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Research Paper

REGULAR DIVISORS OF A SUBMODULE

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Abstract. In this article, we extend the concept of divisors to ideals of Noetherian rings, more generally, to submodules of finitely generated modules over Noetherian rings. For a submodule N of a finitely generated module M over a Noetherian ring, we say a submodule K of M is a regular divisor of N in M if K occurs in a regular prime extension filtration of M over N. We show that a submodule N of M has only a finite number of regular divisors in M. We also show that an ideal $\mathfrak{b}$ is a regular divisor of a non-zero ideal $\mathfrak{a}$ in a Dedekind domain R if and only if $\mathfrak{b}$ contains $\mathfrak{a}$. We characterize regular divisors using some ordered sequences of prime ideals and study their various properties. Lastly, we formulate a method to compute the number of regular divisors of a submodule by solving a combinatorics problem.

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1. Introduction

In [5], the concept of prime ideal factorization is generalized to proper submodules of finitely generated modules over a Noetherian ring. If

$$N = M_0 \overset{\mathfrak{q}_1}{\subset} M_1 \subset \cdots \subset M_{n-1} \overset{\mathfrak{q}_n}{\subset} M_n = M \tag{1}$$

is a filtration of submodules, where for each i, $\mathfrak{q}_i$ is a maximal element in $\text{Ass}(M/M_{i-1})$ and M_i is maximal among the submodules of M such that M_{i-1} is a $\mathfrak{q}_i$-prime submodule of M_i, then we say the generalized prime ideal factorization of N in M, denoted $\mathcal{P}_M(N)$, is $\mathfrak{q}_1 \cdots \mathfrak{q}_n$. We also write $\mathcal{P}_M(N) = \mathfrak{p}_1{}^{r_1} \cdots \mathfrak{p}_k{}^{r_k}$ if $\mathfrak{p}_i$ occurs exactly r_i times in $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$, and $r_1 + \cdots + r_k = n$. In this case, the filtration (1) is called a regular prime extension (RPE) filtration of M over N. We call a submodule which occurs in any RPE filtration of M over N as a regular divisor of N in M. We show that regular divisors extend the concept of divisors to submodules of finitely generated modules over Noetherian rings. If n is an integer, then d is a divisor of n if and only if $d\mathbb{Z}$ is a regular divisor of $n\mathbb{Z}$ in $\mathbb{Z}$. So $n\mathbb{Z}$ has $\prod_{i=1}^{k}(r_i + 1)$ regular divisors if the prime factorization of n is $p_1{}^{r_1} \cdots p_k{}^{r_k}$.

In this paper, we show that a submodule N of M has only a finite number of regular divisors in M. We also formulate a method to compute the number of regular divisors of a submodule.

Throughout this article, we assume that R is a commutative Noetherian ring with identity, M is a finitely generated unitary R-module, and N is a proper submodule of M. For terminology used, the standard reference is [4].

In [3], Lu put forward various useful properties of prime submodules of modules and showed their applications. In [1], a submodule K of M is called a $\mathfrak{p}$-prime extension of N in M if N is a prime submodule of K with $(N : K) = \mathfrak{p}$, and it is denoted as $N \overset{\mathfrak{p}}{\subset} K$. Further, if K is not properly contained in any other $\mathfrak{p}$-prime extensions of N in M, then we say $N \overset{\mathfrak{p}}{\subset} K$ is maximal. If $\mathfrak{p}$ is a maximal element in $\text{Ass}(M/N)$, then a maximal $\mathfrak{p}$-prime extension $N \overset{\mathfrak{p}}{\subset} K$ is called a regular $\mathfrak{p}$-prime extension.

Let $\mathcal{F} : N = M_0 \overset{\mathfrak{p}_1}{\subset} M_1 \subset \cdots \subset M_{n-1} \overset{\mathfrak{p}_n}{\subset} M_n = M$ be a filtration of submodules containing N, where each extension $M_{i-1} \overset{\mathfrak{p}_i}{\subset} M_i$ is a regular $\mathfrak{p}_i$-prime extension. Then $\mathcal{F}$ is called a regular prime extension (RPE) filtration of M over N. Regular prime extension filtration of submodules is introduced and studied in [1].

It is proved that a regular $\mathfrak{p}$-prime extension of a submodule is unique.

Lemma 1.1. [1, Theorem 11] *Let N be a proper submodule of M and $\mathfrak{p}$ be a maximal element in* $\text{Ass}(M/N)$. *Then the submodule* $(N : \mathfrak{p})$ *of M is the unique maximal $\mathfrak{p}$-prime extension of N in M.*

Remark 1.2. Hence, if $\mathfrak{p}$ is a maximal element in $\text{Ass}(M/N)$, then $(N : \mathfrak{p})$ is the regular $\mathfrak{p}$-prime extension of N in M. So the number of regular prime extensions of N in M is exactly equal to the number of maximal elements in $\mathrm{Ass}(M/N)$. Hence, a submodule of M has only a finite number of regular prime extensions in M.

