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Research Paper

RESULTS ON GENERALIZED DERIVATIONS IN PRIME RINGS

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ABSTRACT. A prime ring S with the centre Z and generalised derivations that meet certain algebraic identities is considered. Let's assume that Ψ and Φ are two generalised derivations associated with ψ and φ on S, respectively. In this article, we examine the following identities: (i) $\Psi(a)b - a\Phi(b) \in Z$, (ii) $\Psi(a)b - b\Phi(a) \in Z$, (iii) $\Psi(a)a - b\Phi(b) \in Z$, (iv) $\Psi(a)a - a\Phi(b) \in Z$, (v) $\Psi(a)a - b\Phi(a) \in Z$, for every $a, b \in J$, where J is a non-zero two sided ideal of S. We also provide an example to show that the condition of primeness imposed in the hypotheses of our results is essential.

1. INTRODUCTION

Let S be a ring with center Z. For any $a, b \in S$ the symbol $(a \circ b)$ $[a, b]$ denotes the (anti-) commutator $(ab + ba)$ $ab - ba$. If $aSb = (0)$ (where $a, b \in S$) implies $a = 0$ or $b = 0$, a ring S is said to be a prime ring. The non-zero central elements of a prime ring are not zero divisors. We let $Q_r = Q_r(S)$ (resp. $Q_l = Q_l(S)$) denote the right (resp. left) Martindale ring of quotient

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of S. We define the symmetric Martindale ring of quotient of S as $Q_s = Q_s(S)$. The extended centroid of S is denoted by the ring C. The ring SC is referred to as the S central closure. It is known that $S \subseteq SC \subseteq Q_s \subseteq Q_r$ (and Q_l). If S is a prime ring, it is obvious that SC, Q_s, Q_r (and Q_l) are also prime rings. For further information, we suggest the reader to the book [6]. An additive mapping $\psi : S \to S$ is said to be a derivation, if $\psi(ab) = \psi(a)b + a\psi(b)$ holds for every $a, b \in S$. An additive mapping $\Psi : S \to S$ is said to be a generalized derivation, if there exists a derivation $\psi : S \to S$ such that $\Psi(ab) = \Psi(a)b + a\psi(b)$ holds for every $a, b \in S$. As a result, each derivation is a generalised derivation.

The relationship between the commutativity of a prime ring S and the behaviour of a derivation or extended derivation on S has attracted continued research. Posner [19] initiated the study of such mappings, and he established the relationship between the commutativity of a prime ring S. In [15], Herstein demonstrated that S is commutative if ψ is a non-zero derivation of S and $[\psi(a), \psi(b)] = 0$ for every $a, b \in S$, where S is a 2-torsion free prime ring. After that, Daif [11] improved this result for ideal of semi-prime ring. A semi-prime ring S must be commutative, according to Daif and Bell's proof in [12], if it admits a derivation ψ such that $[\psi(a), \psi(b)] \pm [a, b] = 0$ for every $a, b \in S$. This classical theorem was extended to include the class of generalised derivations by Bell and Rehman in [7]. Later, many authors have studied the action of such types of mappings as derivations, generalized derivations, skew derivations etc. on semi-prime and prime (rings) ideals in different directions. (see [3, 4, 5, 16, 21, 22] where references can be found).

Recently many authors viz. [13, 2] and [14, Theorem 3.4] have gained commutativity of semi-prime and prime rings with derivations satisfying certain algebraic identities. Motivated by these results, in the present article we shall explore the commutativity of ring S satisfying any one of the properties $\Psi(a)b - a\Phi(b) \in Z$, $\Psi(a)b - b\Phi(a) \in Z$, $\Psi(a)a - b\Phi(b) \in Z$, $\Psi(a)a - a\Phi(b) \in Z$, $\Psi(a)a - b\Phi(a) \in Z$, for every $a, b \in J$, where J is a non-zero ideal of S. We also provide an example to demonstrate that the hypothesis of primeness imposed in our results is essential.

