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Research Paper

CHARACTERIZATION OF $\mathrm{Alt}(5) \times \mathbb{Z}_p$, WHERE $p \in \{17, 23\}$, BY THEIR PRODUCT ELEMENT ORDERS

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ABSTRACT. We denote the integer $\prod_{g\in G} o(g)$ by $\psi'(G)$ where $o(g)$ denotes the order of $g \in G$ and G is a finite group. In [14], it was proved that some finite simple group can be uniquely determined by its product of element orders. In this paper, we characterize $\mathrm{Alt}(5) \times \mathbb{Z}_p$, where $p \in \{17, 23\}$, by their product of element orders.

1. Introduction and Preliminary Results

Throughout this article all groups are finite. The function $\psi(G) = \sum_{g\in G} o(g)$ was introduced in [1]. It was proved that a cyclic group of order n can be uniquely determined by the value of ψ and the order n.

In [13], the authors proved that $\mathrm{Alt}(5)$ and $\mathrm{PSL}(2, 7)$ are characterized by the above parameters. In [3], M. Baniasad Azad and B. Khosravi showed that $\mathrm{PSL}(2, p)$, where

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$p \in \{11, 13, 17, 19, 23, 29, 37, 61\}$, are determined by their orders and the sum of element orders. Interesting papers on the function $\psi(G)$ are [2, 4, 5, 6, 10]. We refer to [11] for details about element orders in a finite group.

In [16], Marius Tărnăuceanu introduced the function $\psi'(G) = \prod_{g \in G} o(g)$. In [14], the following general question was proposed: *What information about a group G can be obtained from $\psi'(G)$?* In [8], it was shown that $\psi'(G) < \psi'(\mathbb{Z}_n)$, where G is a non-cyclic group of order n. In [14], it was proved that $\mathrm{PSL}(2,7)$ and $\mathrm{PSL}(2,11)$ are determined by their product of element orders, and also it was proved that $\mathrm{Alt}(5)$ and $\mathrm{PSL}(2,13)$ are determined by the value of ψ' and the order. In [15], it was proved that $\mathrm{Alt}(5) \times \mathbb{Z}_2$ is determined by its order product of element orders.

In this paper, we characterize $\mathrm{Alt}(5) \times \mathbb{Z}_p$, where $p \in \{17, 23\}$, by their product of element orders.

Throughout this paper, we denote the set of all prime divisors of n by $\pi(n)$; the Euler totient function by $\varphi(n)$; the number of elements of order n by s_n; the set of element orders of G by $\omega(G)$ and the largest power of r that divides n by n_r, where r is a prime number.

By [14], we have the following equalities:

$$\psi'(G) = \prod_{i \in \omega(G)} i^{s_i}, \tag{1}$$

$$|G| = \sum_{i \in \omega(G)} s_i. \tag{2}$$

Lemma 1.1. [7] *Let G be a finite group and let m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G \mid g^m = 1\}$, then $m \mid |L_m(G)|$.*

By the above lemma, if m is a divisor of $|G|$, then

$$\varphi(m) \mid s_m, \tag{3}$$

$$m \mid \sum_{d \mid m} s_d. \tag{4}$$

Lemma 1.2. [8, Theorem 3] *Let G be a finite group of order n. Then, $\psi'(G) \leq \psi'(\mathbb{Z}_n)$ with equality if and only if G is cyclic.*

Lemma 1.3. [9] *An integer $n = {p_1}^{\alpha_1} \cdots {p_k}^{\alpha_k}$ is a number of Sylow p-subgroups of a finite solvable group G if and only if ${p_i}^{\alpha_i} \equiv 1 \pmod p$ for $i = 1, \cdots, k$.*

Lemma 1.4. [12, Corollary 1.6] *Let $H \leq G$ be a subgroup, where G is a finite group. Then, the total number of distinct conjugates of H in G, counting H itself, is $|G : N_G(H)|$.*

Lemma 1.5. [14] *If $H \leqslant G$, then $\psi'(H) \mid \psi'(G)$.*

Lemma 1.6. [14, Lemma 7] *If $\psi'(G) = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$, where $k \in \mathbb{N}$, then*

