RESULTS ON GENERALIZED DERIVATIONS IN PRIME RINGS

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ABSTRACT. A prime ring $S$ with the centre $Z$ and generalised derivations that meet certain algebraic identities is considered. Let’s assume that $\Psi$ and $\Phi$ are two generalised derivations associated with $\psi$ and $\varphi$ on $S$, respectively. In this article, we examine the following identities: (i) $\Psi(a)b − a\Phi(b) \in Z$, (ii) $\Psi(a)b − b\Phi(a) \in Z$, (iii) $\Psi(a)a − b\Phi(b) \in Z$, (iv) $\Psi(a)a − a\Phi(b) \in Z$, (v) $\Psi(a)a − b\Phi(a) \in Z$, for every $a, b \in J$, where $J$ is a non-zero two sided ideal of $S$. We also provide an example to show that the condition of primeness imposed in the hypotheses of our results is essential.

1. Introduction

Let $S$ be a ring with center $Z$. For any $a, b \in S$ the symbol $(a \circ b) [a, b]$ denotes the (anti-) commutator $(ab + ba) ab − ba$. If $aSb = (0)$ (where $a, b \in S$) implies $a = 0$ or $b = 0$, a ring $S$ is said to be a prime ring. The non-zero central elements of a prime ring are not zero divisors. We let $Q_r = Q_r(S)$ (resp. $Q_l = Q_l(S)$) denote the right (resp. left) Martindale ring of quotient
of $S$. We define the symmetric Martindale ring of quotient of $S$ as $Q_s = Q_s(S)$. The extended centroid of $S$ is denoted by the ring $C$. The ring $SC$ is referred to as the $S$ central closure. It is known that $S \subseteq SC \subseteq Q_s \subseteq Q_r$ (and $Q_l$). If $S$ is a prime ring, it is obvious that $SC$, $Q_s$, $Q_r$ (and $Q_l$) are also prime rings. For further information, we suggest the reader to the book [6]. An additive mapping $\psi : S \rightarrow S$ is said to be a derivation, if $\psi(ab) = \psi(a)b + a\psi(b)$ holds for every $a, b \in S$. An additive mapping $\Psi : S \rightarrow S$ is said to be a generalized derivation, if there exists a derivation $\psi : S \rightarrow S$ such that $\Psi(ab) = \Psi(a)b + a\psi(b)$ holds for every $a, b \in S$. As a result, each derivation is a generalised derivation.

The relationship between the commutativity of a prime ring $S$ and the behaviour of a derivation or extended derivation on $S$ has attracted continued research. Posner [19] initiated the study of such mappings, and he established the relationship between the commutativity of a prime ring $S$. In [15], Herstein demonstrated that $S$ is commutative if $\psi$ is a non-zero derivation of $S$ and $[\psi(a), \psi(b)] = 0$ for every $a, b \in S$, where $S$ is a 2-torsion free prime ring. After that, Daif [11] improved this result for ideal of semi-prime ring. A semi-prime ring $S$ must be commutative, according to Daif and Bell’s proof in [12], if it admits a derivation $\psi$ such that $[\psi(a), \psi(b)] \pm [a, b] = 0$ for every $a, b \in S$. This classical theorem was extended to include the class of generalised derivations by Bell and Rehman in [6]. Later, many authors have studied the action of such types of mappings as derivations, generalized derivations, skew derivations etc. on semi-prime and prime (rings) ideals in different directions. (see [3, 4, 5, 16, 21, 22] where references can be found).

Recently many authors viz. [13, 2] and [14, Theorem 3.4] have gained commutativity of semi-prime and prime rings with derivations satisfying certain algebraic identities. Motivated by these results, in the present article we shall explore the commutativity of ring $S$ satisfying any one of the properties $\Psi(a)b - a\Phi(b) \in Z$, $\Psi(a)b - b\Phi(a) \in Z$, $\Psi(a)a - b\Phi(b) \in Z$, $\Psi(a)a - a\Phi(b) \in Z$, $\Psi(a)a - b\Phi(a) \in Z$, for every $a, b \in J$, where $J$ is a non-zero ideal of $S$. We also provide an example to demonstrate that the hypothesis of primeness imposed in our results is essential.

