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Research Paper

# SHEFFER STROKE $R_0$−ALGEBRAS

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ABSTRACT. The main objective of this study is to introduce Sheffer stroke $R_0$−algebra (for short, $SR_0$− algebra). Then it is stated that the axiom system of a Sheffer stroke $R_0$−algebra is independent. It is indicated that every Sheffer stroke $R_0$−algebra is $R_0$−algebra but specific conditions are necessarily for the inverse. Afterward, various ideals of a Sheffer stroke $R_0$−algebra are defined, a congruence relation on a Sheffer stroke $R_0$−algebra is determined by the ideal and quotient Sheffer stroke $R_0$−algebra is built via this congruence relation. It is proved that quotient Sheffer stroke $R_0$−algebra constructed by a prime ideal of this algebra is totally ordered and the cardinality is less than or equals to 2. After all, important conclusions are obtained for totally ordered Sheffer stroke $R_0$−algebras by applying various properties of prime ideals.

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## 1. Introduction

The notion of lattice implication algebras was introduced and some properties were examined by Xu [21]. Also, he and Qin presented implicative filters of these algebraic structures and researched some their properties [22]. Then Turunen gave the concept of Boolean deductive system, i.e., Boolean filter of BL-algebras which is the algebraic structure of Hájek' s Basic Logic [4]. Esteva and Godo introduced MTL-algebras which are the algebraic structures of monoidal t-norm based logic, IMTL-algebras as an extension of MTL-algebras, and so, NM-algebras as an extension of IMTL-algebras [3]. $R_0-$algebras were introduced by Wang who suggested a formal deductive system $\mathfrak{L}^*$ for fuzzy propositional calculus ([15], [19] and [20]). Besides, Pei and Wang showed that NM-algebras are categorically isomorphic to $R_0-$algebras [14]. Jun and Liu investigated some filters of $R_0-$algebras and stated that $R_0-$algebras is contributes to the development of the theory of MTL-algebras [5]. Tough these new algebraic structures are different from BL-algebras, lattice implication algebras and MTL-algebras, all these algebras have the implication operator $\longrightarrow$. Therefore, BL-algebras and lattice implication algebras can be generalized to $R_0-$algebras.

The Sheffer stroke (or Sheffer operation) was first introduced by H. M. Sheffer [16]. Since any Boolean formulae or axiom can be stated by means of this operation [6], it draws many researchers' attention. The most important application is to have all diods on the chip forming processor in a computer. Thus, it is simpler and cheaper than to produce different diods for other Boolean operations. Since Sheffer stroke is a commutative, applying to many logical algebras leads to many useful results, and it reducts axiom systems of many algebraic structures. Hence, we replace unary and binary operations with the binary operation called Sheffer stroke. Recently, the mathematicians has widely investigated algebraic structures with Sheffer stroke such as Sheffer stroke non-associative MV-algebras[2], Sheffer stroke BL-algebras and (fuzzy) filters [9], filters of strong Sheffer stroke non-associative MV-algebras [10], Sheffer stroke UP-algberas [12], Sheffer stroke Hilbert algebras [11] and (fuzzy) filters [13]. There also exist authentic studies on Sheffer stroke algebras such as representations of strongly algebraically closed algebras [7], Visser algebras [8], and a shortest 2-basis for Boolean algebra in terms of the Sheffer stroke [18]. Thus, applying Sheffer stroke to $R_0-$algebra provides to obtain more useful system which has fewer axioms.

The setup scheme of the manuscript is as below. In the first section, the historical backgorund and current studies of mentioned structures are presented. In the second section, the basic definitions and notions using throughout the study are presented. In the second section, basic definitions and concepts using throughout the manuscript are given. In the third section, $R_0-$algebras with Sheffer stroke are introduced. In the fourth section, various ideals of these new algberaic structures are defined. In the fifth section, a congruence relation on these algebras is described by means of an ideal. Then quotient Sheffer stroke $R_0-$algebras are constructed by this relation, and totally ordered Sheffer stroke $R_0-$algebras are defined. Also, related concepts are examined, and these results are supported with illustrative examples. In the last section, the conclusions are summarized in detail. Since these conclusions are new and novel in literature, the manuscript contributes to pure mathematics regarding $R_0-$algebras, Sheffer operation and abstract algebra.

## 2. Preliminaries

In this section, we give basic definitions and notions about Sheffer stroke and $R_0-$algebras.

**Definition 2.1.** [1] Let $\mathcal{M} = (M, |)$ be a groupoid. The operation $|$ is said to be a *Sheffer stroke* if it satisfies the following conditions:
(S1) $x|y = y|x$,
(S2) $(x|x)|(x|y) = x$,
(S3) $x|((y|z)|(y|z)) = ((x|y)|(x|y))|z$,
(S4) $(x|((x|x)|(y|y)))|(x|((x|x)|(y|y))) = x$,
for all $x, y, z \in M$.

**Definition 2.2.** [19] Let $M$ be a $(\neg, \wedge, \vee, \longrightarrow)-$type algebra, where $\neg$ is a unary operation, $\wedge$, $\vee$ and $\longrightarrow$ are binary operations. If there is a partial order $\leq$ on $M$, such that $(M, \leq)$ is a bounded distributive lattice, $\wedge$, $\vee$ are infimum and supermum operations with respect to $\leq$, $\neg$ is an order-reversing involution with respect to $\leq$, and the following conditions hold for any $x, y, z \in M$
(R1) $\neg x \longrightarrow \neg y = y \longrightarrow x$,
(R2) $1 \longrightarrow x = x$, $x \longrightarrow x = 1$,
(R3) $y \longrightarrow z \leq (x \longrightarrow y) \longrightarrow (x \longrightarrow z)$,
(R4) $x \longrightarrow (y \longrightarrow z) = y \longrightarrow (x \longrightarrow z)$,
(R5) $x \longrightarrow (y \vee z) = (x \longrightarrow y) \vee (x \longrightarrow z)$, $x \longrightarrow (y \wedge z) = (x \longrightarrow y) \wedge (x \longrightarrow z)$,
(R6) $(x \longrightarrow y) \vee ((x \longrightarrow y) \longrightarrow (\neg x \vee y)) = 1$
where 1 is the largest element of $M$, then $M$ is called a $R_0-$algebra.

**Proposition 2.3.** [19] *Let $M$ be a $R_0-$algebra. Then for all $x, y, z \in M$*
*(P1)* $x \leq y$ *if and only if* $x \longrightarrow y = 1$,
*(P2)* $x \leq y \longrightarrow z$ *if and only if* $y \leq x \longrightarrow z$,
*(P3)* $(x \vee y) \longrightarrow z = (x \longrightarrow z) \wedge (y \longrightarrow z)$, $(x \wedge y) \longrightarrow z = (x \longrightarrow z) \vee (y \longrightarrow z)$,
*(P4) If* $x \leq y$, *then* $z \longrightarrow x \leq z \longrightarrow y$ *and* $y \longrightarrow z \leq x \longrightarrow z$,
*(P5)* $x \longrightarrow y \geq \neg x \vee y$,
*(P6)* $(x \longrightarrow y) \vee (y \longrightarrow x) = 1$, $x \vee y = ((x \longrightarrow y) \longrightarrow y) \wedge ((y \longrightarrow x) \longrightarrow x)$,

*(P7)* $x \longrightarrow (y \longrightarrow x) = 1$, $x \longrightarrow (\neg x \longrightarrow y) = 1$,
*(P8)* $x \longrightarrow y \leq (x \vee z) \longrightarrow (y \vee z)$, $x \longrightarrow y \leq (x \wedge z) \longrightarrow (y \wedge z)$,
*(P9)* $x \longrightarrow y \leq (x \longrightarrow z) \vee (z \longrightarrow y)$,
*(P10) If* $x \leq y$, *then* $x \otimes z \leq y \otimes z$,
*(P11)* $x \otimes y \longrightarrow z = x \longrightarrow (y \longrightarrow z)$, $x \longrightarrow (y \longrightarrow x \otimes y) = 1$,
*(P12)* $x \otimes \neg x = 0$, $1 \otimes x = x$,
*(P13)* $x^n = x^2$ *for all* $n \geq 2$,
*(P14)* $(x \vee y)^n = x^n \vee y^n$ *for all* $n \in \mathbb{N}$
*where* $x \otimes y = \neg(x \longrightarrow \neg y)$, $x^n$ *is inductively defined as follows:* $x^1 = x$, $x^{k+1} = x^k \otimes x$, *for all* $k \in \mathbb{N}$.

**Proposition 2.4.** [14] *Let* $M$ *be a* $R_0-$*algebra. Then* $\neg x = x \longrightarrow 0$, *for all* $x, y, z \in M$.

## 3. Sheffer stroke $\mathbf{R_0}-$algebras

In this section, we introduce Sheffer stroke $R_0-$algebras and present some of properties.