2. **Preliminaries**

We will make use of the following fundamental identities that apply to every $a, b, c \in S$ without explicitly mentioning them:

$$[ab, c] = a[b, c] + [a, c]b,$$

$$[a, bc] = b[a, c] + [a, b]c.$$

Facts 2.1, 2.2, and 2.3 can be verified easily.

Fact 2.1. *Let S be a prime ring and $a \in S$. If $\{az, z\} \subseteq Z$, then $a \in Z$ or $z = 0$.*

Fact 2.2. *Let S be a prime ring and J be a non-zero ideal of S. If for $a, b \in S$ and $aJb = (0)$, then $a = 0$ or $b = 0$. In particular, if $aJ = (0)$, then $a = 0$, also if $Jb = (0)$, then $b = 0$,*

Fact 2.3. *Let S be a prime ring and J a non-zero ideal of S. Suppose that S admits generalized derivation Ψ with associated derivations ψ. If $\Psi(a) = 0$ for every $a \in J$, then $\Psi(a) = 0$ for every $a \in S$, that is $\Psi = 0$.*

Fact 2.4. [2, Lemma 2.2] *If a prime ring S contains a commutative non-zero ideal, then S is commutative.*

Fact 2.5. [1, Lemma 2.5] *Let S be a prime ring and J be a non-zero ideal of S such that*

(i) $[a, b] \in Z$ *for every $a, b \in J$; or*

(ii) $(a \circ b) \in Z$ *for every $a, b \in J$,*

then S is commutative.

Fact 2.6. [10] *Let S be a prime ring and J a non-zero ideal of S. Then, S, J, and Q_r (resp. Q_l) satisfy the same generalized polynomial identities with coefficients in Q_r (resp. Q_l).*

Fact 2.7. [18, Theorem 3] *Let S be a prime ring, then the following statements hold:*

(i) *Every generalized derivation of S can be uniquely extended to Q_r (and Q_l).*

(ii) *Every derivation of S can be uniquely extended to Q_r (and Q_l). Since every derivation is generalized derivation, it follows from (i).*

Fact 2.8. [8, Lemma 2] *Let S be a prime ring and $\Psi : S \to SC$ be an additive map satisfying $\Psi(ab) = \Psi(a)b$ (resp. $\Psi(ab) = a\Psi(b)$) for every $a, b \in S$. Then there exists $q \in Q_r$ (resp. $q \in Q_l$) such that $\Psi(a) = qa$ (resp. $\Psi(a) = aq$) for every $a \in S$.*

Fact 2.9. [17, Theorem L] *Let S be a prime ring with a derivation ψ, J a left ideal of S and n, m two positive integers. Suppose that $[\psi(a^m), a^m]_n = 0$ for every $x \in J$. Then either $\psi = 0$ or S is commutative.*

In [17, Theorem 2], they worked on a non-zero left ideal, but we will take a special case, when J is a non-zero ideal of S, as follows:

Fact 2.10. [17, Theorem 2] *Let S be a prime ring, J a non-zero ideal of S, ψ a derivation of S, and m_i fixed positive integers, where $i \in \{1, ..., 4\}$. If $[\psi(a^{m_1})a^{m_2}, a^{m_3}]_{m_4} = 0$ ($[a^{m_1}\psi(a^{m_2}), a^{m_3}]_{m_4} = 0$) for every $a \in J$, then $\psi = 0$ or S is commutative.*

3. The Main Result

Theorem 3.1 (Main theorem)**.** *Let S be a prime ring and J a non-zero ideal of S. Assume that S admit generalized derivations Ψ and Φ with associated derivations ψ and φ, respectively.*

(1) *If* $\Psi(a)b \pm a\Phi(b) \in Z$ *for every* $a, b \in J$, *then*
   (i) S *is commutative or*
   (ii) $\Psi(a) = aq$ *with* $\psi(a) = [a, q]$ *and* $\Phi(a) = \pm qa$ *with* $\varphi = 0$ *for every* $a \in J$ *and some* $q \in Q_s$.

