathcal{L} = \{a, b, c, 1\}, \leq)$ be a chain where $a \leq b \leq c \leq 1$. Define the operation $\rightsquigarrow$ on $\mathcal{L}$ in Table 5:

$\rightsquigarrow$	a	b	c	1
a	1	1	1	1
b	c	1	1	1
c	a	b	1	1
1	a	b	c	1

Table 5

Then $(\mathcal{L}, \rightsquigarrow, 1)$ is a CKL-algebra (and also KL-algebra) but is not positive, because

$$c \rightsquigarrow (b \rightsquigarrow a) = c \rightsquigarrow c = 1 \neq c = b \rightsquigarrow a = (c \rightsquigarrow b) \rightsquigarrow (c \rightsquigarrow a).$$

Proposition 3.17. *Let $\mathcal{L}$ be a CKL-algebra. Then for any $x, y, z \in \mathcal{L}$, we have*

$$(x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z) \leq x \rightsquigarrow (y \rightsquigarrow z).$$

Proof. Assume $x, y, z \in \mathcal{L}$. Then by Proposition 2.5(ii), $y \leq x \rightsquigarrow y$ and by Proposition 2.5(v) we have $(x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z) \leq y \rightsquigarrow (x \rightsquigarrow z)$ and by (C) we get $(x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z) \leq x \rightsquigarrow (y \rightsquigarrow z)$. □

Next example shows that the converse of equation in Proposition 3.17 does not hold.

Example 3.18. According to Example 3.16, $\mathcal{L}$ is a CKL-algebra but

$$c \rightsquigarrow (b \rightsquigarrow a) = 1 \not\leq c = b \rightsquigarrow a = (c \rightsquigarrow b) \rightsquigarrow (c \rightsquigarrow a).$$

Note. According to Proposition 3.17, if $\mathcal{L}$ is a CKL-algebra, then $\mathcal{L}$ is positive if for any $x, y, z \in \mathcal{L}$ we have

$$x \rightsquigarrow (y \rightsquigarrow z) \leq (x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z).$$

Theorem 3.19. *If $\mathcal{L}$ is a positive L-algebra and $a \in \mathcal{L}$, then a mapping $g_a : \mathcal{L} \to \mathcal{L}$, where for any $x \in \mathcal{L}$, $g_a(x) = a \rightsquigarrow x$ is a modal operator.*

Proof. Let $x, y \in \mathcal{L}$. Then by Theorem 3.14, $\mathcal{L}$ is a CKL-algebra, thus by Proposition 2.5(ii), $x \leq a \rightsquigarrow x = g_a(x)$. So, $x \leq g_a(x)$ and (M_1) holds. Moreover, by assumption we have

$$g_a(x \rightsquigarrow y) = a \rightsquigarrow (x \rightsquigarrow y) = (a \rightsquigarrow x) \rightsquigarrow (a \rightsquigarrow y) = g_a(x) \rightsquigarrow g_a(y).$$

Thus (M_3) holds. For proving (M_2), by (P), we have

$$g_a(g_a(x)) = g_a(a \rightsquigarrow x) = a \rightsquigarrow (a \rightsquigarrow x) = (a \rightsquigarrow a) \rightsquigarrow (a \rightsquigarrow x) = 1 \rightsquigarrow (a \rightsquigarrow x) = a \rightsquigarrow x = g_a(x).$$

Hence, $g_a(x) \in \mathcal{M}od(\mathcal{L})$. □

In the following example we show that the condition positive L-algebra in Theorem 3.19 is necessary.

Example 3.20. According to Example 3.18, clearly $g_c(b \rightsquigarrow a) \not\leq g_c(b) \rightsquigarrow g_c(a)$. Thus (M_3) does not hold, and so $g_c \notin \mathcal{M}od(\mathcal{L})$. Hence, the condition positive L-algebra in Theorem 3.19 is necessary.

Remark 3.21. (i) Clearly, $g_1(x) = x$, and so $g_1(x) = id_{\mathcal{L}}$. Also, in a positive L-algebra, $g_a(x)$ is a homomorphism on $\mathcal{L}$.
(ii) In a positive L-algebra, for $a \in \mathcal{L}$ and $n \in \mathbb{N}$, $g_a^n = g_a$, where $g_a^n(x) = \underbrace{a \rightsquigarrow (a \rightsquigarrow \cdots (a}_{\text{n-times}} \rightsquigarrow x) \cdots)$ and $x \in \mathcal{L}$.