**Definition 3.1.** A Sheffer stroke $R_0-$algebra (briefly, $SR_0-$algebra) is an algebra $(M, \vee, \wedge, |, 0, 1)$ of type $(2, 2, 2, 0, 0)$ satisfying the following properties for all $x, y, z \in M$:
(SR1) $x|(x|x) = 1$,
(SR2) $y|(z|z) \leq (x|(y|y))|((x|(z|z))|(x|(z|z)))$,
(SR3) $x|((y \vee z)|(y \vee z)) = (x|(y|y)) \vee (x|(z|z))$ and $x|((y \wedge z)|(y \wedge z)) = (x|(y|y)) \wedge (x|(z|z))$,
where $(M, \leq)$ is a bounded distributive lattice, $\vee$, $\wedge$ are supremum and infimum with respect to $\leq$, and $|$ is Sheffer stroke on $M$.

Moreover, $1 = 0|0$ is the greatest element and $0 = 1|1$ is the least element of $M$.

**Proposition 3.2.** *The axioms (SR1)-(SR3) are independent.*

*Proof.* (1) Consider a set $M = \{0, 1/2, 1\}$ with the Cayley tables in Table 1. Then (SR2) and (SR3) hold while (SR1) does not, since $1/2|_1(1/2|_1 1/2) = 1/2 \neq 1$.

Table 1. Operation tables for independency of (SR1)

| $\vert_1$ | 0 | 1/2 | 1 |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| 1/2 | 1 | 1/2 | 0 |
| 1 | 1 | 0 | 0 |

| $\vee_1$ | 0 | 1/2 | 1 |
|---|---|---|---|
| 0 | 0 | 1/2 | 1 |
| 1/2 | 1/2 | 1/2 | 1 |
| 1 | 1 | 1 | 1 |

| $\wedge_1$ | 0 | 1/2 | 1 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1/2 | 0 | 1/2 | 1/2 |
| 1 | 0 | 1/2 | 1 |

(2) Consider a set $M = \{0, 1/2, 1\}$ with the Cayley tables in Table 2. Then (SR1) and (SR3) hold but (SR2) does not, because $1/2|_2(0|_2 0) = 1/2 > 0 = 1|_2(0|_2 0) = (1|_2(1/2|_2 1/2))|_2((1|_2(0|_2 0))|_2(1|_2(0|_2 0)))$.

Table 2. Operation tables for independency of (SR2)

| $\vert_2$ | 0 | 1/2 | 1 |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| 1/2 | 1 | 0 | 1/2 |
| 1 | 1 | 1/2 | 0 |

| $\vee_2$ | 0 | 1/2 | 1 |
|---|---|---|---|
| 0 | 0 | 1/2 | 1 |
| 1/2 | 1/2 | 1/2 | 1 |
| 1 | 1 | 1 | 1 |

| $\wedge_2$ | 0 | 1/2 | 1 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1/2 | 0 | 1/2 | 1/2 |
| 1 | 0 | 1/2 | 1 |

(3) Consider a set $M = \{0, 1/2, 1\}$ with Cayley tables in Table 3. Then (SR1) and (SR2) hold whereas (SR3) does not, since $1|((1/2 \wedge_3 1)|(1/2 \wedge_3 1)) = 0 \neq 1 = (1|_3(1/2|_3 1/2)) \wedge_3 (1|_3(1|_3 1))$.

□

Table 3. Operation tables for independency of (SR3)

| $\vert_3$ | 0 | 1/2 | 1 |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| 1/2 | 1 | 0 | 0 |
| 1 | 1 | 0 | 0 |

| $\vee_3$ | 0 | 1/2 | 1 |
|---|---|---|---|
| 0 | 0 | 1/2 | 1 |
| 1/2 | 1/2 | 1/2 | 1 |
| 1 | 1 | 1 | 1 |

| $\wedge_3$ | 0 | 1/2 | 1 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1/2 | 0 | 1/2 | 0 |
| 1 | 0 | 0 | 1 |

**Example 3.3.** Consider a set $M = \{0, a, b, 1\}$ with

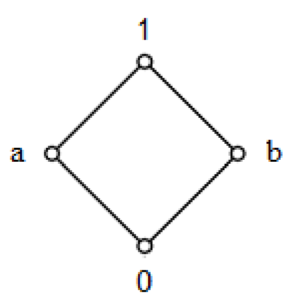


Figure 1. Hasse diagram for $M$

The binary operations $|$, $\vee$ and $\wedge$ on $M$ have the Cayley tables in Table 4:

Then this structure is a $\mathrm{SR}_0-$algebra.

**Example 3.4.** Consider a set $M = \{0, a, b, c, d, e, f, 1\}$ with

The binary operations $|$, $\vee$ and $\wedge$ on $M$ have the Cayley tables in Table 5:

TABLE 4. Cayley tables of the binary operations $|$, $\vee$ and $\wedge$ on $M$

| $\vert$ | 0 | $a$ | $b$ | 1 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| $a$ | 1 | $b$ | 1 | $b$ |
| $b$ | 1 | 1 | $a$ | $a$ |
| 1 | 1 | $b$ | $a$ | 0 |

| $\vee$ | 0 | $a$ | $b$ | 1 |
|---|---|---|---|---|
| 0 | 0 | $a$ | $b$ | 1 |
| $a$ | $a$ | $a$ | 1 | 1 |
| $b$ | $b$ | 1 | $b$ | 1 |
| 1 | 1 | 1 | 1 | 1 |

| $\wedge$ | 0 | $a$ | $b$ | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | $a$ | 0 | $a$ |
| $b$ | 0 | 0 | $b$ | $b$ |
| 1 | 0 | $a$ | $b$ | 1 |

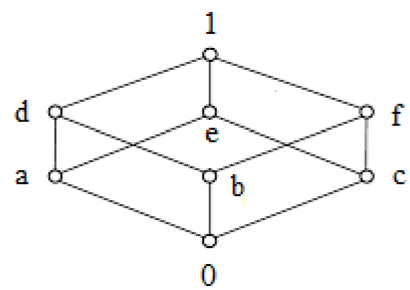


FIGURE 2. Hasse diagram for $M$

Then this structure is a $\mathrm{SR}_0-$algebra.

**Lemma 3.5.** *Let* $(M, \vee, \wedge, |, 0, 1)$ *be a* $SR_0-$*algebra. Then*

(1) $x|((y|(z|z))|(y|(z|z))) = y|((x|(z|z))|(x|(z|z)))$,

(2) $1|(x|x) = x$,

(3) $x|(1|1) = 1$,

(4) $y \leq x|(y|y)$,

(5) $(x|1)|(x|1) = x$,

(6) $(x|y)|(x|y) \leq x$ *and* $(x|y)|(x|y) \leq y$,

(7) $(x|1)|1 = x$,

(8) $x|(x|1) = 1$,

(9) $x \leq (x|y)|y$,

(10) $x \leq y \Leftrightarrow x|(y|y) = 1$,

(11) $x \leq y|(z|z) \Leftrightarrow y \leq x|(z|z)$,

(12) *If* $x \leq y$, *then* $y|z \leq x|z$ *and* $z|(x|x) \leq z|(y|y)$,

(13) $x \vee y \leq (x|x)|(y|y)$,

(14) $x|((y|(x|x))|(y|(x|x))) = 1$,

(15) $x \leq y \Leftrightarrow y|y \leq x|x$,

(16) *if* $x \leq y$ *and* $z \leq t$, *then* $y|t \leq x|z$,

Table 5. Cayley tables of the binary operations $|$, $\vee$ and $\wedge$

| $\mid$ | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | 1 |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $a$ | 1 | $f$ | 1 | 1 | $f$ | $f$ | 1 | $f$ |
| $b$ | 1 | 1 | $e$ | 1 | $e$ | 1 | $e$ | $e$ |
| $c$ | 1 | 1 | 1 | $d$ | 1 | $d$ | $d$ | $d$ |
| $d$ | 1 | $f$ | $e$ | 1 | $c$ | $f$ | $e$ | $c$ |
| $e$ | 1 | $f$ | 1 | $d$ | $f$ | $b$ | $d$ | $b$ |
| $f$ | 1 | 1 | $e$ | $d$ | $e$ | $d$ | $a$ | $a$ |
| 1 | 1 | $f$ | $e$ | $d$ | $c$ | $b$ | $a$ | 0 |

| $\vee$ | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | 1 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | 1 |
| $a$ | $a$ | $a$ | $d$ | $e$ | $d$ | $e$ | 1 | 1 |
| $b$ | $b$ | $d$ | $b$ | $f$ | $d$ | 1 | $f$ | 1 |
| $c$ | $c$ | $e$ | $f$ | $c$ | 1 | $e$ | $f$ | 1 |
| $d$ | $d$ | $d$ | $d$ | 1 | $d$ | 1 | 1 | 1 |
| $e$ | $e$ | $e$ | 1 | $e$ | 1 | $e$ | 1 | 1 |
| $f$ | $f$ | 1 | $f$ | $f$ | 1 | 1 | $f$ | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

| $\wedge$ | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | 1 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | $a$ | 0 | 0 | $a$ | $a$ | 0 | $a$ |
| $b$ | 0 | 0 | $b$ | 0 | $b$ | 0 | $b$ | $b$ |
| $c$ | 0 | 0 | 0 | $c$ | 0 | $c$ | $c$ | $c$ |
| $d$ | 0 | $a$ | $b$ | 0 | $d$ | $a$ | $b$ | $d$ |
| $e$ | 0 | $a$ | 0 | $c$ | $a$ | $e$ | $c$ | $e$ |
| $f$ | 0 | 0 | $b$ | $c$ | $b$ | $c$ | $f$ | $f$ |
| 1 | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | 1 |