(2) S *is commutative or* $\Psi = \Phi = 0$ *if, for every* $a, b \in J$, *satisfies any one of the following*
   (i) $\Psi(a)b \pm b\Phi(a) \in Z$,
   (ii) $\Psi(a)a \pm b\Phi(b) \in Z$,
   (iii) $\Psi(a)a \pm a\Phi(b) \in Z$,
   (iv) $\Psi(a)a \pm b\Phi(a) \in Z$.

We need some auxiliary lemmas in order to prove our main theorem.

Throughout this section, S is a prime ring and J a non-zero ideal of S such that S admit generalized derivations Ψ and Φ with associated derivations ψ and φ, respectively.

Lemma 3.2. *If* $\Psi(a)b - a\Phi(b) \in Z$ *for every* $a, b \in J$, *then*
   (i) S *is commutative or*
   (ii) $\Psi(a) = aq$ *with* $\psi(a) = [a, q]$ *and* $\Phi(a) = qa$ *with* $\varphi = 0$ *for every* $a \in J$ *and some* $q \in Q_s$.

Proof. Assume that

$$\Psi(a)b - a\Phi(b) \in Z \tag{1}$$

for every $a, b \in J$. Replacing b by bw in (1), where $w \in J$, we have

$$(\Psi(a)b - a\Phi(b))w - ab\varphi(w) \in Z. \tag{2}$$

for every $a, b, w \in J$. Using (1) in (2), we get $[ab\varphi(w), w] = 0$. Putting $a = b = w$ in the last relation, we obtain $w^2[\varphi(w), w] = 0$, and so $\varphi = 0$ or S is commutative, by Fact 2.10. In case S is commutative, as desired. Now, in case

$$\varphi = 0, \tag{3}$$

as desired. Now, by using (3) in (2), we see that $(\Psi(a)b - a\Phi(b))w \in Z$, and by using (1) and Fact 2.1 in the last relation, we find that $\Psi(a)b - a\Phi(b) = 0$ or $w \in Z$. If $w \in Z$, then $J \subseteq Z$, hence S is commutative, by Fact 2.4. Now, if

$$\Psi(a)b - a\Phi(b) = 0 \tag{4}$$

for every $a, b \in J$. Then by [9, p. 200], there exists $q \in Q_s$ such that

$$\Psi(a) = aq \tag{5}$$

for every $a \in J$ and

$$\Phi(a) = qa \tag{6}$$

for every $a \in J$. On other hand, from definition of Ψ, we have $\Psi(ab) = \Psi(a)b + a\psi(b)$, and by using (5) in the last relation, we arrive at $abq = aqb + a\psi(b)$, that is, $a(bq - qb - \psi(b)) = 0$, hence $J(bq-qb-\psi(b)) = (0)$, thus $bq-qb-\psi(b) = 0$, by Fact 2.2, this implies that $\psi(b) = [b, q]$ for every $b \in J$ and some $q \in Q_s$, as desired. □

Corollary 3.3. *If $\varphi \neq 0$ and $\Psi(a)b - a\Phi(b) \in Z$ for every $a, b \in J$, then S is commutative.*

Lemma 3.4. *If $\Psi(a)b - b\Phi(a) \in Z$ for every $a, b \in J$, then S is commutative or $\Psi = \Phi = 0$.*

Proof. Assume that

$$\Psi(a)b - b\Phi(a) \in Z \tag{7}$$

for every $a, b \in J$.