Proposition 3.22. *Let $\mathcal{L}$ be an L-algebra and $x \in \mathcal{L}$. Then for $x, y \in \mathcal{L}$ we have*
(i) g_a is isotone.
(ii) If $\mathcal{L}$ is a KL-algebra, then $x \leq g_a(x) \leq g_a^2(x) \leq \cdots$.
(iii) If $\mathcal{L}$ is a CKL-algebra, then $g_a(x) \rightsquigarrow g_a(y) \leq g_a(x \rightsquigarrow y)$.
(iv) If $\mathcal{L}$ is a CKL-algebra, then $x \rightsquigarrow y \leq g_a^n(x) \rightsquigarrow g_a^n(y)$, for any $n \in \mathbb{N}$.
(v) $Im(g_a^n) \subseteq Im(g_a^{n-1}) \subseteq \cdots \subseteq Im(g_a^2) \subseteq Im(g_a)$.
(vi) $Fix(g_a) \subseteq Fix(g_a^2) \subseteq \cdots \subseteq Fix(g_a^n)$, where $Fix(g_a) := \{x \in \mathcal{L} \mid g_a(x) = x\}$.
(vii) $\ker(g_a) \subseteq \ker(g_a^2) \subseteq \cdots \subseteq \ker(g_a^n)$, where $\ker(g_a) := \{x \in \mathcal{L} \mid g_a(x) = 1\}$.
(viii) $Fix(g_a^n) \subseteq Im(g_a^n)$.
(ix) $Fix(g_a^n) \cap \ker(g_a^n) = \{1\}$.

Proof. (i) Assume $x, y \in \mathcal{L}$ such that $x \leq y$. Then by Proposition 2.2, $a \rightsquigarrow x \leq a \rightsquigarrow y$ and so $g_a(x) \leq g_a(y)$.
(ii) Since $\mathcal{L}$ is a KL-algebra, by Proposition 2.3(i), $x \leq a \rightsquigarrow x \leq a \rightsquigarrow (a \rightsquigarrow x) \leq \cdots$. Thus, $x \leq g_a(x) \leq g_a^2(x) \leq \cdots$.
(iii) By Proposition 3.17, the proof is clear.
(iv) By Proposition 2.5(viii), $x \rightsquigarrow y \leq (a \rightsquigarrow x) \rightsquigarrow (a \rightsquigarrow y) \leq (a \rightsquigarrow (a \rightsquigarrow x)) \rightsquigarrow (a \rightsquigarrow (a \rightsquigarrow y)) \leq \cdots$. Thus, $x \rightsquigarrow y \leq g_a(x) \rightsquigarrow g_a(y) \leq g_a^2(x) \rightsquigarrow g_a^2(y) \leq \cdots$.
(v) The proof is straightforward.
(vi) Let $x \in \mathrm{Fix}(g_a)$. Then $g_a(x) = a \rightsquigarrow x = x$. Also,

$$g_a^2(x) = a \rightsquigarrow (a \rightsquigarrow x) = a \rightsquigarrow x = x \qquad \text{since } x \in \mathrm{Fix}(g_a)$$

So, $\mathrm{Fix}(g_a) \subseteq \mathrm{Fix}(g_a^2)$.
(vii) Let $x \in \ker(g_a)$. Then $g_a(x) = a \rightsquigarrow x = 1$. Also,

$$g_a^2(x) = a \rightsquigarrow (a \rightsquigarrow x) = a \rightsquigarrow 1 = 1 \qquad \text{by (L1) and since } x \in \ker(g_a)$$

So, $\ker(g_a) \subseteq \ker(g_a^2)$.
$(viii)$ The proof is straightforward.
(ix) Clearly, $\{1\} \subseteq \mathrm{Fix}(g_a^n) \cap \ker(g_a^n)$. Assume $x \in \mathrm{Fix}(g_a^n) \cap \ker(g_a^n)$. Then $g_a^n(x) = x$ and $g_a^n(x) = 1$. Thus $x = 1$, and so $\mathrm{Fix}(g_a^n) \cap \ker(g_a^n) \subseteq \{1\}$. □

Proposition 3.23. *Let $m \in \mathcal{M}od(\mathcal{L})$. Then $Fix(m) = Im(m)$.*

Proof. If $x \in \mathrm{Fix}(m)$, then $x = m(x) \in \mathrm{Im}(m)$, whence $\mathrm{Fix}(m) \subseteq \mathrm{Im}(m)$. Conversely, suppose $y \in \mathrm{Im}(m)$. Then there exists $x \in \mathcal{L}$ such that $m(x) = y$. Since $m \in \mathcal{M}od(\mathcal{L})$, by (M_2) we have $m(y) = m(m(x)) = m(x) = y$. Thus $y \in \mathrm{Fix}(m)$. Hence, $\mathrm{Fix}(m) = \mathrm{Im}(m)$. □