2. Preliminaries

We will make use of the following fundamental identities that apply to every $a, b, c \in S$ without explicitly mentioning them:

$$[ab, c] = a[b, c] + [a, c]b,$$

$$[a, bc] = b[a, c] + [a, b]c.$$  

Facts 2.1, 2.2 and 2.3 can be verified easily.

**Fact 2.1.** Let $S$ be a prime ring and $a \in S$. If $\{az, z\} \subseteq Z$, then $a \in Z$ or $z = 0$. 


Fact 2.2. Let $S$ be a prime ring and $J$ be a non-zero ideal of $S$. If for $a, b \in S$ and $aJb = (0)$, then $a = 0$ or $b = 0$. In particular, if $aJ = (0)$, then $a = 0$, also if $Jb = (0)$, then $b = 0$.

Fact 2.3. Let $S$ be a prime ring and $J$ a non-zero ideal of $S$. Suppose that $S$ admits generalized derivation $\Psi$ with associated derivations $\psi$. If $\Psi(a) = 0$ for every $a \in J$, then $\Psi(a) = 0$ for every $a \in S$, that is $\Psi = 0$.

Fact 2.4. [2, Lemma 2.2] If a prime ring $S$ contains a commutative non-zero ideal, then $S$ is commutative.

Fact 2.5. [1, Lemma 2.5] Let $S$ be a prime ring and $J$ a non-zero ideal of $S$ such that
(i) $[a, b] \in Z$ for every $a, b \in J$; or
(ii) $(a \circ b) \in Z$ for every $a, b \in J$,
then $S$ is commutative.

Fact 2.6. [10] Let $S$ be a prime ring and $J$ a non-zero ideal of $S$. Then, $S$, $J$, and $Q_r$ (resp. $Q_l$) satisfy the same generalized polynomial identities with coefficients in $Q_r$ (resp. $Q_l$).

Fact 2.7. [18, Theorem 3] Let $S$ be a prime ring, then the following statements hold:
(i) Every generalized derivation of $S$ can be uniquely extended to $Q_r$ (and $Q_l$).
(ii) Every derivation of $S$ can be uniquely extended to $Q_r$ (and $Q_l$). Since every derivation is generalized derivation, it follows from (i).

Fact 2.8. [8, Lemma 2] Let $S$ be a prime ring and $\Psi : S \to SC$ be an additive map satisfying $\Psi(ab) = \Psi(a)b$ (resp. $\Psi(ab) = a\Psi(b)$) for every $a, b \in S$. Then there exists $q \in Q_r$ (resp. $q \in Q_l$) such that $\Psi(a) = qa$ (resp. $\Psi(a) = aq$) for every $a \in S$.

Fact 2.9. [17, Theorem L] Let $S$ be a prime ring with a derivation $\psi$, $J$ a left ideal of $S$ and $n, m$ two positive integers. Suppose that $[\psi(a^m), a^n] = 0$ for every $x \in J$. Then either $\psi = 0$ or $S$ is commutative.

In [14, Theorem 2], they worked on a non-zero left ideal, but we will take a special case, when $J$ is a non-zero ideal of $S$, as follows:

Fact 2.10. [17, Theorem 2] Let $S$ be a prime ring, $J$ a non-zero ideal of $S$, $\psi$ a derivation of $S$, and $m_i$ fixed positive integers, where $i \in \{1, \ldots, 4\}$. If $[\psi(a^{m_1})a^{m_2}, a^{m_3}]_{m_4} = 0$ ($[a^{m_1}\psi(a^{m_2}), a^{m_3}]_{m_4} = 0$) for every $a \in J$, then $\psi = 0$ or $S$ is commutative.