Case (I): Suppose that $J \cap Z \neq (0)$. Replacing b by z in (7), where $0 \neq z \in J \cap Z$, we have $\Psi(a) - \Phi(a) \in Z$, by Fact 2.1, that is, $(\Psi - \Phi)(a) \in Z$. Putting $H = \Psi - \Phi$ with $\varphi = \psi - \varphi$ in the last relation, we get

$$H(a) \in Z \tag{8}$$

for every $a \in J$. Note that H is a generalized derivation of S with associated derivation φ. Now, replacing a by ab in (8), where $b \in J$, we obtain

$$H(a)b + a\varphi(b) \in Z \tag{9}$$

for every $a, b \in J$. By using (8) in (9), we see that $[a\varphi(b), b] = 0$. Putting $a = b$ in the last relation, we find that $b[\varphi(b), b] = 0$, hence $\varphi = 0$ or S is commutative, by Fact 2.10. In case S is commutative, as desired. Now, in case $\varphi = 0$. Using the last relation in (9), we conclude that

$$H(a)b \in Z. \tag{10}$$

for every $a, b \in J$. By using (8) in (10) and by Fact 2.1, we get $H(a) = 0$ or $b \in Z$. If $b \in Z$ for every $b \in J$, then $J \subseteq Z$, and by using Fact 2.4 in the last relation, we see that S is commutative. Now, in case $H(a) = 0$, then $H = 0$, by Fact 2.3, that is, $\Psi - \Phi = 0$, hence

$$\Psi = \Phi. \tag{11}$$

Using (11) in (7), we obtain $\Psi(a)b - b\Psi(a) \in Z$, and so

$$[\Psi(a), b] \in Z \tag{12}$$

for every $a, b \in J$. Replacing b by $\Psi(a)b$ in (12), we get $\Psi(a)[\Psi(a), b] \in Z$, and by using (12) and Fact 2.1 in the last expression, we have $\Psi(a) \in Z$ or $[\Psi(a), b] = 0$. Note that $\Psi(a) \in Z$ if and only if $[\Psi(a), b] = 0$, and so

$$\Psi(a) \in Z \tag{13}$$

for every $a \in J$. Now, the same as in Eq. (8), we get S is commutative or $\Psi = 0$. In case S is commutative, as desired. Now, in case $\Psi = 0$. Using the last relation in (11), we arrive at $\Psi = \Phi = 0$, as desired.

Case (II): Suppose that $J \cap Z = (0)$. Since $a, b \in J$ in (7) and by using the our assumption in Case (II), we get

$$\Psi(a)b - b\Phi(a) = 0 \tag{14}$$

for every $a, b \in J$. Left multiplying (14) by t, where $t \in J$, we have

$$t\Psi(a)b - tb\Phi(a) = 0 \tag{15}$$

for every $t, a, b \in J$. Replacing b by tb in (14), where $t \in J$, we see that

$$\Psi(a)tb - tb\Phi(a) = 0 \tag{16}$$

for every $t, a, b \in J$. Subtracting (15) from (16), we obtain $\Psi(a)tb - t\Psi(a)b = 0$ this implies that $(\Psi(a)t - t\Psi(a))b = 0$, and so $(\Psi(a)t - t\Psi(a))J = (0)$ and by Fact 2.2, we conclude that $\Psi(a)t - t\Psi(a) = 0$, hence $[\Psi(a), t] = 0$, that is,

$$\Psi(a) \in Z(R). \tag{17}$$

Now, the same as in Eq. (13), we get S is commutative or $\Psi = 0$. In case S is commutative, as desired. Now, in case $\Psi = 0$, as desired. On other hand, since $\Psi = 0$, and by using the last relation in (14), gives $b\Phi(a) = 0$, that is, $J\Phi(a) = (0)$ and by Fact 2.2, we find that $\Phi(a) = 0$, and by Fact 2.3, we get $\Phi = 0$, as desired. □

Lemma 3.5. *If $\Psi(a)a - b\Phi(b) \in Z$ for every $a, b \in J$, then S is commutative or $\Psi = \Phi = 0$.*

Proof. Assume that

$$\Psi(a)a - b\Phi(b) \in Z \tag{18}$$

for every $a, b \in J$. Putting $b = 0$ in (18), we get

$$\Psi(a)a \in Z \tag{19}$$

for every $a \in J$. By linearizing (18), we obtain

$$\Psi(a)b + \Psi(b)a \in Z \tag{20}$$

for every $a, b \in J$.