Theorem 3.24. *If $m \in \mathcal{M}od(\mathcal{L})$, then $Fix(m)$ is closed under the operation $\rightsquigarrow$ and so $\langle Fix(m), \rightsquigarrow, 1\rangle$ is an L-algebra.*

Proof. Consider $x, y \in \mathrm{Fix}(m)$. Then $m(x) = x$ and $m(y) = y$. Since $m \in \mathcal{M}od(\mathcal{L})$, by (M_1),

$$x \rightsquigarrow y \leq m(x \rightsquigarrow y) \leq m(x) \rightsquigarrow m(y) = x \rightsquigarrow y.$$

Thus, $m(x \rightsquigarrow y) = m(x) \rightsquigarrow m(y) = x \rightsquigarrow y$. So, $x \rightsquigarrow y \in \mathrm{Fix}(m)$. Hence, $\mathrm{Fix}(m)$ is closed under the operation $\rightsquigarrow$. It is straightforward to prove $\langle \mathrm{Fix}(m), \rightsquigarrow, 1\rangle$ is an L-algebra. □

In the following example we show that $\mathrm{Fix}(g_a)$ and $\ker(g_a)$ are not ideal of $\mathcal{L}$, in general.

Example 3.25. Let $(\mathcal{L} = \{a, b, c, 1\}, \leq)$ be a poset where $a, b, c \leq 1$. Define the operation $\rightsquigarrow$ on $\mathcal{L}$ in Table 6:

$\rightsquigarrow$	a	b	c	1
a	1	b	a	1
b	a	1	c	1
c	a	b	1	1
1	a	b	c	1

Table 6

Then $(\mathcal{L}, \rightsquigarrow, 1)$ is an L-algebra. Then $\mathrm{Fix}(g_a) = \{b, 1\}$ is an ideal of $\mathcal{L}$ but $\mathrm{Fix}(g_c) = \{a, b, 1\}$ is not an ideal of $\mathcal{L}$ since $a \in \mathrm{Fix}(g_c)$ and $a \rightsquigarrow c = a \in \mathrm{Fix}(g_c)$, but $c \notin \mathrm{Fix}(g_c)$. So (I_2) does not hold.

In addition, $\ker(g_c) = \{c, 1\}$ which is not an ideal of $\mathcal{L}$, since $c \not\leq a \rightsquigarrow c = a$ and so (I_4) does not hold.

Proposition 3.26. *If $\mathcal{L}$ is a positive L-algebra, then $\ker(g_a)$ is an ideal of $\mathcal{L}$.*

Proof. By Theorem 3.14, $\mathcal{L}$ is a CKL-algebra and so for proving that $\ker(g_a)$ is an ideal of $\mathcal{L}$, it is enough to show (I_1) and (I_2) hold. For this, clearly, $g_a(1) = a \rightsquigarrow 1 = 1$, by $(L1)$. Thus $1 \in \ker(g_a)$. Assume $x, x \rightsquigarrow y \in \ker(g_a)$. Then $g_a(x) = g_a(x \rightsquigarrow y) = 1$. Since $\mathcal{L}$ is positive, by $(L1)$ we have

$$1 = g_a(x \rightsquigarrow y) = a \rightsquigarrow (x \rightsquigarrow y) = (a \rightsquigarrow x) \rightsquigarrow (a \rightsquigarrow y) = 1 \rightsquigarrow (a \rightsquigarrow y) = a \rightsquigarrow y.$$

Hence, $g_a(y) = 1$ and so $y \in \ker(g_a)$. Therefore, $\ker(g_a)$ is an ideal of $\mathcal{L}$. □

Proposition 3.27. *If $\mathcal{L}$ is a positive L-algebra, then $Im(g_a)$, $Fix(g_a)$ and $\ker(g_a)$ are closed under $\rightsquigarrow$.*

Proof. Assume $x, y \in \mathrm{Im}(g_a)$. Then there are $b, c \in \mathcal{L}$ such that $g_a(b) = x$ and $g_a(c) = y$. Then by (P), we have

$$x \rightsquigarrow y = g_a(b) \rightsquigarrow g_a(c) = (a \rightsquigarrow b) \rightsquigarrow (a \rightsquigarrow c) = a \rightsquigarrow (b \rightsquigarrow c) = g_a(b \rightsquigarrow c).$$