(17) $y|(z|z) \leq (z|(x|x))|((y|(x|x))|(y|(x|x)))$,

(18) $(x|y)|(x|y) \leq z \Leftrightarrow x \leq y|(z|z)$,

(19) $(x|(z|z))|(x|(z|z)) \leq (x|(y|y))|(y|(z|z))$,

(20) $(x|(y|y))|(y|y) = (y|(x|x))|(x|x)$,

(21) $x \vee y = (x|(y|y))|(y|y)$ *and* $x \wedge y = (x|(x|(y|y)))|(x|(x|(y|y)))$,

(22) $(x|(y|y)) \vee (y|(x|x)) = 1$,

(23) $(x|(y|y)) \vee ((x|(y|y))|(((x|x) \vee y)|((x|x) \vee y))) = 1$,

(24) $x|y = ((x|y)|y)|y$ *and*

(25) $((x|(z|z))|(x|(z|z))) \vee ((y|(z|z))|(y|(z|z))) = ((x \vee y)|(z|z))|((x \vee y)|(z|z))$,

*for all* $x, y, z \in M$.

*Proof.* The properties (1) through (20) follow from (S1)-(S3), Definition 3.1 and each other.

(21) It is known that $x \leq (x|(y|y))|(y|y)$ and $y \leq (x|(y|y))|(y|y)$ from (9) and (4), respectively. Then $(x|(y|y))|(y|y)$ is an upper bound of $x$ and $y$. Let $x, y \leq z$. Thus,

$(x|(y|y))|(y|y) \leq (z|(y|y))|(y|y) = (y|(z|z))|(z|z) = 1|(z|z) = z$ from (12), (20) and (2), respectively. Hence, $(x|(y|y))|(y|y)$ is a supremum of $x$ and $y$, i.e., $x \vee y = (x|(y|y))|(y|y)$. In a similar way, $x \wedge y = (x|(x|(y|y)))|(x|(x|(y|y)))$.

(22)

$$\begin{aligned}(x|(y|y)) \vee (y|(x|x)) &= ((x|(y|y))|((y|(x|x))|(y|(x|x))))|((y|(x|x))|(y|(x|x))) \\ &= (y|(((x|x)|(x|(y|y)))|((x|x)|(x|(y|y)))))|((y|(x|x))|(y|(x|x))) \\ &= (y|(x|x))|((y|(x|x))|(y|(x|x))) \\ &= 1\end{aligned}$$

from (21), (1), (S1), (S2) and (SR1), respectively.

(23)

$$\begin{aligned}&(x|(y|y)) \vee ((x|(y|y))|(((x|x) \vee y)|((x|x) \vee y))) \\ &= ((x|(y|y))|(((x|(y|y))|(((x|x) \vee y)|((x|x) \vee y)))|((x|(y|y))|(((x|x) \vee y)|((x|x)\vee \\ &\quad y)))))|(((x|(y|y))|(((x|x) \vee y)|((x|x) \vee y)))|((x|(y|y))|(((x|x) \vee y)|((x|x) \vee y)))) \\ &= ((((x|(y|y))|(x|(y|y)))|((x|(y|y))|(x|(y|y))))|(((x|x) \vee y)|((x|x) \vee y)))| \\ &\quad (((x|(y|y))|(((x|x) \vee y)|((x|x) \vee y)))|((x|(y|y))|(((x|x) \vee y)|((x|x) \vee y)))) \\ &= ((x|(y|y))|(((x|x) \vee y)|((x|x) \vee y)))|(((x|(y|y))|(((x|x) \\ &\quad \vee y)|((x|x) \vee y)))|((x|(y|y))|(((x|x) \vee y)|((x|x) \vee y)))) \\ &= 1\end{aligned}$$

from (21), (S3), (S2) and (SR1), respectively.

(24) It is known from (9) that $x|y \leq ((x|y)|y)|y$. Since

$$\begin{aligned}(((x|y)|y)|y)|((x|y)|(x|y)) &= (((x|y)|((y|y)|(y|y)))|((y|y)|(y|y)))|((x|y)|(x|y)) \\ &= (((y|y)|((x|y)|(x|y)))|((x|y)|(x|y)))|((x|y)|(x|y)) \\ &= ((x|((y|(y|y))|(y|(y|y))))|((x|y)|(x|y)))|((x|y)|(x|y)) \\ &= (x|y)|((x|y)|(x|y)) \\ &= 1\end{aligned}$$

from (1)-(3), (20), (S1), (S2) and (SR1), it follows from (10) that $((x|y)|y)|y \leq x|y$. Thus, $x|y = ((x|y)|y)|y$.

(25)

$$\begin{aligned}
&((x|(z|z))|(x|(z|z))) \vee ((y|(z|z))|(y|(z|z))\\
&= (y|(z|z))|((y|(z|z))|((x|(z|z))|(x|(z|z))))\\
&= ((x|(z|z)) \wedge (y|(z|z)))|((x|(z|z)) \wedge (y|(z|z)))\\
&= (((z|z)|((x|x)|(x|x))) \wedge ((z|z)|((y|y)|(y|y))))|(((z|z)|((x|x)|(x|x))) \wedge ((z|z)|((y|y)|(y|y))))\\
&= ((z|z)|(((x|x) \wedge (y|y))|((x|x) \wedge (y|y))))|((z|z)|(((x|x) \wedge (y|y))|((x|x) \wedge (y|y))))\\
&= ((z|z)|((y|(x|x))|(x|x)))|((z|z)|((y|(x|x))|(x|x)))\\
&= ((z|z)|((x|(y|y))|(y|y)))|((z|z)|((x|(y|y))|(y|y)))\\
&= ((x \vee y)|(z|z))|((x \vee y)|(z|z))
\end{aligned}$$

from (20), (21), (S1), (S2) and (SR3). □

**Theorem 3.6.** *Let $(M, \vee, \wedge, |, 0, 1)$ be a $SR_0-$algebra. If $x \longrightarrow y := x|(y|y)$ and $\neg x := x|x$, then $(M, \vee, \wedge, \neg, \longrightarrow, 1)$ is a $R_0-$algebra.*

*Proof.* It is obvious that $(M, \leq)$ is a bounded distributive lattice, $\vee$, $\wedge$ are supremum and infimum with respect to $\leq$, and 1 is the greatest element of $M$. Also, $\neg$ is an order-reversing involution with respect to $\leq$ from Lemma 3.5 (15).

(R1): $\neg x \longrightarrow \neg y = (x|x)|((y|y)|(y|y)) = y|(x|x) = y \longrightarrow x$ from (S1) and (S2).

(R2): $1 \longrightarrow x = 1|(x|x) = x$ and $x \longrightarrow x = x|(x|x) = 1$ from Lemma 3.5 (2) and (SR1), respectively.

(R3): $y \longrightarrow z = y|(z|z) \leq (x|(y|y))|((x|(z|z))|(x|(z|z))) = (x \longrightarrow y) \longrightarrow (x \longrightarrow z)$ from (SR2).

(R4): $x \longrightarrow (y \longrightarrow z) = x|((y|(z|z))|(y|(z|z))) = y|((x|(z|z))|(x|(z|z))) = y \longrightarrow (x \longrightarrow z)$ from Lemma 3.5 (1).

(R5): $x \longrightarrow (y \vee z) = x|((y \vee z)|(y \vee z)) = (x|(y|y)) \vee (x|(z|z)) = (x \longrightarrow y) \vee (x \longrightarrow z)$ and similarly $x \longrightarrow (y \wedge z) = (x \longrightarrow y) \wedge (x \longrightarrow z)$ from (SR3).

(R6): $(x \longrightarrow y) \vee ((x \longrightarrow y) \longrightarrow (\neg x \vee y)) = (x|(y|y)) \vee ((x|(y|y))|(((x|x) \vee y)|((x|x) \vee y))) = 1$ from Lemma 3.5 (23). □

**Example 3.7.** Given the $SR_0-$algebra $M$ in Example 3.3. Then a $R_0-$algebra defined by the $SR_0-$algebra has the Cayley tables in Table 6.