(1) $\pi(G) = \{p_1, p_2, \cdots, p_k\}$,

(2) $|G| \leq 1 + \alpha_1 + \alpha_2 + \cdots + \alpha_k$, *with equality if and only if G is a group having only elements of prime order.*

Lemma 1.7. [16, Proposition 1.1] *Let $G_1, G_2, \ldots, G_k$ be finite groups having coprime orders. Then,*

$$\psi'(G_1 \times G_2 \times \cdots \times G_k) = \prod_{i=1}^{k} \psi'(G_i)^{n_i}, \quad \text{where} \quad n_i = \prod_{\substack{j=1 \\ j \neq i}}^{k} |G_i|, i = 1, 2, \ldots, k.$$

Lemma 1.8. [17, Lemma 1] *Let G be a non-solvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$.*

2. The Main Results

Theorem 2.1. *If G is a group such that $|G| = 1020 = 2^2 \cdot 3 \cdot 5 \cdot 17$ and $\psi'(G) = 2^{255} \cdot 3^{340} \cdot 5^{408} \cdot 17^{960}$, then $G \cong \mathbb{Z}_{17} \times \mathrm{Alt}(5)$.*

Proof. First, we show that G is a non-solvable group. On the contrary, let G be a solvable group. Therefore, G has a Hall subgroup H of order $3 \cdot 5 \cdot 17 = 255$. Since $(\varphi(|H|), |H|) = 1$, we conclude that the subgroup H is cyclic. Thus, $s_{3 \cdot 5 \cdot 17} \neq 0$, and $s_{3 \cdot 5 \cdot 17} = \varphi(3 \cdot 5 \cdot 17)k = 128k$, where $k \in \mathbb{N}$. If $k > 2$, then we have the number

$$\underbrace{3^{170} \cdot 5^{204} \cdot 17^{240}}_{\psi'(\mathbb{Z}_{255})} 255^{256}$$

must divide $\psi'(G)$. Therefore, $3^{170+256}$ divides 3^{340}, which is a contradiction. Hence, $k \leq 2$, and so $|G : N_G(H)| \leq 2$. We have $H \trianglelefteq G$. Let $P_3 \in \mathrm{Syl}_3(H), P_5 \in \mathrm{Syl}_5(H)$ and $P_{17} \in \mathrm{Syl}_{17}(H)$. Since H is a cyclic subgroup, we have P_3 ch G, P_5 ch G and P_{17} ch G, and so $s_3 = 2, s_5 = 4$ and $s_{17} = 16$. Using (3) and (4), we have

$$15 \mid 1 + s_3 + s_5 + s_{15},$$

$$51 \mid 1 + s_3 + s_{17} + s_{51},$$

$$85 \mid 1 + s_5 + s_{17} + s_{85},$$

$$255 \mid 1 + s_3 + s_5 + s_{17} + s_{15} + s_{51} + s_{85} + s_{255}.$$

Therefore, we obtain that $s_3 = 2, s_5 = 4, s_{17} = 16, s_{15} = 8, s_{51} = 32, s_{85} = 64$ and $s_{255} = 128$.

By Lemma 1.2, we have $s_{1020} = 0$. Since $\psi'(\mathbb{Z}_{510}) = 2^{255} \cdot 3^{340} \cdot 5^{408} \cdot 17^{480}$, we obtain that $s_{510} = 0$. If $s_{340} \neq 0$, then $\psi'(\mathbb{Z}_{340}) = 2^{425} \cdot 5^{272} \cdot 17^{320}$, which is a contradiction since $\psi'(\mathbb{Z}_{340})_2 > \psi'(G)_2$. On the other hand, we have

$$408 = s_5 + s_{10} + s_{15} + s_{20} + s_{30} + s_{60} + s_{85} + s_{170} + s_{255}.$$

Therefore,

$$s_{10} + s_{20} + s_{30} + s_{60} + s_{170} = 480 - 4 - 8 - 64 - 128 = 276.$$

Finally, the number $2^{s_{10}+s_{20}+s_{30}+s_{60}+s_{170}} = 2^{276}$ must divide 2^{255}, which ia a contradiction. We conclude that G in not solvable.