Hence, $x \rightsquigarrow y \in \mathrm{Im}(g_a)$. Also, suppose $x, y \in \mathrm{Fix}(g_a)$. Then $g_a(x) = x$ and $g_a(y) = y$. Thus by (P), we get

$$x \rightsquigarrow y = g_a(x) \rightsquigarrow g_a(y) = (a \rightsquigarrow x) \rightsquigarrow (a \rightsquigarrow y) = a \rightsquigarrow (x \rightsquigarrow y) = g_a(x \rightsquigarrow y).$$

Hence, $x \rightsquigarrow y \in \mathrm{Fix}(g_a)$. In addition, if $x, y \in \ker(g_a)$, then by (P) and $(L1)$ we have

$$g_a(x \rightsquigarrow y) = a \rightsquigarrow (x \rightsquigarrow y) = (a \rightsquigarrow x) \rightsquigarrow (a \rightsquigarrow y) = g_a(x) \rightsquigarrow g_a(y) = 1 \rightsquigarrow 1 = 1.$$

Hence, $x \rightsquigarrow y \in \ker(g_a)$. □

Proposition 3.28. *If $\mathcal{L}$ is a positive L-algebra, then for any $a, b \in \mathcal{L}$ we have*
(i) $g_a \circ g_b = g_b \circ g_a$,
(ii) $g_a \circ g_a = g_a$,
(iii) $g_1 \circ g_a = g_a = g_a \circ g_1$,
(iv) if $a \leq b$, then $g_b \leq g_a$ and $g_a \circ g_b = g_a$, where $g_b(x) \leq g_a(x)$, for any $x \in \mathcal{L}$.

Proof. (i) Let $x \in \mathcal{L}$. Then by Theorem 3.14 and (C), we have

$$(g_a \circ g_b)(x) = g_a(g_b(x)) = g_a(b \rightsquigarrow x) = a \rightsquigarrow (b \rightsquigarrow x) = b \rightsquigarrow (a \rightsquigarrow x) = g_b(g_a(x)).$$

(ii) Let $x \in \mathcal{L}$. Then by Theorem 3.14, (C) and $(L1)$, we have

$$(g_a \circ g_a)(x) = g_a(a \rightsquigarrow x) = a \rightsquigarrow (a \rightsquigarrow x) = (a \rightsquigarrow a) \rightsquigarrow (a \rightsquigarrow x) = 1 \rightsquigarrow (a \rightsquigarrow x) = a \rightsquigarrow x = g_a(x).$$

(iii) Assume $x \in \mathcal{L}$. Then by $(L1)$, we have

$$(g_1 \circ g_a)(x) = 1 \rightsquigarrow (a \rightsquigarrow x) = a \rightsquigarrow x = g_a(x).$$

(iv) Consider $a \leq b$. Then by Theorem 3.14 and Proposition 2.5(v) we have $b \rightsquigarrow x \leq a \rightsquigarrow x$, and so $g_b(x) \leq g_a(x)$. Also, for $x \in \mathcal{L}$, if $a \leq b$, then $a \rightsquigarrow b = 1$ and by (P) and $(L1)$ we have

$$(g_a \circ g_b)(x) = a \rightsquigarrow (b \rightsquigarrow x) = (a \rightsquigarrow b) \rightsquigarrow (a \rightsquigarrow x) = 1 \rightsquigarrow (a \rightsquigarrow x) = a \rightsquigarrow x = g_a(x).$$

□

Corollary 3.29. *Assume $\mathcal{L}$ is a chain positive L-algebra. Then $\mathcal{G} = \{g_a \mid a \in \mathcal{L}\}$ is a commutative monoid under the composition of mapping with zero element g_1.*

Proof. By Proposition 3.28, clearly, the composition is commutative, idempotent and has g_1 as a neutral element. Now, we prove g_1 is unique. For this, assume there exists $x \in \mathcal{L}$ such that $g_x \circ g_a = g_a$, for all $a \in \mathcal{L}$. Then by Proposition 3.28(iii) we have

$$g_1 = g_1 \circ g_x = g_x \circ g_1 = g_x.$$

So, g_1 is unique. Now, we prove that the composition is associative. Suppose $a, b, c \in \mathcal{L}$. Since $\mathcal{L}$ is a chain, assume $b \leq c$. Then by Proposition 3.28(iv), $g_b \circ g_c = g_b$, and so $g_a \circ (g_b \circ g_c) = g_a \circ g_b$. We have the following cases:

Case 1. If $a \leq b$, then by Proposition 3.28(iv), $g_a \circ g_b = g_a$. On the other side, we have

$$(g_a \circ g_b) \circ g_c = g_a \circ g_c = g_c,$$

since $a \leq b \leq c$. So, associativity holds in this case.