**Theorem 3.8.** *Let $(M, \vee, \wedge, \neg, \longrightarrow, 1)$ be a $R_0-$algebra such that $\neg x = x \longrightarrow \neg x$ and $(x \longrightarrow y) \longrightarrow x = x$ for all $x, y \in M$. If $x|y := x \longrightarrow \neg y$, then $(M, \vee, \wedge, |, 0, 1)$ is a $SR_0-$algebra.*

Table 6. Cayley tables of the binary operations $\longrightarrow$, $\vee$ and $\wedge$ on $M$ in Example 3.7

| $\longrightarrow$ | 0 | $a$ | $b$ | 1 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| $a$ | $b$ | 1 | $b$ | 1 |
| $b$ | $a$ | $a$ | 1 | 1 |
| 1 | 0 | $a$ | $b$ | 1 |

| $\vee$ | 0 | $a$ | $b$ | 1 |
|---|---|---|---|---|
| 0 | 0 | $a$ | $b$ | 1 |
| $a$ | $a$ | $a$ | 1 | 1 |
| $b$ | $b$ | 1 | $b$ | 1 |
| 1 | 1 | 1 | 1 | 1 |

| $\wedge$ | 0 | $a$ | $b$ | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | $a$ | 0 | $a$ |
| $b$ | 0 | 0 | $b$ | $b$ |
| 1 | 0 | $a$ | $b$ | 1 |

*Proof.* It is clear that $(M, \leq)$ is a bounded distributive lattice, $\vee$ and $\wedge$ are supremum and infimum with respect to $\leq$.

It is firstly shown that $|$ is Sheffer stroke on $M$.

(S1): $x|y = x \longrightarrow \neg y = x \longrightarrow (y \longrightarrow 0) = y \longrightarrow (x \longrightarrow 0) = y \longrightarrow \neg x = y|x$ from Proposition 2.4 and (R4).

(S2): $(x|x)|(x|y) = (x \longrightarrow \neg x) \longrightarrow \neg(x \longrightarrow \neg y) = \neg x \longrightarrow \neg(x \longrightarrow \neg y) = (x \longrightarrow \neg y) \longrightarrow x = x$ from (R1).

(S3):

$$\begin{aligned}
x|((y|z)|(y|z)) &= x \longrightarrow \neg\neg(y \longrightarrow \neg z) \\
&= x \longrightarrow (((y \longrightarrow \neg z) \longrightarrow 0) \longrightarrow 0) \\
&= ((y \longrightarrow \neg z) \longrightarrow 0) \longrightarrow (x \longrightarrow 0) \\
&= \neg(y \longrightarrow \neg z) \longrightarrow \neg x \\
&= x \longrightarrow (y \longrightarrow \neg z) \\
&= x \longrightarrow (z \longrightarrow \neg y) \\
&= z \longrightarrow (x \longrightarrow \neg y) \\
&= ((x|y)|(x|y))|z
\end{aligned}$$

from Proposition 2.4, (R4), (R1) and (S1).

(S4): It is obtained from (S2) that $(x|((x|x)|(y|y)))|(x|((x|x)|(y|y))) = x$.

Moreover, $1 = 0|0$ is the greatest element and $0 = 1|1$ is the least element of $M$.

(SR1): $x|(x|x) = x \longrightarrow \neg\neg x = x \longrightarrow ((x \longrightarrow 0) \longrightarrow 0) = (x \longrightarrow 0) \longrightarrow (x \longrightarrow 0) = 1$ from Proposition 2.4, (R4) and (R2).

(SR2): $y|(z|z) = y \longrightarrow z \leq (x \longrightarrow y) \longrightarrow (x \longrightarrow z) = (x|(y|y))|((x|(z|z))|(x|(z|z)))$ from Proposition 2.4, (R4), (R1) and (R3).

(SR3): $x|((y \vee z)|(y \vee z)) = x \longrightarrow (y \vee z) = (x \longrightarrow y) \vee (x \longrightarrow z) = (x|(y|y)) \vee (x|(z|z))$ and similarly $x|((y \wedge z)|(y \wedge z)) = (x|(y|y)) \wedge (x|(z|z))$ from Proposition 2.4, (R4), (R1) and (R5).

□

**Example 3.9.** Consider a $R_0-$algebra $(M, \vee, \wedge, \neg, \longrightarrow, 1)$ which has Cayley tables in Table 7. Then a $SR_0-$algebra defined by the $R_0-$algebra is the $SR_0-$algebra $M$ in Example 3.4.

TABLE 7. Cayley tables of the binary operations $\longrightarrow$, $\vee$ and $\wedge$ on $M$ in Example 3.9

| $\longrightarrow$ | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | 1 |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $a$ | $f$ | 1 | $f$ | $f$ | 1 | 1 | $f$ | 1 |
| $b$ | $e$ | $e$ | 1 | $e$ | 1 | $e$ | 1 | 1 |
| $c$ | $d$ | $d$ | $d$ | 1 | $d$ | 1 | 1 | 1 |
| $d$ | $c$ | $e$ | $f$ | $c$ | 1 | $e$ | $f$ | 1 |
| $e$ | $b$ | $d$ | $b$ | $f$ | $d$ | 1 | $f$ | 1 |
| $f$ | $a$ | $a$ | $d$ | $e$ | $d$ | $e$ | 1 | 1 |
| 1 | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | 1 |

| $\vee$ | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | 1 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | 1 |
| $a$ | $a$ | $a$ | $d$ | $e$ | $d$ | $e$ | 1 | 1 |
| $b$ | $b$ | $d$ | $b$ | $f$ | $d$ | 1 | $f$ | 1 |
| $c$ | $c$ | $e$ | $f$ | $c$ | 1 | $e$ | $f$ | 1 |
| $d$ | $d$ | $d$ | $d$ | 1 | $d$ | 1 | 1 | 1 |
| $e$ | $e$ | $e$ | 1 | $e$ | 1 | $e$ | 1 | 1 |
| $f$ | $f$ | 1 | $f$ | $f$ | 1 | 1 | $f$ | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

| $\wedge$ | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | 1 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | $a$ | 0 | 0 | $a$ | $a$ | 0 | $a$ |
| $b$ | 0 | 0 | $b$ | 0 | $b$ | 0 | $b$ | $b$ |
| $c$ | 0 | 0 | 0 | $c$ | 0 | $c$ | $c$ | $c$ |
| $d$ | 0 | $a$ | $b$ | 0 | $d$ | $a$ | $b$ | $d$ |
| $e$ | 0 | $a$ | 0 | $c$ | $a$ | $e$ | $c$ | $e$ |
| $f$ | 0 | 0 | $b$ | $c$ | $b$ | $c$ | $f$ | $f$ |
| 1 | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | 1 |

**Definition 3.10.** Let $(M, \vee_M, \wedge_M, |_M, 0_M, 1_M)$ and $(N, \vee_N, \wedge_N, |_N, 0_N, 1_N)$ be two $SR_0-$ algebras. Then the set $M \times N$ is the Cartesian product of $M$ and $N$, the operations $|_{M\times N}$, $\vee_{M\times N}$, $\wedge_{M\times N}$ and the partial order $\leq_{M\times N}$ on $M \times N$ are defined by $(x_1, y_1)|_{M\times N}(x_2, y_2) = (x_1|_M x_2, y_1|_N y_2)$, $(x_1, y_1) \vee_{M\times N} (x_2, y_2) = (x_1 \vee_M x_2, y_1 \vee_N y_2)$, $(x_1, y_1) \wedge_{M\times N} (x_2, y_2) = (x_1 \wedge_M x_2, y_1 \wedge_N y_2)$ and $(x_1, y_1) \leq_{M\times N} (x_2, y_2) = (x_1 \leq_M x_2, y_1 \leq_N y_2)$, respectively. Also, $0_{M\times N} = (0_M, 0_N)$ and $1_{M\times N} = (1_M, 1_N)$.

**Theorem 3.11.** *Let $(M, \vee_M, \wedge_M, |_M, 0_M, 1_M)$ and $(N, \vee_N, \wedge_N, |_N, 0_N, 1_N)$ be two $SR_0-$ algebras. Then $(M \times N, \vee_{M\times N}, \wedge_{M\times N}, |_{M\times N}, 0_{M\times N}, 1_{M\times N})$ is a $SR_0-$algebra.*

## 4. Ideals of $SR_0-$algebras

In this section, we give some types of ideals on a $SR_0$-algebra. Unless indicated otherwise, $M$ represents a $SR_0$-algebra.

**Definition 4.1.** Let $M$ be a SR$_0$-algebra. Then a nonempty subset $I \subseteq M$ is called an ideal of $M$ if it satisfies

(sI1) $0 \in I$,

(sI2) $y \in I$ and $(x|(y|y))|(x|(y|y)) \in I$ imply $x \in I$.

**Example 4.2.** For the SR$_0$-algebra $M$ in Example 3.3, the subsets $\{0\}$, $\{0, a\}$, $\{0, b\}$ and $M$ are ideals of $M$.

**Proposition 4.3.** *Let $I$ be a nonempty subset of a $SR_0-$algebra $M$. Then $I$ is an ideal of $M$ if and only if the following hold:*

*(sI3) $y \in I$ and $x \leq y$ imply $x \in I$,*

*(sI4) $x \in I$ and $y \in I$ imply $(x|x)|(y|y) \in I$.*

*Proof.* ($\Rightarrow$) Let $I$ be an ideal of $M$.

(sI3): Let $y \in I$ and $x \leq y$. Then $x|(y|y) = 1$ and so $(x|(y|y))|(x|(y|y)) = 1|1 = 0 \in I$ from Lemma 3.5 (10) and (sI1). Thus, $x \in I$ from (sI2).