3. The Main Result

Theorem 3.1 (Main theorem). Let $S$ be a prime ring and $J$ a non-zero ideal of $S$. Assume that $S$ admit generalized derivations $\Psi$ and $\Phi$ with associated derivations $\psi$ and $\varphi$, respectively.
(1) If \( \Psi(a)b \pm a\Phi(b) \in Z \) for every \( a, b \in J \), then
   (i) \( S \) is commutative or
   (ii) \( \Psi(a) = aq \) with \( \psi(a) = [a, q] \) and \( \Phi(a) = qa \) with \( \varphi = 0 \) for every \( a \in J \) and some \( q \in Q_s \).

(2) \( S \) is commutative or \( \Psi = \Phi = 0 \) if, for every \( a, b \in J \), satisfies any one of the following
   (i) \( \Psi(a)b \pm b\Phi(a) \in Z \),
   (ii) \( \Psi(a)a \pm b\Phi(b) \in Z \),
   (iii) \( \Psi(a)a \pm a\Phi(b) \in Z \),
   (iv) \( \Psi(a)a \pm b\Phi(a) \in Z \).

We need some auxiliary lemmas in order to prove our main theorem.

Throughout this section, \( S \) is a prime ring and \( J \) a non-zero ideal of \( S \) such that \( S \) admit generalized derivations \( \Psi \) and \( \Phi \) with associated derivations \( \psi \) and \( \varphi \), respectively.

**Lemma 3.2.** If \( \Psi(a)b - a\Phi(b) \in Z \) for every \( a, b \in J \), then
   (i) \( S \) is commutative or
   (ii) \( \Psi(a) = aq \) with \( \psi(a) = [a, q] \) and \( \Phi(a) = qa \) with \( \varphi = 0 \) for every \( a \in J \) and some \( q \in Q_s \).

**Proof.** Assume that

\[ \Psi(a)b - a\Phi(b) \in Z \]

for every \( a, b \in J \). Replacing \( b \) by \( bw \) in (1), where \( w \in J \), we have

\[ (\Psi(a)b - a\Phi(b))w - ab\varphi(w) \in Z. \]

for every \( a, b, w \in J \). Using (1) in (2), we get \([ab\varphi(w), w] = 0\). Putting \( a = b = w \) in the last relation, we obtain \( w^2[\varphi(w), w] = 0 \), and so \( \varphi = 0 \) or \( S \) is commutative, by Fact 2.10. In case \( S \) is commutative, as desired. Now, in case

\[ \varphi = 0, \]

as desired. Now, by using (3) in (2), we see that \( (\Psi(a)b - a\Phi(b))w \in Z \), and by using (1) and Fact 2.1 in the last relation, we find that \( \Psi(a)b - a\Phi(b) = 0 \) or \( w \in Z \). If \( w \in Z \), then \( J \subseteq Z \), hence \( S \) is commutative, by Fact 2.4. Now, if

\[ \Psi(a)b - a\Phi(b) = 0 \]

for every \( a, b \in J \). Then by [8, p. 200], there exists \( q \in Q_s \) such that

\[ \Psi(a) = aq \]

(5)
for every $a \in J$ and

$$
\Phi(a) = qa
$$

for every $a \in J$. On other hand, from definition of $\Psi$, we have $\Psi(ab) = \Psi(a)b + a\psi(b)$, and by using (5) in the last relation, we arrive at $abq = aqb + a\psi(b)$, that is, $a(bq - qb - \psi(b)) = 0$, hence $J(bq - qb - \psi(b)) = (0)$, thus $bq - qb - \psi(b) = 0$, by Fact 2.2, this implies that $\psi(b) = [b, q]$ for every $b \in J$ and some $q \in Q_s$, as desired. \[
\]

**Corollary 3.3.** If $\varphi \neq 0$ and $\Psi(a)b - a\Phi(b) \in Z$ for every $a, b \in J$, then $S$ is commutative.

**Lemma 3.4.** If $\Psi(a)b - b\Phi(a) \in Z$ for every $a, b \in J$, then $S$ is commutative or $\Psi = \Phi = 0$.

**Proof.** Assume that

$$
\Psi(a)b - b\Phi(a) \in Z
$$

for every $a, b \in J$.