Case (I): Suppose that $J \cap Z \neq (0)$. From (20), we have

$$[\Psi(a)b + \Psi(b)a, s] = 0 \tag{21}$$

for every $a, b, s \in J$. Replacing b by bs in (21), we see that

$$[\Psi(a)bs + \Psi(bs)a, s] = 0 \tag{22}$$

for every $a, b, s \in J$. Right multiplying (21) by s, we find that

$$[\Psi(a)bs - \Psi(b)as, s] = 0 \tag{23}$$

for every $a, b, s \in J$. Comparing (22) and (23), we infer that

$$[\Psi(bs)a - \Psi(b)as, s] = 0$$

for every $a, b, s \in J$. It implies that

$$[\Psi(b)sa + b\psi(s)a - \Psi(b)as, s] = 0$$

for every $a, b, s \in J$. That is,

$$[\Psi(b)[s, a] + b\psi(s)a, s] = 0$$

for every $a, b, s \in J$. Putting $a = z$ in the last relation, where $0 \neq z \in J \cap Z$, we deduce that $[b\psi(s), s]z = 0$ for every $b, s \in J$. Since the non-zero central elements of a prime ring are not zero divisors, we get $[b\psi(s), s] = 0$ for every $b, s \in J$. Again, putting $b = z$ in the last relation, where $0 \neq z \in J \cap Z$, we have $z[\psi(s), s] = 0$ for every $s \in J$. It follows that $[\psi(s), s] = 0$ for every $s \in J$, and by Fact 2.9, we get $\psi = 0$ or S is commutative. In case S is commutative, as desired. Now, if $\psi = 0$, then from definition of Ψ, we obtain $\Psi(ab) = \Psi(a)b$, and by Fact 2.8, we see that

$$\Psi(a) = qa \tag{24}$$

for every $a \in J$ and some $q \in Q_r$. Using (24) in (19), we find that $qa^2 \in Z$. Putting $a = z$ in the last relation, where $0 \neq z \in J \cap Z$, we arrive at $qz^2 \in Z$. That is $[qz^2, t] = 0$ for every $t \in S$. Hence $z^2[q, r] = 0$ for every $r \in S$, and by using Fact 2.6 in the last relation, we deduce that $z^2[q, t] = 0$ for every $t \in Q_r$. Since $z \neq 0$, we have $[q, t] = 0$ for every $t \in Q_r$. That is

$$q \in C. \tag{25}$$

Now, from (20), we obtain

$$[\Psi(a)b + \Psi(b)a, t] = 0$$

for every $t \in S$ and $a, b \in J$. Using Facts 2.6 and 2.7 in the last relation, we get

$$[\Psi(a)b + \Psi(b)a, t] = 0$$

for every $a, b, t \in Q_r$. That is

$$\Psi(a)b + \Psi(b)a \in C$$

for every $a, b \in Q_r$. Using (24) in the last relation, we see that $qab + qba \in C$, it implies that $q(a \circ b) \in C$, and by using (25) and Fact 2.1 in the last relation, we get two cases: $a \circ b \in C$ or $q = 0$. In case $a \circ b \in C$, then by Fact 2.5(ii), we obtain Q_r is commutative, hence S is commutative, as desired. Now, assume that R is not commutative. Now, in case $q = 0$, then from (24), we get $\Psi(a) = 0$ and by Fact 2.3, we obtain $\Psi = 0$.