(sI4): Let $x$ and $y$ be any elements of $I$. Since

$$\begin{aligned}
&(((((x|x)|(y|y))|(y|y))|(((x|x)|(y|y))|(y|y)))|(x|x))|\\
&(((((x|x)|(y|y))|(y|y))|(((x|x)|(y|y))|(y|y)))|(x|x))\\
&= ((((y|((x|x)|(x|x)))|((x|x)|(x|x)))|((y|((x|x)|(x|x)))|((x|x)|(x|x))))|(x|x))|\\
&\quad ((((y|((x|x)|(x|x)))|((x|x)|(x|x)))|((y|((x|x)|(x|x)))|((x|x)|(x|x))))|(x|x))\\
&= ((((y|x)|x)|((y|x)|x))|(x|x))|((((y|x)|x)|((y|x)|x))|(x|x))\\
&= ((y|x)|((x|(x|x))|(x|(x|x))))|((y|x)|((x|(x|x))|(x|(x|x))))\\
&= 1|1\\
&= 0
\end{aligned}$$

from Lemma 3.5 (3) and (20), (S2)-(S3), (SR1) and (sI1), it is obtained from (sI2) that $(((x|x)|(y|y))|(y|y))|(((x|x)|(y|y))|(y|y)) \in I$. Hence, $(x|x)|(y|y) \in I$ from (sI2).

($\Leftarrow$) Let $I$ be a nonempty subset of $M$ satisfying (1) and (2). Assume that $x \in I$. Since $0 \leq x$ for all $x \in M$, we have from (1) that $0 \in I$. Let $y$ and $(x|(y|y))|(x|(y|y))$ be any elements of $I$. Then $x \vee y = (x|(y|y))|(y|y) = (((x|(y|y))|(x|(y|y)))|((x|(y|y))|(x|(y|y))))|(y|y) \in I$ from Lemma 3.5 (21), (S2) and (sI4), respectively. Since $x \leq x \vee y$, it follows from (sI3) that $x \in I$.

□

**Lemma 4.4.** *Let $I$ be a nonempty subset of a $SR_0-$algebra $M$. Then $I$ is an ideal of $M$ if and only if*

*(sI5) $0 \in I$,*

*(sI6) $x \in I$ and $y \in I$ imply $x \vee y \in I$.*

*Proof.* Let $I$ be an ideal of $M$. Then (sI5) is obvious from (sI1). Assume that $x \in I$ and $y \in I$. $x \vee y \in I$. Since $(x|x)|(y|y) \in I$ and $x \vee y \leq (x|x)|(y|y)$ from (sI4) and Lemma 3.5 (13), it is obtained from (sI3) that $x \vee y \in I$.

Conversely, let $I$ be a nonempty subset of $M$ satisfying (sI5) and (sI6). Then (sI1) is obvious from (sI5). Suppose that $y \in I$ and $(x|(y|y))|(x|(y|y)) \in I$. Then

$$\begin{aligned} x \vee y &= (x|(y|y))|(y|y) \\ &= (x|(((y|y)|(y|y))|((y|y)|(y|y))))|(y|y) \\ &= (((x|(y|y))|(x|(y|y)))|(y|y))|(y|y) \\ &= ((x|(y|y))|(x|(y|y))) \vee y \in I \end{aligned}$$

from Lemma 3.5 (21), (S1), (S2) and (sI6). Since $x \in x \vee y$, it follows from (sI3) that $x \in I$. Thus, $I$ is an ideal of $M$. □

**Theorem 4.5.** *The family $\mathcal{I}_M$ of all ideals of a SR$_0-$algebra $M$ forms a complete lattice.*

*Proof.* Let $\{I_i : i \in J\}$ be a family of ideals of a SR$_0-$algebra $M$. Since $0 \in I_i$, for all $i \in J$, we have $0 \in \bigcup_{i \in J} I_i$ and $0 \in \bigcap_{i \in J} I_i$.

(1) Assume that $y \in \bigcap_{i \in J} I_i$ and $(x|(y|y))|(x|(y|y)) \in \bigcap_{i \in J} I_i$. Then $y \in I_i$ and $(x|(y|y))|(x|(y|y)) \in I_i$, for all $i \in J$. Since $I_i$ is an ideal of $M$, for all $i \in J$, it is obtained that $x \in I_i$, for all $i \in J$ which implies that $x \in \bigcap_{i \in J} I_i$.

(2) Let $\gamma$ be the family of all ideals of $M$ containing $\bigcup_{i \in J} I_i$. Thus, $\bigcap \gamma$ is an ideal of $M$ by (1).

If $\bigwedge_{i \in J} I_i = \bigcap_{i \in J} I_i$ and $\bigvee_{i \in J} I_i = \bigcap \gamma$, then $(\mathcal{I}_M, \bigvee, \bigwedge)$ is a complete lattice. □

**Definition 4.6.** Let $I$ be an ideal of a SR$_0-$algebra $M$. $I$ is called a prime ideal of $M$ if $x \wedge y \in I$ implies $x \in I$ or $y \in I$, for all $x, y \in M$.

**Example 4.7.** Consider the SR$_0$-algebra $M$ in Example 3.4. Then $\{0, a, c, e\}$ is a prime ideal of $M$ while $\{0, a\}$ is not since $d$ and $e$ are not in $I$ when $d \wedge e = a \in I$.

**Proposition 4.8.** *Let $I$ be an ideal of a SR$_0-$algebra $M$. Then $I$ is a prime ideal of $M$ if and only if $x \in I$ or $x|x \in I$, for all $x \in M$.*

*Proof.* (⇒) Let $I$ be a prime ideal of $M$. Since $x \wedge (x|x) = 0 \in I$ from Lemma 3.5 (21), (S2), (SR1) and (sI1), it follows that $x \in I$ or $x|x \in I$, for all $x \in M$.

(⇐) Let $I$ be an ideal of $M$ such that $x \in I$ or $x|x \in I$ for all $x \in M$, and $x \wedge y \in I$. Suppose that $x \notin I$ for some $x \in M$. Then $x|x \in I$. Since

$((x|(y|y))|((x|x)|(x|x)))|((x|(y|y))|((x|x)|(x|x))) = (x|(x|(y|y)))|(x|(x|(y|y))) = x \wedge y \in I$ from (S1), (S2) and Lemma 3.5 (21), it is obtained from (sI2) that $x|(y|y) \in I$. Since $y \leq x|(y|y)$ from Lemma 3.5 (4), we have $y \in I$ from (sI3).

Also, assume that $y \notin I$ for some $y \in M$. Then $y|y \in I$. Since $(x|(y|y))|(x|(y|y)) \leq y|y$ from Lemma 3.5 (4) and (15), we get from (sI3) that $(x|(y|y))|(x|(y|y)) \in I$. Since $(x|(((x|(y|y))|(x|(y|y)))|((x|(y|y))|(x|(y|y)))))|(x|(((x|(y|y))|(x|(y|y)))|((x|(y|y))|(x|(y|y))))) = (x|(x|(y|y)))|(x|(x|(y|y))) = x \wedge y \in I$ from (S2) and Lemma 3.5 (21), it follows from (sI2) that $x \in I$. □

**Proposition 4.9.** *Let $I$ be an ideal of a $SR_0-$algebra $M$. Then $I$ is a prime ideal of $M$ if and only if $(x|(y|y))|(x|(y|y)) \in I$ or $(y|(x|x))|(y|(x|x)) \in I$, for all $x, y \in M$.*

*Proof.* Let $I$ be a prime ideal of $M$. Since

$$\begin{aligned}
&((x|(y|y))|(x|(y|y))) \wedge ((y|(x|x))|(y|(x|x)))\\
&= (((x|(y|y))|(x|(y|y)))|(((x|(y|y))|(x|(y|y)))|(y|(x|x))))|\\
&\quad (((x|(y|y))|(x|(y|y)))|(((x|(y|y))|(x|(y|y)))|(y|(x|x))))\\
&= (((x|(y|y))|(x|(y|y)))|(x|(((y|y)|(y|(x|x)))|((y|y)|(y|(x|x))))))|\\
&\quad (((x|(y|y))|(x|(y|y)))|(x|(((y|y)||(y|(x|x)))|((y|y)|(y|(x|x))))))|\\
&= ((x|(y|y))|((x|(y|y))|(x|(y|y))))|((x|(y|y))|((x|(y|y))|(x|(y|y))))\\
&= 1|1\\
&= 0 \in I
\end{aligned}$$

from Lemma 3.5 (21), (S1)-(S3), (SR1) and (sI1), it is obtained that $(x|(y|y))|(x|(y|y)) \in I$ or $(y|(x|x))|(y|(x|x)) \in I$, for all $x, y \in M$.