**Case (I):** Suppose that $J \cap Z \neq (0)$. Replacing $b$ by $z$ in (1), where $0 \neq z \in J \cap Z$, we have $\Psi(a) - \Phi(a) \in Z$, by Fact 2.1, that is, $(\Psi - \Phi)(a) \in Z$. Putting $H = \Psi - \Phi$ with $\varphi = \psi - \varphi$ in the last relation, we get

$$
H(a) \in Z
$$

for every $a \in J$. Note that $H$ is a generalized derivation of $S$ with associated derivation $\varphi$.

Now, replacing $a$ by $ab$ in (8), where $b \in J$, we obtain

$$
H(a)b + a\varphi(b) \in Z
$$

for every $a, b \in J$. By using (8) in (9), we see that $[a\varphi(b), b] = 0$. Putting $a = b$ in the last relation, we find that $b[\varphi(b), b] = 0$, hence $\varphi = 0$ or $S$ is commutative, by Fact 2.10. In case $S$ is commutative, as desired. Now, in case $\varphi = 0$. Using the last relation in (9), we conclude that

$$
H(a)b \in Z.
$$

for every $a, b \in J$. By using (8) in (10) and by Fact 2.1, we get $H(a) = 0$ or $b \in Z$. If $b \in Z$ for every $b \in J$, then $J \subseteq Z$, and by using Fact 2.4 in the last relation, we see that $S$ is commutative. Now, in case $H(a) = 0$, then $H = 0$, by Fact 2.3, that is, $\Psi - \Phi = 0$, hence

$$
\Psi = \Phi.
$$

Using (11) in (7), we obtain $\Psi(a)b - b\Psi(a) \in Z$, and so

$$
[H(a), b] \in Z
$$
for every $a, b \in J$. Replacing $b$ by $\Psi(a)b$ in (12), we get $\Psi(a)[\Psi(a), b] \in Z$, and by using (12) and Fact 2.1 in the last expression, we have $\Psi(a) \in Z$ or $[\Psi(a), b] = 0$. Note that $\Psi(a) \in Z$ if and only if $[\Psi(a), b] = 0$, and so

$$\Psi(a) \in Z$$

for every $a \in J$. Now, the same as in Eq. (8), we get $S$ is commutative or $\Psi = 0$. In case $S$ is commutative, as desired. Now, in case $\Psi = 0$, Using the last relation in (11), we arrive at $\Psi = \Phi = 0$, as desired.

**Case (II):** Suppose that $J \cap Z = (0)$. Since $a, b \in J$ in (7) and by using the our assumption in Case (II), we get

$$\Psi(a)b - b\Phi(a) = 0$$

for every $a, b \in J$. Left multiplying (14) by $t$, where $t \in J$, we have

$$t\Psi(a)b - tb\Phi(a) = 0$$

for every $t, a, b \in J$. Replacing $b$ by $tb$ in (14), where $t \in J$, we see that

$$\Psi(a)tb - tb\Phi(a) = 0$$

for every $t, a, b \in J$. Subtracting (15) from (16), we obtain $\Psi(a)tb - t\Psi(a)b = 0$ this implies that $(\Psi(a)t - t\Psi(a))b = 0$, and so $(\Psi(a)t - t\Psi(a))J = (0)$ and by Fact 2.2, we conclude that $\Psi(a)t - t\Psi(a) = 0$, hence $[\Psi(a), t] = 0$, that is,

$$\Psi(a) \in Z(R).$$

Now, the same as in Eq. (13), we get $S$ is commutative or $\Psi = 0$. In case $S$ is commutative, as desired. Now, in case $\Psi = 0$, as desired. On other hand, since $\Psi = 0$, and by using the last relation in (14), gives $b\Phi(a) = 0$, that is, $J\Phi(a) = (0)$ and by Fact 2.2, we find that $\Phi(a) = 0$, and by Fact 2.3, we get $\Phi = 0$, as desired. □

**Lemma 3.5.** If $\Psi(a)a - b\Phi(b) \in Z$ for every $a, b \in J$, then $S$ is commutative or $\Psi = \Phi = 0$.

**Proof.** Assume that

$$\Psi(a)a - b\Phi(b) \in Z$$

for every $a, b \in J$. Putting $b = 0$ in (18), we get

$$\Psi(a)a \in Z$$

for every $a \in J$. By linearizing (18), we obtain

$$\Psi(a)b + \Psi(b)a \in Z$$
for every $a, b \in J$.