Case 2. If $b \leq a$, then by Proposition 3.28(iv), $g_a \circ g_b = g_b$. On the other side, we have

$$(g_a \circ g_b) \circ g_c = g_b \circ g_c = g_b,$$

since $b \leq a, c$. So, associativity holds in this case. Hence, in these cases associativity holds. The proof of other case is similar. Therefore, $\mathcal{G}$ is a commutative monoid. □

For $a \in \mathcal{L}$, consider a mapping $k_a : \mathcal{L} \to \mathcal{L}$, where for any $x \in \mathcal{L}$, $k_a(x) = x \rightsquigarrow a$.

Proposition 3.30. *The following statements hold:*

(i) $k_a(1) = a$ and $k_a(a) = 1$, for $a \in \mathcal{L}$.

(ii) If $\mathcal{L}$ is bounded, then $k_a(0) = 1$ and $k_0(a) = a'$,

(iii) $k_1(x) = 1$, for any $x \in \mathcal{L}$,

(iv) if $a \leq b$, then $k_a \leq k_b$,

(v) if $\mathcal{L}$ is a KL-algebra and $x \leq y$, then $k_a(y) \leq k_a(x)$, and so $k_a(1) \leq k_a(b)$, for any $a, b \in \mathcal{L}$.

Proof. The proof is straightforward. □

Proposition 3.31. *Assume $\mathcal{L}$ is a CKL-algebra and $a, x, y \in \mathcal{L}$. Then the following statements hold:*
(i) For any natural number $n \in \mathbb{N}$, and $a \in \mathcal{L}$, we have

$$k_a^n = \begin{cases} k_a & n \text{ is odd} \\ k_a^2 & n \text{ is even} \end{cases}$$

where $k_a^n(x) = (((x \rightsquigarrow \underbrace{a) \rightsquigarrow a) \cdots a}_{n-times})$.
(ii) $k_a^2(x) \rightsquigarrow k_a(y) = x \rightsquigarrow k_a(y)$,
(iii) $y \rightsquigarrow k_a^2(x) = k_a(x) \rightsquigarrow k_a(y)$ and $k_a^2(x) \rightsquigarrow k_a^2(y) = x \rightsquigarrow k_a^2(y)$,
(iv) the mapping k_a^2 is isotone.

Proof. By Proposition 2.5, the proof is straightforward. □

Theorem 3.32. *Consider $\mathcal{L}$ is a CKL-algebra, $h \in \mathcal{M}od(\mathcal{L})$ and $a \in \mathcal{L}$. Then $h \leq k_a^2$ if and only if $h(a) = a$.*

Proof. Assume $h \leq k_a^2$. Then for any $x \in \mathcal{L}$, $h(x) \leq k_a^2(x)$, and so $h(x) \leq (x \rightsquigarrow a) \rightsquigarrow a$. Since $a \in \mathcal{L}$, we have $h(a) \leq (a \rightsquigarrow a) \rightsquigarrow a$, and so by $(L1)$, $h(a) \leq a$. Also, since $h \in \mathcal{M}od(\mathcal{L})$, by (M_1), $a \leq h(a)$. Hence, $h(a) = a$.

Conversely, by Proposition 2.5(iii), $x \leq (x \rightsquigarrow a) \rightsquigarrow a$ and since $h \in \mathcal{M}od(\mathcal{L})$, we get

$x \leq (x \rightsquigarrow a) \rightsquigarrow a \Rightarrow h(x) \leq h((x \rightsquigarrow a) \rightsquigarrow a)$ by Proposition 2.5(iii) and Remark 3.2(ii)

$\Rightarrow h(x) \leq h(h(x \rightsquigarrow a) \rightsquigarrow h(a))$ by (M_3) and by Proposition 3.4(i)

$\Rightarrow h(x) \leq (x \rightsquigarrow a) \rightsquigarrow h(a)$ by Proposition 3.4(ii)

$\Rightarrow h(x) \leq (x \rightsquigarrow a) \rightsquigarrow a$ since $h(a) = a$

Hence, for any $x \in \mathcal{L}$, $h(x) \leq k_a^2(x)$, and so $h \leq k_a^2$. □

Theorem 3.33. *Let $a \in \mathcal{L}$ where $\mathcal{L}$ is a CKL-algebra. Then the following statements are equivalent:*
(i) k_a^2 is an identity map.
(ii) k_a is an injective map.
(iii) k_a is a surjective map.