Now, we will prove that $\Phi = 0$. Using (19) in (18), we have

$$b\Phi(b) \in Z \tag{26}$$

for every $b \in J$. Putting $b = z$ in (26), where $0 \neq z \in J \cap Z$, and by Fact 2.1, we find that

$$\Phi(z) \in Z. \tag{27}$$

By linearizing (26), we obtain

$$a\Phi(b) + b\Phi(a) \in Z \tag{28}$$

for every $a, b \in J$. From (28), we have

$$[a\Phi(b) + b\Phi(a), s] = 0 \tag{29}$$

for every $a, b, s \in J$. Replacing b by sb in (29), we see that

$$[a\Phi(sb) + sb\Phi(a), s] = 0 \tag{30}$$

for every $a, b, s \in J$. Left multiplying (29) by s, we find that

$$[sa\Phi(b) + sb\Phi(a), s] = 0 \tag{31}$$

for every $a, b, s \in J$. Subtracting (31) from (30), we infer that

$$[a\Phi(sb) - sa\Phi(b), s] = 0$$

for every $a, b, s \in J$. Putting $b = a = z$ in the last relation, where $0 \neq z \in J \cap Z$, we deduce that

$$[z\Phi(zs) - sz\Phi(z), s] = 0$$

for every $s \in J$. That is

$$[z\Phi(z)s + z^2\varphi(s) - sz\Phi(z), s] = 0$$

for every $s \in J$. Using (27) in the last relation, we arrive at $[z^2\varphi(s), s] = 0$ for every $s \in J$. Hence $z^2[\varphi(s), s] = 0$, and so $[\varphi(s), s] = 0$, and by using Fact 2.9 in the last relation, we get $\varphi = 0$ or S is commutative. But, R is not commutative as in assumption, and so $\varphi = 0$. By using the last relation in definition of Φ, we obtain $\Phi(ab) = \Phi(a)b$, and by Fact 2.8, we see that $\Phi(a) = qa$ for every $a \in J$ and some $q \in Q_r$. Now, the same as in (24), we get S is commutative or $\Phi = 0$. But R is not commutative, and so $\Phi = 0$, as desired.

Case (II): Suppose that $J \cap Z = (0)$. Using this assumption in (19), we deduce that

$$\Psi(a)a = 0 \tag{32}$$

for every $a \in J$. By linearizing (32), we obtain

$$\Psi(a)b + \Psi(b)a = 0. \tag{33}$$

for every $a, b \in J$. Replacing b by ab in (33) and using (32), we get

$$\Psi(ab)a = 0. \tag{34}$$

for every $a, b \in J$. Again, replacing b by ba in (34), we have

$$0 = \Psi(aba)a = (\Psi(ab)a + ab\psi(a))a$$

for every $a, b \in J$. Using (34) in the last relation, we see that $ab\psi(a)a = 0$ this implies that $\psi(a)ab\psi(a)a = 0$ which leads to $\psi(a)aJ\psi(a)a = (0)$, so $\psi(a)a = 0$, that is $[\psi(a)a, a] = 0$, thus $[\psi(a), a]a = 0$, hence $\psi = 0$ or S is commutative, by Fact 2.10. In case S is commutative, as desired. Now, if $\psi = 0$. Then by using (34) and definition of Ψ in (34), we infer that $\Psi(a)ba = 0$ so $\Psi(a)Ja = (0)$ this implies that $\Psi(a) = 0$ or $a = 0$. But $J \neq 0$, and so $\Psi(a) = 0$, and by Fact 2.3, we get

$$\Psi = 0, \tag{35}$$

as desired. Now, putting $a = 0$ in (18) and since $J \cap Z = (0)$, we get

$$b\Phi(b) = 0. \tag{36}$$

$b \in J$. By linearizing (36), we get

$$a\Phi(b) + b\Phi(a) = 0. \tag{37}$$

$a, b \in J$. Replacing a by ab in (37) and using (36), we deduce that

$$b\Phi(ab) = 0. \tag{38}$$

$a, b \in J$. Replacing a by ab in (38), we obtain $0 = b\Phi(abb) = (b\Phi(ab))b + bab\varphi(b)$, and by using (38) in the last relation, we infer that $0 = bab\varphi(b)$. Again, replacing a by b in the last relation, this gives $0 = b^3\varphi(b)$, and so $b^3[\varphi(b), b] = 0$, hence $\varphi = 0$ or S is commutative, by Fact 2.10. S is commutative, as desired. Now, if $\varphi = 0$. Then by using definition of Φ in (38), we find that