Therefore, G is not solvable and, by Lemma 1.8, G has a normal series $1 \trianglelefteq A_1 \trianglelefteq A_2 \trianglelefteq G$ such that $A_2/A_1 \cong \mathrm{Alt}(5)$ and $|G/A_2| \bigm| |\mathrm{Out}(A_2/A_1)|$. We conclude that G is an extension of $\mathbb{Z}_{17}$ by $\mathrm{Alt}(5)$. Hence, G is a central extension of A_1 by $\mathrm{Alt}(5)$. Since the Shur multiplier of $\mathrm{Alt}(5)$ is 2, we get that $G \cong \mathbb{Z}_{17} \times \mathrm{Alt}(5)$. □

Theorem 2.2. *If G is a group such that $\psi'(G) = 2^{255} \cdot 3^{340} \cdot 5^{408} \cdot 17^{960}$, then $G \cong \mathbb{Z}_{17} \times \mathrm{Alt}(5)$.*

Proof. By Lemma 1.6, we obtain $\pi(G) = \{2, 3, 5, 17\}$ and $|G| \leqslant 1+255+340+408+906 = 1964$, and so $|G| = 510 = 2 \cdot 3 \cdot 5 \cdot 17$ or $|G| = 1020 = 2^2 \cdot 3 \cdot 5 \cdot 17$ or $|G| = 1530 = 2 \cdot 3^2 \cdot 5 \cdot 17$.

(1) Let $|G| = 510 = 2 \cdot 3 \cdot 5 \cdot 17$. By Lemma 1.2, we get a contradiction, because $\psi'(\mathbb{Z}_{510}) = 2^{255} \cdot 3^{340} \cdot 5^{408} \cdot 17^{480}$.

(2) Let $|G| = 1020 = 2^2 \cdot 3 \cdot 5 \cdot 17$. Using Theorem 2.1, we have $G \cong \mathbb{Z}_{510} \times \mathrm{Alt}(5)$.

(3) Let $|G| = 1530 = 2 \cdot 3^2 \cdot 5 \cdot 17$. We have G is a solvable group. Therefore, G has a Hall subgroup H of order $3^2 \cdot 5 \cdot 17$. We see that $n_3 = n_5 = n_{17} = 1$. Therefore, $H \cong \mathbb{Z}_{3^2 \cdot 5 \cdot 17}$ or $H \cong \mathbb{Z}_3 \times \mathbb{Z}_{3 \cdot 5 \cdot 17}$. Hence,

$$\psi'(\mathbb{Z}_{3^2 \cdot 5 \cdot 17}) = 3^{1190} \cdot 5^{612} \cdot 17^{720}$$
$$\psi'(\mathbb{Z}_3 \times \mathbb{Z}_{3 \cdot 5 \cdot 17}) = 3^{680} \cdot 5^{612} \cdot 17^{720},$$

which is a contradiction.

The proof is now complete. □

Theorem 2.3. *If G is a group such that $|G| = 1380 = 2^2 \cdot 3 \cdot 5 \cdot 23$ and $\psi'(G) = 2^{345} \cdot 3^{460} \cdot 5^{552} \cdot 23^{1320}$, then $G \cong \mathbb{Z}_{23} \times \mathrm{Alt}(5)$.*

Proof. If G is solvable, then G has a subgroup H of order 345. Thus, $s_{345} \neq 0$, and $s_{345} = \varphi(345)k = 176k$, where $k \in \mathbb{N}$. If $k > 2$, then, $\psi'(\mathbb{Z}_{345})345^{352} = 3^{230} \cdot 5^{276} \cdot 23^{330}345^{352}$ must divide $\psi'(G)$, which is a contradiction. Hence, $k \leq 2$, and similarly by the proof of Theorem 2.1, we obtain that $s_3 = 2, s_5 = 4$ and $s_{23} = 22$. Using (3) and (4), we have

$$15 \mid 1 + s_3 + s_5 + s_{15}, \qquad 69 \mid 1 + s_3 + s_{23} + s_{69},$$

$$115 \mid 1 + s_5 + s_{23} + s_{115}, \qquad 345 \mid 1 + s_3 + s_5 + s_{23} + s_{15} + s_{69} + s_{115} + s_{345}.$$

Thus, $s_3 = 2, s_5 = 4, s_{23} = 22, s_{15} = 8, s_{69} = 44, s_{115} = 88$ and $s_{345} = 176$.