Conversely, let $I$ be an ideal of $M$ such that $(x|(y|y))|(x|(y|y)) \in I$ or $(y|(x|x))|(y|(x|x)) \in I$, for all $x, y \in M$. Assume that $x \wedge y \in I$. Since $(x|(((x|(y|y))|(x|(y|y)))|((x|(y|y))|(x|(y|y)))))|(x|(((x|(y|y))|(x|(y|y)))|((x|(y|y))|(x|(y|y))))) = (x|(x|(y|y)))|(x|(x|(y|y))) = x \wedge y \in I$ from (S2) and Lemma 3.5 (21), it follows from (sI2) that $x \in I$. Similarly, $y \in I$ since $x \wedge y = y \wedge x$. □

## 5. Quotient $\mathbf{SR_0}-$algebra via ideals

In this section, we introduce a quotient $SR_0-$algebra via ideals and present some properties.

Let $I$ be an ideal of a $SR_0-$algebra $M$. A binary relation $\sim_I$ on $M$ is defined by

$$x \sim_I y \quad \Leftrightarrow \quad (x|(y|y))|(x|(y|y)) \in I \quad \textit{and} \quad (y(x|x))|(y|(x|x)) \in I, \tag{1}$$

for all $x, y \in M$.

**Definition 5.1.** If $x\beta y$ implies $x|z\beta y|z$, $x\vee z\beta y\vee z$ and $x\wedge z\beta y\wedge z$, for all $x,y,z\in M$, then the equivalence relation $\beta$ is called a congruence relation on $M$.

**Example 5.2.** Consider the $\mathrm{SR}_0-$algebra $M$ in Example 3.3. Then $\beta = \{(0,0),(a,a),(b,b),(1,1),(0,$
$a),(a,0),(b,1),(1,b)\}$ is a congruence relation on $M$.

**Lemma 5.3.** *An equivalence relation $\beta$ is a congruence relation on $M$ if and only if $x\beta y$ and $a\beta b$ imply $x|a\beta y|b$, $x\vee a\beta y\vee b$ and $x\wedge a\beta y\wedge b$, for all $x,y,a,b\in M$.*

*Proof.* Let $\beta$ be a congruence relation on $M$ and $x,y,a$ and $b$ be any elements of $M$ such that $x\beta y$ and $a\beta b$. Since $x|a\beta y|a$ and $y|a\beta y|b$ from (S1), it follows from the transitivity of $\beta$ that $x|a\beta y|b$, for all $x,y,a,b\in M$. It is obtained from Lemma 3.5 (21) that $x\vee a\beta y\vee b$ and $x\wedge a\beta y\wedge b$, for all $x,y,a,b\in M$.

Conversely, let $\beta$ be an equivalence relation on $M$ such that $x\beta y$ and $a\beta b$ imply $x|a\beta y|b$, $x\vee a\beta y\vee b$ and $x\wedge a\beta y\wedge b$, for all $x,y,a,b\in M$. Assume that $x,y$ and $z$ be arbitrary elements of $M$ such that $x\beta y$. Since $z\beta z$, we get $x|z\beta y|z$, $x\vee z\beta y\vee z$ and $x\wedge z\beta y\wedge z$, for all $x,y,z\in M$. Then $\beta$ is a congruence relation on $M$. □

**Lemma 5.4.** *Let $I$ be an ideal of a $SR_0-$algebra $M$ and $\sim_I$ be defined as the statement (1). Then $\sim_I$ is a congruence relation on $M$.*

*Proof.* • Reflexivity is obvious from (SR1).

• Symmetry: let $x$ and $y$ be any elements of $M$ such that $x \sim_I y$. Since $(x|(y|y))|(x|(y|y)),(y|(x|x))|(y|(x|x))\in I$, it is clear that $y\sim_I x$.

• Transitivity: let $x,y$ and $z$ be any elements of $M$ such that $x\sim_I y$ and $y\sim_I z$. Then $(x|(y|y))|(x|(y|y)),(y|(x|x))|(y|(x|x)),(y|(z|z))|(y|(z|z)),(z|(y|y))|(z|(y|y))\in I$. Since

$$\begin{aligned}(x|(y|y))|(y|(z|z)) &= (((x|(y|y))|(x|(y|y)))|((x|(y|y))|(x|(y|y))))|\\ &\quad (((y|(z|z))|(y|(z|z)))|((y|(z|z))|(y|(z|z))))\in I\end{aligned}$$

and $(x|(z|z))|(x|(z|z))\le(x|(y|y))|(y|(z|z))$ from (S2), (sI4) and Lemma 3.5 (19), we get from (sI3) that $(x|(z|z))|(x|(z|z))\in I$. Similarly, $(z|(x|x))|(z|(x|x))\in I$. So, $x\sim_I z$.

Thus, $\sim_I$ is an equivalence relation on $M$.

Let $x,y,a$ and $b$ be any elements of $M$ such that $x \sim_I y$ and $a \sim_I b$. Then $(x|(y|y))|(x|(y|y)),(y|(x|x))|(y|(x|x)),(a|(b|b))|(a|(b|b)),(b|(a|a))|(b|(a|a))\in I$.

(1) Since

$$\begin{aligned} y|(x|x) &= (x|x)|((y|y)|(y|y)) \\ &\leq (a|((x|x)|(x|x)))|((a|((y|y)|(y|y)))|(a|((y|y)|(y|y)))) \\ &= (x|a)|((y|a)|(y|a)) \end{aligned}$$

from (S1), (S2) and (SR2), it is obtained from Lemma 3.5 (15) that

$$((x|a)|((y|a)|(y|a)))|((x|a)|((y|a)|(y|a))) \leq (y|(x|x))|(y|(x|x)).$$

By (sI3), $((x|a)|((y|a)|(y|a)))|((x|z)|((y|z)|(y|z))) \in I$, and similarly, $((y|a)|((x|a)|(x|a)))|((y|a)|((x|a)|(x|a))) \in I$. Thus, $x|a \sim_I y|a$.

(2) By substituting $[x := a]$, $[y := b]$ and $[a := y]$ in 1, simultaneously, we obtain from (S1) that $y|a \sim_I y|b$.

Hence, $x|a \sim_I y|b$ from the transitivity of $\sim_I$. Also, $x \vee a \sim_I y \vee b$ and $x \wedge a \sim_I y \wedge b$ from Lemma 3.5 (21). □

**Theorem 5.5.** *Let $I$ be an ideal of a $SR_0-$algebra $M$ and $\sim_I$ be a congruence relation on $M$ defined by $I$. Then $(M/\sim_I, \vee_{\sim_I}, \wedge_{\sim_I}, |_{\sim_I}, [0]_{\sim_I}, [1]_{\sim_I})$ is a $SR_0-$algebra where the quotient set $M/\sim_I = \{[x]_{\sim_I} : x \in M\}$, the binary operations $|_{\sim_I}$, $\vee_{\sim_I}$ and $\wedge_{\sim_I}$ are defined by $[x]_{\sim_I}|_{\sim_I}[y]_{\sim_I} = [x|y]_{\sim_I}$, $[x]_{\sim_I} \vee_{\sim_I} [y]_{\sim_I} = [x \vee y]_{\sim_I}$ and $[x]_{\sim_I} \wedge_{\sim_I} [y]_{\sim_I} = [x \wedge y]_{\sim_I}$, $[0]_{\sim_I}$ is the least element and $[1]_{\sim_I}$ is the greatest element of $M/\sim_I$, respectively. Moreover, the partial order $\leq$ on $M/\sim_I$ is defined by $[x]_{\sim_I} \leq [y]_{\sim_I} \Leftrightarrow (x|(y|y))|(x|(y|y)) \in I$, for any $x, y \in M$.*

*Proof.* Let a relation $\leq$ on $M/\sim_I$ be defined by $[x]_{\sim_I} \leq [y]_{\sim_I} \Leftrightarrow (x|(y|y))|(x|(y|y)) \in I$, for any $x, y \in M$.

- Since $(x|(x|x))|(x|(x|x)) = 1|1 = 0 \in I$ from (SR1) and (sI1), we have $[x]_{\sim_I} \leq [x]_{\sim_I}$.
- Let $[x]_{\sim_I} \leq [y]_{\sim_I}$ and $[y]_{\sim_I} \leq [x]_{\sim_I}$. Then $(x|(y|y))|(x|(y|y)) \in I$ and $(y|(x|x))|(y|(x|x)) \in I$. So, $x \sim_I y$ which implies $[x]_{\sim_I} = [y]_{\sim_I}$.
- Let $[x]_{\sim_I} \leq [y]_{\sim_I}$ and $[y]_{\sim_I} \leq [z]_{\sim_I}$. Then $(x|(y|y))|(x|(y|y)) \in I$ and $(y|(z|z))|(y|(z|z)) \in I$. Since $(x|(y|y))|(y|(z|z)) = (((x|(y|y))|(x|(y|y)))|((x|(y|y))|(x|(y|y))))|(((y|(z|z))|(y|(z|z)))|((y|(z|z))|(y|(z|z)))) \in I$ and $(x|(z|z))|(x|(z|z)) \leq (x|(y|y))|(y|(z|z))$ from (S2), (sI4) and Lemma 3.5 (19), it follows from (sI3) that $(x|(z|z))|(x|(z|z)) \in I$. Thus, $[x]_{\sim_I} \leq [z]_{\sim_I}$.