Proof. $(i) \Rightarrow (ii)$. Let k_a^2 be an identity map. Let $x, y \in \mathcal{L}$ such that $k_a(x) = k_a(y)$. Then $x \rightsquigarrow a = y \rightsquigarrow a$, and so

$$x = k_a^2(x) = (x \rightsquigarrow a) \rightsquigarrow a = (y \rightsquigarrow a) \rightsquigarrow a = k_a^2(y) = y.$$

Hence, k_a is an injective map on $\mathcal{L}$.
$(ii) \Rightarrow (iii)$. For any $x, y \in \mathcal{L}$, we have $k_a((x \rightsquigarrow a) \rightsquigarrow a) = k_a(x)$ by Proposition 2.5(xii). Since k_a is an injective map on $\mathcal{L}$, it follows that $(x \rightsquigarrow a) \rightsquigarrow a = x$. Moreover, we know that $\mathrm{Im}(k_a) \subseteq \mathcal{L}$. Let $y \in \mathcal{L}$. Then $k_a(y \rightsquigarrow a) = (y \rightsquigarrow a) \rightsquigarrow a = y$ and so $y \in \mathrm{Im}(k_a)$. Hence, $\mathcal{L} = \mathrm{Im}(k_a)$. Therefore, k_a is a surjective map on $\mathcal{L}$.
$(iii) \Rightarrow (i)$. Using Proposition 2.5(iii), we have $x \leq (x \rightsquigarrow a) \rightsquigarrow a = k_a^2(x)$ for any $x \in \mathcal{L}$. Since k_a is a surjective map, for any $y \in \mathcal{L}$, there exists $x \in \mathcal{L}$ such that $k_a(x) = y$, i.e., $x \rightsquigarrow a = y$. It follows from (C), Proposition 2.5(xii) and $(L1)$ that

$$\begin{aligned} k_a^2(y) \rightsquigarrow y &= ((y \rightsquigarrow a) \rightsquigarrow a) \rightsquigarrow (x \rightsquigarrow a) = x \rightsquigarrow (((y \rightsquigarrow a) \rightsquigarrow a) \rightsquigarrow a) \\ &= x \rightsquigarrow (y \rightsquigarrow a) = y \rightsquigarrow (x \rightsquigarrow a) = y \rightsquigarrow y = 1, \end{aligned}$$

that is, $k_a^2(y) \leq y$ for all $y \in \mathcal{L}$. Hence, $k_a^2(y) = y$ for all $y \in \mathcal{L}$. Therefore, k_a^2 is an identity map. □

Theorem 3.34. *Assume $\mathcal{L}$ is a CKL-algebra. Then k_a^2 is a modal operator.*

Proof. Let $a \in \mathcal{L}$. Then by Proposition 2.5(iii), we have $x \leq (x \rightsquigarrow a) \rightsquigarrow a = k_a^2(x)$, and so (M_1) holds. Also, by Proposition 2.5(xii) we have

$$k_a^2(k_a^2(x)) = k_a^2((x \rightsquigarrow a) \rightsquigarrow a) = (((x \rightsquigarrow a) \rightsquigarrow a) \rightsquigarrow a) \rightsquigarrow a = (x \rightsquigarrow a) \rightsquigarrow a = k_a^2(x).$$

Thus, (M_2) holds. Finally for proving (M_3) we have

$$\begin{aligned} & [((x \rightsquigarrow y) \rightsquigarrow a) \rightsquigarrow a] \rightsquigarrow [((x \rightsquigarrow a) \rightsquigarrow a) \rightsquigarrow ((y \rightsquigarrow a) \rightsquigarrow a)] \\ &= ((x \rightsquigarrow a) \rightsquigarrow a) \rightsquigarrow [(((x \rightsquigarrow y) \rightsquigarrow a) \rightsquigarrow a) \rightsquigarrow ((y \rightsquigarrow a) \rightsquigarrow a)] \quad \text{by (C)} \\ &= ((x \rightsquigarrow a) \rightsquigarrow a) \rightsquigarrow [(y \rightsquigarrow a) \rightsquigarrow ((((x \rightsquigarrow y) \rightsquigarrow a) \rightsquigarrow a) \rightsquigarrow a)] \quad \text{by (C)} \\ &= ((x \rightsquigarrow a) \rightsquigarrow a) \rightsquigarrow [(y \rightsquigarrow a) \rightsquigarrow ((x \rightsquigarrow y) \rightsquigarrow a)] \quad \text{by Proposition 2.5(xii)} \\ &= (y \rightsquigarrow a) \rightsquigarrow [((x \rightsquigarrow a) \rightsquigarrow a) \rightsquigarrow ((x \rightsquigarrow y) \rightsquigarrow a)] \quad \text{by (C)} \\ &= (y \rightsquigarrow a) \rightsquigarrow [(x \rightsquigarrow y) \rightsquigarrow (((x \rightsquigarrow a) \rightsquigarrow a) \rightsquigarrow a)] \quad \text{by (C)} \\ &= (y \rightsquigarrow a) \rightsquigarrow [(x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow a)] \quad \text{by Proposition 2.5(xii)} \\ &= (y \rightsquigarrow a) \rightsquigarrow [(y \rightsquigarrow x) \rightsquigarrow (y \rightsquigarrow a)] \quad \text{by (L2)} \\ &= (y \rightsquigarrow x) \rightsquigarrow [(y \rightsquigarrow a) \rightsquigarrow (y \rightsquigarrow a)] \quad \text{by (C)} \\ &= (y \rightsquigarrow x) \rightsquigarrow 1 \quad \text{by (L1)} \\ &= 1. \quad \text{by (L1)} \end{aligned} \tag{1}$$