**Case (I):** Suppose that $J \cap Z \neq (0)$. From (20), we have
\[
[\Psi(a)b + \Psi(b)a, s] = 0
\]
for every $a, b, s \in J$. Replacing $b$ by $bs$ in (21), we see that
\[
[\Psi(a)bs + \Psi(bs)a, s] = 0
\]
for every $a, b, s \in J$. Right multiplying (21) by $s$, we find that
\[
[\Psi(a)bs - \Psi(b)as, s] = 0
\]
for every $a, b, s \in J$. Comparing (22) and (23), we infer that
\[
[\Psi(bs)a - \Psi(bsa), s] = 0
\]
for every $a, b, s \in J$. It implies that
\[
[\Psi(b)s + \Psi(s)a, s] = 0
\]
for every $a, b, s \in J$. That is,
\[
[\Psi(s)a + b\psi(s)a - \Psi(b)as, s] = 0
\]
for every $a, b, s \in J$. Putting $a = z$ in the last relation, where $0 \neq z \in J \cap Z$, we deduce that
\[
[b\psi(s), s]z = 0
\]
for every $b, s \in J$. Since the non-zero central elements of a prime ring are not zero divisors, we get $[b\psi(s), s] = 0$ for every $b, s \in J$. Again, putting $b = z$ in the last relation, where $0 \neq z \in J \cap Z$, we have $z[\psi(s), s] = 0$ for every $s \in J$. It follows that $[\psi(s), s] = 0$ for every $s \in J$, and by Fact 2.9, we get $\psi = 0$ or $S$ is commutative. In case $S$ is commutative, as desired. Now, if $\psi = 0$, then from definition of $\Psi$, we obtain $\Psi(ab) = \Psi(a)b$, and by Fact 2.8, we see that
\[
\Psi(a) = qa
\]
for every $a \in J$ and some $q \in Q_r$. Using (24) in (19), we find that $qa^2 \in Z$. Putting $a = z$ in the last relation, where $0 \neq z \in J \cap Z$, we arrive at $qz^2 \in Z$. That is $[qz^2, t] = 0$ for every $t \in S$. Hence $z^2[q, r] = 0$ for every $r \in S$, and by using Fact 2.6 in the last relation, we deduce that $z^2[q, t] = 0$ for every $t \in Q_r$. Since $z \neq 0$, we have $[q, t] = 0$ for every $t \in Q_r$. That is
\[
q \in C.
\]
Now, from (20), we obtain
\[
[\Psi(a)b + \Psi(b)a, t] = 0
\]
for every $t \in S$ and $a, b \in J$. Using Facts 2.6 and 2.7 in the last relation, we get
\[
[\Psi(a)b + \Psi(b)a, t] = 0
\]
for every $a, b, t \in Q_r$. That is

$$\Psi(a)b + \Psi(b)a \in C$$

for every $a, b \in Q_r$. Using (24) in the last relation, we see that $qab + qba \in C$, it implies that

$q(a \circ b) \in C$, and by using (25) and Fact 2.3 in the last relation, we get two cases: $a \circ b \in C$

or $q = 0$. In case $a \circ b \in C$, then by Fact 2.5(ii), we obtain $Q_r$ is commutative, hence $S$ is

commutative, as desired. Now, assume that $R$ is not commutative. Now, in case $q = 0$, then from (24), we get $\Psi(a) = 0$ and by Fact 2.3, we obtain $\Psi = 0$.