Hence, the relation $\leq$ is a partial otder on $M/\sim_I$.

- Since $x \leq x \vee y$ and $xy \leq x \vee y$, we have from Lemma 3.5 (10) and (sI1) that $(x|((x \vee y)|(x \vee y)))|(x|((x \vee y)|(x \vee y))) = 1|1 = 0 \in I$ and $(y|((x \vee y)|(x \vee y)))|(y|((x \vee y)|(x \vee y))) = 1|1 = 0 \in I$. Then $[x]_{\sim_I} \leq [x \vee y]_{\sim_I} = [x]_{\sim_I} \vee_{\sim_I} [y]_{\sim_I}$ and $[y]_{\sim_I} \leq [x \vee y]_{\sim_I} = [x]_{\sim_I} \vee_{\sim_I} [y]_{\sim_I}$. Thus, $[x \vee y]_{\sim_I} = [x]_{\sim_I} \vee_{\sim_I} [y]_{\sim_I}$ is an upper bound of $[x]_{\sim_I}$ and $[y]_{\sim_I}$. Assume that $[x]_{\sim_I} \leq [z]_{\sim_I}$ and $[y]_{\sim_I} \leq [z]_{\sim_I}$. So, $(x|(z|z))|(x|(z|z)) \in I$ and $(y|(z|z))|(y|(z|z)) \in I$. Since $((x \vee y)|(z|z))|((x \vee$

$y)|(z|z)) = ((x|(z|z))|(x|(z|z))) \vee ((y|(z|z))|(y|(z|z))) \in I$ from Lemma 3.5 (25) and (sI6), we have $[x]_{\sim_I} \vee_{\sim_I} [y]_{\sim_I} = [x \vee y]_{\sim_I} \leq [z]_{\sim_I}$ which means that $[x]_{\sim_I} \vee_{\sim_I} [y]_{\sim_I} = [x \vee y]_{\sim_I}$ is a supremum of $[x]_{\sim_I}$ and $[y]_{\sim_I}$. Similarly, $[x]_{\sim_I} \wedge_{\sim_I} [y]_{\sim_I} = [x \wedge y]_{\sim_I}$ is an infimum of $[x]_{\sim_I}$ and $[y]_{\sim_I}$.

- $[x]_{\sim_I} \vee_{\sim_I} ([y]_{\sim_I} \wedge_{\sim_I} [z]_{\sim_I}) = [x \vee (y \wedge z)]_{\sim_I} = [(x \vee y) \wedge (x \vee z)]_{\sim_I} = ([x]_{\sim_I} \vee_{\sim_I} [y]_{\sim_I}) \wedge_{\sim_I} ([x]_{\sim_I} \vee_{\sim_I} [z]_{\sim_I})$, and similarly, $([x]_{\sim_I} \wedge_{\sim_I} [y]_{\sim_I}) \vee_{\sim_I} [z]_{\sim_I} = ([x]_{\sim_I} \vee_{\sim_I} [z]_{\sim_I}) \wedge_{\sim_I} ([y]_{\sim_I} \vee_{\sim_I} [z]_{\sim_I})$, $[x]_{\sim_I} \wedge_{\sim_I} ([y]_{\sim_I} \vee_{\sim_I} [z]_{\sim_I}) = ([x]_{\sim_I} \wedge_{\sim_I} [y]_{\sim_I}) \vee_{\sim_I} ([x]_{\sim_I} \wedge_{\sim_I} [z]_{\sim_I})$ and $([x]_{\sim_I} \vee_{\sim_I} [y]_{\sim_I}) \wedge_{\sim_I} [z]_{\sim_I} = ([x]_{\sim_I} \wedge_{\sim_I} [z]_{\sim_I}) \vee_{\sim_I} ([y]_{\sim_I} \wedge_{\sim_I} [z]_{\sim})$, for all $[x]_{\sim_I}, [y]_{\sim_I}, [z]_{\sim_I} \in M/\sim_I$.
- Since $(0|(x|x))|(0|(x|x)) = ((x|x)|(1|1))|((x|x)|(1|1)) = 1|1 = 0 \in I$ and $(x|(1|1))|(x|(1|1)) = 1|1 = 0 \in I$ from (S1), Lemma 3.5 (3) and (sI1), we have $[0]_{\sim_I} \leq [x]_{\sim_I}$ and $[x]_{\sim_I} \leq [1]_{\sim_I}$, for all $[x]_{\sim_I} \in M/\sim_I$, respectively.

Hence, $(M/\sim_I, \leq)$ is a bounded distributive lattice.

(S1): $[x]_{\sim_I}|_{\sim_I}[y]_{\sim_I} = [x|y]_{\sim_I} = [y|x]_{\sim_I} = [y]_{\sim_I}|_{\sim_I}[x]_{\sim_I}$,

(S2): $([x]_{\sim_I}|_{\sim_I}[x]_{\sim_I})|_{\sim_I}([x]_{\sim_I}|_{\sim_I}[y]_{\sim_I}) = [(x|x)|(x|y)]_{\sim_I} = [x]_{\sim_I}$,

(S3):

$$\begin{aligned}
[x]_{\sim_I}|_{\sim_I}(([y]_{\sim_I}|_{\sim_I}[z]_{\sim_I})|_{\sim_I}([y]_{\sim_I}|_{\sim_I}[z]_{\sim_I})) &= [x|((y|z)|(y|z))]_{\sim_I} \\
&= [((x|y)|(x|y))|z]_{\sim_I} \\
&= (([x]_{\sim_I}|_{\sim_I}[y]_{\sim_I})|_{\sim_I}([x]_{\sim_I}|_{\sim_I}[y]_{\sim_I}))|_{\sim_I}[z]_{\sim_I},
\end{aligned}$$

(S4):

$([x]_{\sim_I}|_{\sim_I}(([x]_{\sim_I}|_{\sim_I}[x]_{\sim_I})|_{\sim_I}([y]_{\sim_I}|_{\sim_I}[y]_{\sim_I})))|_{\sim_I}([x]_{\sim_I}|_{\sim_I}(([x]_{\sim_I}|_{\sim_I}[x]_{\sim_I})|_{\sim_I}([y]_{\sim_I}|_{\sim_I}[y]_{\sim_I})))$
$= [(x|((x|x)|(y|y)))|(x|((x|x)|(y|y)))]_{\sim_I} = [x]_{\sim_I}$,

(SR1): $[x]_{\sim_I}|_{\sim_I}([x]_{\sim_I}|_{\sim_I}[x]_{\sim_I}) = [x|(x|x)]_{\sim_I} = [1]_{\sim_I}$,

(SR2): Since $((y|(z|z))|(((x|(y|y))|((x|(z|z))|(x|(z|z))))|((x|(y|y))|((x|(z|z))|(x|(z|z))))))|$ $((y|(z|z))|(((x|(y|y))|((x|(z|z))|(x|(z|z))))|((x|(y|y))|((x|(z|z))|(x|(z|z)))))) = 1|1 = 0 \in I$ from Lemma 3.5 (10) and (sI1), it follows that

$$\begin{aligned}
[y]_{\sim_I}|_{\sim_I}([z]_{\sim_I}|_{\sim_I}[z]_{\sim_I}) &= [y|(z|z)]_{\sim_I} \\
&\leq [(x|(y|y))|((x|(z|z))|(x|(z|z)))]_{\sim_I} \\
&= ([x]_{\sim_I}|_{\sim_I}([y]_{\sim_I}|_{\sim_I}[y]_{\sim_I}))|_{\sim_I}(([x]_{\sim_I}|_{\sim_I}([z]_{\sim_I} \\
&\quad |_{\sim_I}[z]_{\sim_I}))|_{\sim_I}([x]_{\sim_I}|_{\sim_I}([z]_{\sim_I}|_{\sim_I}[z]_{\sim_I}))),
\end{aligned}$$

(SR3): $[x]_{\sim_I}|_{\sim_I}(([y]_{\sim_I} \vee_{\sim_I} [z]_{\sim_I})|_{\sim_I}([y]_{\sim_I} \vee_{\sim_I} [z]_{\sim_I})) = [x|((y \vee z)|(y \vee z))]_{\sim_I} = [(x|(y|y)) \vee (x|(z|z))]_{\sim_I} = ([x]_{\sim_I}|_{\sim_I}([y]_{\sim_I}|_{\sim_I}[y]_{\sim_I})) \vee_{\sim_I} ([x]_{\sim_I}|_{\sim_I}([z]_{\sim_I}|_{\sim_I}[z]_{\sim_I}))$ and $[x]_{\sim_I}|_{\sim_I}(([y]_{\sim_I} \wedge_{\sim_I} [z]_{\sim_I})|_{\sim_I}([y]_{\sim_I} \wedge_{\sim_I} [z]_{\sim_I})) = ([x]_{\sim_I}|_{\sim_I}([y]_{\sim_I}|_{\sim_I}[y]_{\sim_I})) \wedge_{\sim_I} ([x]_{\sim_I}|_{\sim_I}([z]_{\sim_I}|_{\sim_I}[z]_{\sim_I}))$, for all $[x]_{\sim_I}, [y]_{\sim_I}, [z]_{\sim_I} \in M/\sim_I$.