Hence, $((x \rightsquigarrow y) \rightsquigarrow a) \rightsquigarrow a \leq ((x \rightsquigarrow a) \rightsquigarrow a) \rightsquigarrow ((y \rightsquigarrow a) \rightsquigarrow a)$. Then

$$k_a^2(x \rightsquigarrow y) = ((x \rightsquigarrow y) \rightsquigarrow a) \rightsquigarrow a \leq ((x \rightsquigarrow a) \rightsquigarrow a) \rightsquigarrow ((y \rightsquigarrow a) \rightsquigarrow a) = k_a^2(x) \rightsquigarrow k_a^2(y).$$

Therefore, $k_a^2(x) \in \mathcal{M}od(\mathcal{L})$. □

Theorem 3.35. *Let $\mathcal{L}$ be a CKL-algebra. Then $\ker(k_a^2)$ is an ideal of $\mathcal{L}$.*

Proof. Clearly, by $(L1)$, we have

$$k_a^2(1) = (1 \rightsquigarrow a) \rightsquigarrow a = a \rightsquigarrow a = 1,$$

thus, $1 \in \ker(k_a^2)$. Assume $x, x \rightsquigarrow y \in \ker(k_a^2)$. Then

$$\begin{aligned}
(y \rightsquigarrow a) \rightsquigarrow a &= 1 \rightsquigarrow ((y \rightsquigarrow a) \rightsquigarrow a) && \text{by (L1)}\\
&= (((x \rightsquigarrow y) \rightsquigarrow a) \rightsquigarrow a) \rightsquigarrow ((y \rightsquigarrow a) \rightsquigarrow a) && \text{since } x \rightsquigarrow y \in \ker(k_a^2)\\
&= (y \rightsquigarrow a) \rightsquigarrow [(((x \rightsquigarrow y) \rightsquigarrow a) \rightsquigarrow a) \rightsquigarrow a] && \text{by (C)}\\
&= (y \rightsquigarrow a) \rightsquigarrow ((x \rightsquigarrow y) \rightsquigarrow a) && \text{Proposition 2.5(xii)}\\
&= (y \rightsquigarrow a) \rightsquigarrow [1 \rightsquigarrow ((x \rightsquigarrow y) \rightsquigarrow a)] && \text{by (L1)}\\
&= (y \rightsquigarrow a) \rightsquigarrow [((x \rightsquigarrow a) \rightsquigarrow a) \rightsquigarrow ((x \rightsquigarrow y) \rightsquigarrow a)] && \text{since } x \in \ker(k_a^2)\\
&= (y \rightsquigarrow a) \rightsquigarrow [(x \rightsquigarrow y) \rightsquigarrow (((x \rightsquigarrow a) \rightsquigarrow a) \rightsquigarrow a)] && \text{by (C)}\\
&= (y \rightsquigarrow a) \rightsquigarrow [(x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow a)] && \text{Proposition 2.5(xii)}\\
&= (x \rightsquigarrow y) \rightsquigarrow [(y \rightsquigarrow a) \rightsquigarrow (x \rightsquigarrow a)] && \text{by (C)}\\
&= 1. \quad \text{Proposition 2.5(vii)}
\end{aligned}$$

Hence, $k_a^2(y) = (y \rightsquigarrow a) \rightsquigarrow a = 1$, and so $y \in \ker(k_a^2)$. Therefore, $\ker(k_a^2)$ is an ideal of $\mathcal{L}$. □