Now, we will prove that $\Phi = 0$. Using (19) in (18), we have

$$b\Phi(b) \in Z$$

for every $b \in J$. Putting $b = z$ in (26), where $0 \neq z \in J \cap Z$, and by Fact 2.1, we find that

$$\Phi(z) \in Z.$$

By linearizing (26), we obtain

$$a\Phi(b) + b\Phi(a) \in Z$$

for every $a, b \in J$. From (28), we have

$$[a\Phi(b) + b\Phi(a), s] = 0$$

for every $a, b, s \in J$. Replacing $b$ by $sb$ in (29), we see that

$$[a\Phi(sb) + sb\Phi(a), s] = 0$$

for every $a, b, s \in J$. Left multiplying (29) by $s$, we find that

$$[sa\Phi(b) + sb\Phi(a), s] = 0$$

for every $a, b, s \in J$. Subtracting (31) from (30), we infer that

$$[a\Phi(sb) - sa\Phi(b), s] = 0$$

for every $a, b, s \in J$. Putting $b = a = z$ in the last relation, where $0 \neq z \in J \cap Z$, we deduce that

$$[z\Phi(zs) - sz\Phi(z), s] = 0$$

for every $s \in J$. That is

$$[z\Phi(z)s + z^2\varphi(s) - sz\Phi(z), s] = 0$$

for every $s \in J$. Using (27) in the last relation, we arrive at $[z^2\varphi(s), s] = 0$ for every $s \in J$. Hence $z^2[\varphi(s), s] = 0$, and so $[\varphi(s), s] = 0$, and by using Fact 2.3 in the last relation, we get $\varphi = 0$ or $S$ is commutative. But, $R$ is not commutative as in assumption, and so $\varphi = 0$. By using the last relation in definition of $\Phi$, we obtain $\Phi(ab) = \Phi(a)b$, and by Fact 2.8, we see
that $\Phi(a) = qa$ for every $a \in J$ and some $q \in Q_r$. Now, the same as in (24), we get $S$ is commutative or $\Phi = 0$. But $R$ is not commutative, and so $\Phi = 0$, as desired.

**Case (II):** Suppose that $J \cap Z = (0)$. Using this assumption in (19), we deduce that

$$\Psi(a)a = 0$$

for every $a \in J$. By linearizing (32), we obtain

$$\Psi(a)b + \Psi(b)a = 0.$$  

for every $a, b \in J$. Replacing $b$ by $ab$ in (33) and using (32), we get

$$\Psi(ab)a = 0.$$  

for every $a, b \in J$. Again, replacing $b$ by $ba$ in (34), we have

$$0 = \Psi(aba)a = (\Psi(ab)a + ab\psi(a))a$$

for every $a, b \in J$. Using (34) in the last relation, we see that $ab\psi(a)a = 0$ this implies that $\psi(a)ab\psi(a)a = 0$ which leads to $\psi(a)aj\psi(a)a = 0$, so $\psi(a)a = 0$, that is $[\psi(a), a] = 0$, thus $[\psi(a), a]a = 0$, hence $\psi = 0$ or $S$ is commutative, by Fact 2.10. In case $S$ is commutative, as desired. Now, if $\psi = 0$. Then by using (34) and definition of $\Psi$ in (34), we infer that $\Psi(ab)a = 0$ so $\Psi(a)Ja = (0)$ this implies that $\Psi(a) = 0$ or $a = 0$. But $J \neq 0$, and so $\Psi(a) = 0$, and by Fact 2.3, we get

$$\Psi = 0,$$

as desired. Now, putting $a = 0$ in (18) and since $J \cap Z = (0)$, we get

$$b\Phi(b) = 0.$$  

$b \in J$. By linearizing (36), we get

$$a\Phi(b) + b\Phi(a) = 0.$$  

for every $a, b \in J$. Replacing $a$ by $ab$ in (37) and using (36), we deduce that