Therefore, $(M/\sim_I, \vee_{\sim_I}, \wedge_{\sim_I}, |_{\sim_I}, [0]_{\sim_I}, [1]_{\sim_I})$ is a $\mathrm{SR}_0-$algebra. □

**Example 5.6.** Consider the $\mathrm{SR}_0-$algebra $M$ in Example 3.4. For an ideal $I = \{0, b\}$ of $M$, $\sim_I = \{(0,0),(a,a),(b,b),(c,c),(d,d),(e,e),(f,f),(1,1),(0,b),(b,0),(e,1),(1,e),(a,d),(d,a),$ $(f,c),(c,f)\}$ is a congruence relation on $M$. Then $(M/\sim_I, \vee_{\sim_I}, \wedge_{\sim_I}, |_{\sim_I}, [0]_{\sim_I}, [1]_{\sim_I})$ is a $\mathrm{SR}_0-$algebra with the Hasse diagram in Figure 3 where $M/\sim_I = \{[0]_{\sim_I}, [a]_{\sim_I}, [c]_{\sim_I}, [1]_{\sim_I}\}$ and the binary operations $|_{\sim_I}$, $\vee_{\sim_I}$ and $\wedge_{\sim_I}$ on $M/\sim_I$ have Cayley tables in Table 8.

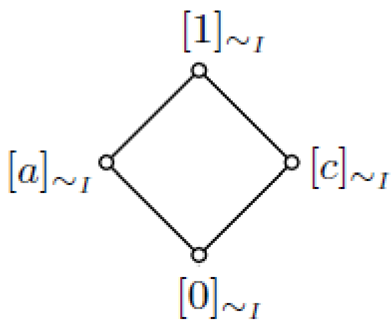


FIGURE 3. Hasse diagram for $M/\sim_I$

TABLE 8. Cayley tables of the binary operations $|_{\sim_I}$, $\vee_{\sim_I}$ and $\wedge_{\sim_I}$ on $M/\sim_I$ in Example 5.6

| $\vert_{\sim_I}$ | $[0]_{\sim_I}$ | $[a]_{\sim_I}$ | $[c]_{\sim_I}$ | $[1]_{\sim_I}$ |
|---|---|---|---|---|
| $[0]_{\sim_I}$ | $[1]_{\sim_I}$ | $[1]_{\sim_I}$ | $[1]_{\sim_I}$ | $[1]_{\sim_I}$ |
| $[a]_{\sim_I}$ | $[1]_{\sim_I}$ | $[c]_{\sim_I}$ | $[1]_{\sim_I}$ | $[c]_{\sim_I}$ |
| $[c]_{\sim_I}$ | $[1]_{\sim_I}$ | $[1]_{\sim_I}$ | $[a]_{\sim_I}$ | $[a]_{\sim_I}$ |
| $[1]_{\sim_I}$ | $[1]_{\sim_I}$ | $[c]_{\sim_I}$ | $[a]_{\sim_I}$ | $[0]_{\sim_I}$ |

| $\vee_{\sim_I}$ | $[0]_{\sim_I}$ | $[a]_{\sim_I}$ | $[c]_{\sim_I}$ | $[1]_{\sim_I}$ |
|---|---|---|---|---|
| $[0]_{\sim_I}$ | $[0]_{\sim_I}$ | $[a]_{\sim_I}$ | $[c]_{\sim_I}$ | $[1]_{\sim_I}$ |
| $[a]_{\sim_I}$ | $[a]_{\sim_I}$ | $[a]_{\sim_I}$ | $[1]_{\sim_\mu}$ | $[1]_{\sim_I}$ |
| $[c]_{\sim_I}$ | $[c]_{\sim_I}$ | $[1]_{\sim_I}$ | $[c]_{\sim_\mu}$ | $[1]_{\sim_I}$ |
| $[1]_{\sim_I}$ | $[1]_{\sim_I}$ | $[1]_{\sim_I}$ | $[1]_{\sim_I}$ | $[1]_{\sim_I}$ |

| $\wedge_{\sim_I}$ | $[0]_{\sim_I}$ | $[a]_{\sim_I}$ | $[c]_{\sim_I}$ | $[1]_{\sim_I}$ |
|---|---|---|---|---|
| $[0]_{\sim_I}$ | $[0]_{\sim_I}$ | $[0]_{\sim_I}$ | $[0]_{\sim_I}$ | $[0]_{\sim_I}$ |
| $[a]_{\sim_I}$ | $[0]_{\sim_I}$ | $[a]_{\sim_I}$ | $[0]_{\sim_I}$ | $[a]_{\sim_I}$ |
| $[c]_{\sim_I}$ | $[0]_{\sim_I}$ | $[0]_{\sim_I}$ | $[c]_{\sim_I}$ | $[c]_{\sim_I}$ |
| $[1]_{\sim_I}$ | $[0]_{\sim_I}$ | $[a]_{\sim_I}$ | $[c]_{\sim_I}$ | $[1]_{\sim_I}$ |

**Theorem 5.7.** *Let $I$ be an ideal of a $SR_0-$algebra $M$. Then $I$ is a prime ideal of $M$ if and only if $M/\sim_I$ is a totally ordered $SR_0-$algebra and $\mid M/\sim_I \mid \leq 2$.*

*Proof.* ($\Rightarrow$) Let $I$ be a prime ideal of a $\mathrm{SR}_0-$algebra $M$. Then $M/\sim_I$ is a $\mathrm{SR}_0-$ algebra from Theorem 5.5. Since $(x|(y|y))|(x|(y|y)) \in I$ or $(y(x|x))|(y|(x|x)) \in I$ by Proposition 4.9, $[x]_{\sim_I} \leq [y]_{\sim_I}$ or $[y]_{\sim_I} \leq [x]_{\sim_I}$, for all $x, y \in M$. Thus, $M/\sim_I$ is a totally ordered $\mathrm{SR}_0-$algebra.

Assume that $| M/\sim_I| > 2$. Let $[x]_{\sim_I} \in M/\sim_I$ such that $[0]_{\sim_I} < [x]_{\sim_I} < [1]_{\sim_I}$. Since $I$ is a prime ideal of $M$, $x \in I$ or $x|x \in I$ from Proposition 4.8. Suppose that $x|x \in I$. Since $(1|(x|x))|(1|(x|x)) = x|x \in I$ and $(x|(1|1))|(x|(1|1)) = 1|1 = 0 \in I$ from Lemma 3.5 (2)-(3) and (sI1), we have $x \sim_I 1$. So, $[x]_{\sim_I} = [1]_{\sim_I}$ which is a contradiction. Hence, $| M/\sim_I| \leq 2$.

($\Leftarrow$) Let $M/\sim_I$ be a totally ordered SR$_0-$algebra. Then $[x]_{\sim_I} \leq [y]_{\sim_I}$ or $[y]_{\sim_I} \leq [x]_{\sim_I}$ which mean that $(x|(y|y))|(x|(y|y)) \in I$ or $(y(x|x))|(y|(x|x)) \in I$, for all $x, y \in M$. Thus, $I$ is a prime ideal of a SR$_0-$algebra $M$ by Proposition 4.9. □

**Corollary 5.8.** *Let $I$ be an ideal of a SR$_0-$algebra $M$. Then $M/\sim_I$ is a totally ordered SR$_0-$algebra if and only if $x \in I$ or $x|x \in I$, for all $x \in M$.*

*Proof.* It follows from Proposition 4.8 and Theorem 5.7. □

## 6. Conclusion

In this paper, a R$_0-$algebra with Sheffer stroke, Cartesian product, some ideals, a congruence relation and quotient structures are introduced. Then it is stated that the axiom system of Sheffer stroke R$_0-$algebras (briefly, SR$_0-$algebras) is independent. It is also shown that a SR$_0-$algebra is a R$_0-$algebra under the conditions $x \longrightarrow y = x|(y|y)$ and $\neg x = x|x$ but special conditions are necessary for the inverse, and these statements are supported by giving illustrative examples. It is demonstrated that a Cartesian product of two SR$_0-$algebras is a SR$_0-$algebra. Afterward, various ideals and their features are studied on SR$_0-$algebras. Moreover, a congruence relation on these algebraic structures is defined by the ideal and quotient SR$_0-$algebra is built by means of this relation. Indeed, it is proved that an ideal of a SR$_0-$algebra is prime if and only if the quotient structure is a totally ordered SR$_0-$algebra, and its cardinality is less that or equals to two. At the end of the study, the new and novel results are given on aforementioned concepts. These results are important to develop relations between Sheffer stroke algebras and related notions.

In future works, we are planning to study fuzziness, neutrosophy, plithogeny, and various filters on Sheffer stroke R$_0-$algebras. Therefore, new bridges can be constructed among abstract algebra, plithogeny, logic, probability and statistics.In this section, we give basic definitions and notions about Sheffer stroke and R$_0-$algebras.

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