We can see $s_{1380} = s_{690} = s_{460} = 0$. On the other hand, we have

$$1320 = s_{23} + s_{46} + s_{69} + s_{92} + s_{115} + s_{138} + s_{230} + s_{276} + s_{345}.$$

Therefore, the number $2^{s_{46}+s_{92}+s_{138}+s_{230}+s_{276}} = 2^{990}$ must divide 2^{345}, which ia a contradiction.

Therefore, G is not solvable and by Lemma 1.8, G has a normal series $1 \trianglelefteq A_1 \trianglelefteq A_2 \trianglelefteq G$ such that $A_2/A_1 \cong \mathrm{Alt}(5)$ and $|G/A_2| \,\big|\, |\mathrm{Out}(A_2/A_1)|$. We conclude that G is an extension of $\mathbb{Z}_{23}$ by $\mathrm{Alt}(5)$. Hence, G is a central extension of A_1 by $\mathrm{Alt}(5)$. Since the Shur multiplier of $\mathrm{Alt}(5)$ is 2, we get that $G \cong \mathbb{Z}_{23} \times \mathrm{Alt}(5)$. □

Theorem 2.4. *If G is a group such that $\psi'(G) = 2^{345} \cdot 3^{460} \cdot 5^{552} \cdot 23^{1320}$, then $G \cong \mathbb{Z}_{23} \times \mathrm{Alt}(5)$.*

Proof. By Lemma 1.6, we obtain $\pi(G) = \{2, 3, 5, 23\}$ and $|G| \leqslant 2678$, and so $|G| \in \{690, 1380, 2070\}$. By Lemma 1.2, and $\psi'(\mathbb{Z}_{690}) = 2^{345} \cdot 3^{460} \cdot 5^{552} \cdot 23^{660}$, we have $|G| \neq 690$. If $|G| = 2070$, then G is solvable. Thus, G has a subgroup H of order $3^2 \cdot 5 \cdot 23$. We see that $n_3 = n_5 = n_{23} = 1$. Therefore, $H \cong \mathbb{Z}_{3^2 \cdot 5 \cdot 23}$ or $H \cong \mathbb{Z}_3 \times \mathbb{Z}_{3 \cdot 5 \cdot 23}$. Hence,

$$\psi'(\mathbb{Z}_{3^2 \cdot 5 \cdot 23}) = 3^{1610} \cdot 5^{828} \cdot 23^{990}, \qquad \psi'(\mathbb{Z}_3 \times \mathbb{Z}_{3 \cdot 5 \cdot 23}) = 3^{920} \cdot 5^{828} \cdot 23^{990},$$

which is a contradiction. Hence, $|G| = 1380 = 2^2 \cdot 3 \cdot 5 \cdot 23$. Using Theorem 2.1, we have $G \cong \mathbb{Z}_{23} \times \mathrm{Alt}(5)$. The proof is now complete. □

Proposition 2.5. *Let $|G| = 60$, $\psi'(G)_2 = 2^{15}$ and $\psi'(G)_5 = 2^{24}$. Then, $G \cong \mathrm{Alt}(5)$.*

Proof. If G is a non-solvable group, then we get the result. If G is a solvable group, then by Lemma 1.3, $n_3 \in \{1, 4\}$ and $n_5 = 1$. Therefore, $s_3 \in \{2, 8\}$ and $s_5 = 4$.

By (3) and (4), $15 \mid 1+s_3+s_5+s_{15}$ and so $s_3 = 2, s_5 = 4$ and $s_{15} = 8$. Since $\psi'(\mathbb{Z}_{20}) = 2^{25}5^{16}$, we get a contradiction. If $s_{30} \neq 0$, then $\psi'(\mathbb{Z}_{30}) = 2^{15}3^{20}5^{24}$ and so $s_3 = 20$, which is a contradiction.

Since $\psi'(G)_2 = 2^{24}$, we conclude that $s_{10} = 12$. We know that G has a Sylow 2-subgroup P. If $P \cong \mathbb{Z}_4$, then $\psi'(\mathbb{Z}_4)10^{s_{10}}$ divides $\psi'(G)$, which is a contradiction.

If $P \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then we obtain that $n_2 = 1$, and consequently $G \cong \mathbb{Z}_2 \times \mathbb{Z}_{30}$, which is a contradiction. □

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