$$b\Phi(a)b = 0. \tag{39}$$

$a, b \in J$. Right multiplying (37) by b, we get

$$a\Phi(b)b + b\Phi(a)b = 0$$

$a, b \in J$. Using (39) in the last relation, we have $a\Phi(b)b = 0$, that is, $J\Phi(b)b = (0)$ and since $J \neq 0$ we obtain

$$\Phi(b)b = 0. \tag{40}$$

$b \in J$. By linearizing (40), then

$$\Phi(a)b + \Phi(b)a = 0. \tag{41}$$

$a, b \in J$. Replacing a by ba in (41) and using (40), we get $\Phi(ba)b = 0$ and since $\varphi = 0$ we get $\Phi(b)ab = 0$, that is, $\Phi(b)Jb = (0)$, and so $\Phi(b) = 0$ or $b = 0$. Since $J \neq 0$, we deduce $\Phi(b) = 0$, and by Fact 2.3 we obtain $\Phi = 0$, as desired. □

Lemma 3.6. *If $\Psi(a)a - a\Phi(b) \in Z$ for every $a, b \in J$, then S is commutative or $\Psi = \Phi = 0$.*

Proof. Assume that

$$\Psi(a)a - a\Phi(b) \in Z \tag{42}$$

for every $a, b \in J$. Putting $b = 0$ in (42), we get

$$\Psi(a)a \in Z \tag{43}$$

for every $a \in J$. In case $J \cap Z \neq (0)$ or $J \cap Z = (0)$ using the same trick in Lemma 3.5 in Eq. (19) or (32) respectively, we get S is commutative or $\Psi = 0$. In case S is commutative, as desired. Now, in case $\Psi = 0$, as desired. On other hand, using fact that $\Psi = 0$ in (42), we have

$$a\Phi(b) \in Z. \tag{44}$$

for every $a, b \in J$.

Case (I): Suppose that $J \cap Z \neq (0)$. Replacing a by z in (44), where $0 \neq z \in J \cap Z$, we see that $z\Phi(b) \in Z$ so $\Phi(b) \in Z$, and by using the last relation in (44), we find that $a \in Z$ or $\Phi(b) = 0$, by Fact 2.1. If $a \in Z$, for every $a \in J$, then $J \subseteq Z$, hence S is commutative by Fact 2.4. Now, if $\Phi(b) = 0$, then by Fact 2.3, we obtain $\Phi = 0$, as desired.

Case (II): Suppose that $J \cap Z = (0)$. Using this assumption in (44), we get $a\Phi(b) = 0$, so $J\Phi(b) = (0)$ and by using Fact 2.2 in the last relation, we find that $\Phi(b) = 0$, and by Fact 2.3, we infer that $\Phi = 0$, as desired. □

Lemma 3.7. *If $\Psi(a)a - b\Phi(a) \in Z$ for every $a, b \in J$, then S is commutative or $\Psi = \Phi = 0$.*

Proof. See the proof of Lemma 3.6. $\square$

We provide an example to demonstrate that the primeness requirement in our results is not unnecessary.

Example 3.8. Let $S = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$ and $J = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in \mathbb{Z} \right\}$. We have S is a ring and J is a non-zero ideal of S. Define $\Psi = 0 = \psi$ and $\Phi = \varphi : S \to S$ by $\varphi \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ is a (generalized) derivation of S. All the identities of all our results are satisfied on J, but S is non-commutative and is not prime with $\Phi \neq 0 \neq \varphi$. So the condition of primeness is essential.

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