Proposition 3.36. *Assume $\mathcal{L}$ is a KL-algebra. If $\mathcal{I}$ and $\mathcal{J}$ are two ideals of $\mathcal{L}$ such that $\mathcal{I} \cap \mathcal{J} = \{1\}$, then $k_x^2(y) = k_y^2(x) = 1$ for all $x \in \mathcal{I}$ and $y \in \mathcal{J}$.*

Proof. Let $\mathcal{I}$ and $\mathcal{J}$ be two ideals of $\mathcal{L}$ such that $\mathcal{I} \cap \mathcal{J} = \{1\}$. Suppose $x \in \mathcal{I}$ and $y \in \mathcal{J}$. By (I_3), we have $(x \rightsquigarrow y) \rightsquigarrow y \in \mathcal{I}$ and by Proposition 2.3, $y \leq (x \rightsquigarrow y) \rightsquigarrow y$, and so $(x \rightsquigarrow y) \rightsquigarrow y \in \mathcal{J}$. Then $(x \rightsquigarrow y) \rightsquigarrow y \in \mathcal{I} \cap \mathcal{J} = \{1\}$, and so $k_y^2(x) = 1$. By the similar way we can prove that $k_x^2(y) = 1$. □

We define the implication $\to$ on $\mathcal{R}(\mathcal{L}) = \{k_a \mid a \in \mathcal{L}\}$ as follows:

$$\to: \mathcal{R}(\mathcal{L}) \times \mathcal{R}(\mathcal{L}) \to \mathcal{R}(\mathcal{L}), \quad \to (k_a, k_b) \mapsto k_a(x) \rightsquigarrow k_b(x).$$

If $\mathcal{L}$ is a positive L-algebra, then, we have

$$(k_a \to k_b)(x) = k_a(x) \rightsquigarrow k_b(x) = (x \rightsquigarrow a) \rightsquigarrow (x \rightsquigarrow b) = x \rightsquigarrow (a \rightsquigarrow b) = k_{a \rightsquigarrow b}(x).$$

Hence, $k_a \to k_b \in \mathcal{R}(\mathcal{L})$.

Note. Define an order " $\lessdot$ " on $\mathcal{R}(\mathcal{L})$ as follows:
For any $k_a, k_b \in \mathcal{R}(\mathcal{L})$

$$k_a \leq k_b \;\Leftrightarrow\; (k_a \to k_b)(x) = k_1(x),$$

for all $x \in \mathcal{L}$.

Obviously, if $\mathcal{L}$ is a positive L-algebra, then $(\mathcal{R}(\mathcal{L}), \leq)$ is a partially ordered set. Since

$$(k_a \to k_a)(x) = k_a(x) \rightsquigarrow k_a(x) = (x \rightsquigarrow a) \rightsquigarrow (x \rightsquigarrow a) = 1 = x \rightsquigarrow 1 = k_1(x).$$

So, $\leq$ is reflexive. Also, if $k_a \leq k_b$ and $k_b \leq k_a$, then

$$k_a(a) \leq k_b(a) \Rightarrow a \rightsquigarrow a = 1 \leq a \rightsquigarrow b \;\Rightarrow a \rightsquigarrow b = 1 \;\Rightarrow a \leq b,$$

$$k_b(b) \leq k_a(b) \Rightarrow b \rightsquigarrow b = 1 \leq b \rightsquigarrow a \;\Rightarrow b \rightsquigarrow a = 1 \;\Rightarrow b \leq a,$$

and so $a = b$. Thus, $k_a = k_b$. Now, if $k_a \leq k_b$ and $k_b \leq k_c$, then for any $x \in \mathcal{L}$ we have $x \rightsquigarrow a \leq x \rightsquigarrow b$ and $x \rightsquigarrow b \leq x \rightsquigarrow c$. Thus $x \rightsquigarrow a \leq x \rightsquigarrow c$. Hence, $k_a \leq k_c$. Therefore, $(\mathcal{R}(\mathcal{L}), \leq)$ is a partially ordered set.

Theorem 3.37. *If $\mathcal{L}$ is a positive L-algebra, then $(\mathcal{R}(\mathcal{L}), \to, k_1)$ is a dual BCK-algebra.*

Proof. The proof is clear, see [10]. □

Conclusion

In this paper, the notion of modal operators on L-algebras is introduced and some important properties of this operator are investigated. In order for the kernel of modal operator to be ideal, what conditions are required, is investigated. Relations between modal operator and endomorphism of L-algebras are studied. Also, notion of positive L-algebra is defined and a characterization of positive L-algebra is established. Finally, it is shown a map k_a is a modal operator and it was proved that the set of all k_a on a positive L-algebra makes a dual BCK-algebra.

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