$$b\Phi(ab) = 0.$$  

for every $a, b \in J$. Replacing $a$ by $ab$ in (38), we obtain $0 = b\Phi(abb) = (b\Phi(ab))b + bab\varphi(b)$, and by using (38) in the last relation, we infer that $0 = bab\varphi(b)$. Again, replacing $a$ by $b$ in the last relation, this gives $0 = b^3\varphi(b)$, and so $b^3[\varphi(b), b] = 0$, hence $\varphi = 0$ or $S$ is commutative, by Fact 2.10. $S$ is commutative, as desired. Now, if $\varphi = 0$. Then by using definition of $\Phi$ in (38), we find that

$$b\Phi(a)b = 0.$$
$a, b \in J$. Right multiplying (37) by $b$, we get

$$a\Phi(b)b + b\Phi(a) = 0$$

$a, b \in J$. Using (39) in the last relation, we have $a\Phi(b)b = 0$, that is, $J\Phi(b)b = (0)$ and since $J \neq 0$ we obtain

(40) \hspace{1cm} \Phi(b)b = 0.

$b \in J$. By linearizing (40), then

(41) \hspace{1cm} \Phi(a)b + \Phi(b)a = 0.

$a, b \in J$. Replacing $a$ by $ba$ in (41) and using (40), we get $\Phi(ba)b = 0$ and since $\phi = 0$ we get $\Phi(bab) = 0$, that is, $\Phi(b)Jb = (0)$, and so $\Phi(b) = 0$ or $b = 0$. Since $J \neq 0$, we deduce $\Phi(b) = 0$, and by Fact 2.3 we obtain $\Phi = 0$, as desired. □

Lemma 3.6. If $\Psi(a)a - a\Phi(b) \in Z$ for every $a, b \in J$, then $S$ is commutative or $\Psi = \Phi = 0$.

Proof. Assume that

(42) \hspace{1cm} \Psi(a)a - a\Phi(b) \in Z

for every $a, b \in J$. Putting $b = 0$ in (42), we get

(43) \hspace{1cm} \Psi(a)a \in Z

for every $a \in J$. In case $J \cap Z \neq (0)$ or $J \cap Z = (0)$ using the same trick in Lemma 3.5 in Eq. (19) or (32) respectively, we get $S$ is commutative or $\Psi = 0$. In case $S$ is commutative, as desired. Now, in case $\Psi = 0$, as desired. On other hand, using fact that $\Psi = 0$ in (42), we have

(44) \hspace{1cm} a\Phi(b) \in Z.

for every $a, b \in J$.

Case (I): Suppose that $J \cap Z \neq (0)$. Replacing $a$ by $z$ in (44), where $0 \neq z \in J \cap Z$, we see that $z\Phi(b) \in Z$ so $\Phi(b) \in Z$, and by using the last relation in (44), we find that $a \in Z$ or $\Phi(b) = 0$, by Fact 2.1. If $a \in Z$, for every $a \in J$, then $J \subseteq Z$, hence $S$ is commutative by Fact 2.4. Now, if $\Phi(b) = 0$, then by Fact 2.3, we obtain $\Phi = 0$, as desired.

Case (II): Suppose that $J \cap Z = (0)$. Using this assumption in (44), we get $a\Phi(b) = 0$, so $J\Phi(b) = (0)$ and by using Fact 2.2 in the last relation, we find that $\Phi(b) = 0$, and by Fact 2.3, we infer that $\Phi = 0$, as desired. □

Lemma 3.7. If $\Psi(a)a - b\Phi(a) \in Z$ for every $a, b \in J$, then $S$ is commutative or $\Psi = \Phi = 0$. 
Proof. See the proof of Lemma 3.6. □

We provide an example to demonstrate that the primeness requirement in our results is not unnecessary.

**Example 3.8.** Let \( S = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{Z} \right\} \) and \( J = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in \mathbb{Z} \right\} \). We have \( S \) is a ring and \( J \) is a non-zero ideal of \( S \). Define \( \Psi = 0 = \psi \) and \( \Phi = \varphi : S \to S \) by \( \varphi \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \) is a (generalized) derivation of \( S \). All the identities of all our results are satisfied on \( J \), but \( S \) is non-commutative and is not prime with \( \Phi \neq 0 \neq \varphi \). So the condition of primeness is essential.

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**References**


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