Lemma 1.3. [1, Proposition 14] *Let* $N = M_0 \overset{\mathfrak{p}_1}{\subset} M_1 \subset \cdots \subset M_{n-1} \overset{\mathfrak{p}_n}{\subset} M_n = M$ *be a filtration of submodules such that each* $M_{i-1} \overset{\mathfrak{p}_i}{\subset} M_i$ *is a maximal* $\mathfrak{p}_i$*-prime extension. Then* $\mathrm{Ass}(M/M_{i-1}) = \{\mathfrak{p}_i, \ldots, \mathfrak{p}_n\}$ *for* $1 \le i \le n$*. In particular,* $\mathrm{Ass}(M/N) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$.

Since regular prime extensions are maximal prime extensions, we have that $\mathrm{Ass}(M/N)$ is precisely the set of prime ideals occurring in any RPE filtration of M over N.

The following lemma shows the uniqueness of the length of RPE filtrations.

Lemma 1.4. [1, Theorem 22] *For a proper submodule N of M, the number of times a prime ideal $\mathfrak{p}$ occurs in any RPE filtration of M over N is unique, and hence, any two RPE filtrations of M over N have the same length.*

Definition 1.5. Let N be a proper submodule of M with $\mathrm{Ass}(M/N) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$, where $\mathfrak{p}_i$'s are distinct. Then we write $\mathcal{P}_M(N) = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_k}^{r_k}$ if, for each i, $\mathfrak{p}_i$ occurs exactly r_i times in an RPE filtration of M over N.

If $\mathcal{P}_M(N) = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_k}^{r_k}$, as a product of ideals, it is possible that ${\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_k}^{r_k} = {\mathfrak{p}_1}^{s_1} \cdots {\mathfrak{p}_k}^{s_k}$ with $r_i \neq s_i$ for some i. But $\mathcal{P}_M(N) \neq {\mathfrak{p}_1}^{s_1} \cdots {\mathfrak{p}_k}^{s_k}$ as per our definition. In [5], $\mathcal{P}_M(N)$ is called the generalized prime ideal factorization of N in M and its various properties are studied.

We prove that a subchain of an RPE filtration is also an RPE filtration using the following lemma.

Lemma 1.6. [2, Lemma 2.8] *If* $N \overset{\mathfrak{p}}{\subset} K$ *is a regular* $\mathfrak{p}$*-prime extension in* M *and* L *is any submodule of* M*, then* $N \cap L \overset{\mathfrak{p}}{\subset} K \cap L$ *is a regular* $\mathfrak{p}$*-prime extension in* L *when* $N \cap L \neq K \cap L$.

Proposition 1.7. *If* $N = M_0 \overset{\mathfrak{p}_1}{\subset} M_1 \subset \cdots \subset M_{n-1} \overset{\mathfrak{p}_n}{\subset} M_n = M$ *is an RPE filtration of* M *over* N*, then* $M_i \overset{\mathfrak{p}_{i+1}}{\subset} M_{i+1} \subset \cdots \subset M_{j-1} \overset{\mathfrak{p}_j}{\subset} M_j$ *is an RPE filtration of* M_j *over* M_i *for every* $0 \le i < j \le n$*, and therefore,* $\mathcal{P}_{M_j}(M_i) = \mathfrak{p}_{i+1} \cdots \mathfrak{p}_j$ *and* $\mathrm{Ass}(M_j/M_i) = \{\mathfrak{p}_{i+1}, \ldots, \mathfrak{p}_j\}$.

Proof. For $i < n$,

$$M_i \overset{\mathfrak{p}_{i+1}}{\subset} M_{i+1} \subset \cdots \subset M_{n-1} \overset{\mathfrak{p}_n}{\subset} M_n = M \tag{2}$$

is an RPE filtration since M_{k+1} is a regular $\mathfrak{p}_{k+1}$-prime extension of M_k in M for $k = i, \ldots, n-1$. Let $i < j \le n$. Then intersecting (2) with M_j, we get a chain

$$M_i \overset{\mathfrak{p}_{i+1}}{\subset} M_{i+1} \subset \cdots \subset M_{j-1} \overset{\mathfrak{p}_j}{\subset} M_j. \tag{3}$$

By Lemma 1.6, (3) is an RPE filtration of M_j over M_i, and hence, $\mathcal{P}_{M_j}(M_i) = \mathfrak{p}_{i+1} \cdots \mathfrak{p}_j$ and by Lemma 1.3, $\mathrm{Ass}(M_j/M_i) = \{\mathfrak{p}_{i+1}, \ldots, \mathfrak{p}_j\}$. □

2. Regular Divisors of a Submodule

In this section, we define regular divisors of a submodule N in M and study its properties.

Definition 2.1. A submodule K of M is called a regular divisor of N in M if there exists an RPE filtration $N = N_0 \overset{\mathfrak{p}_1}{\subset} N_1 \subset \cdots \subset N_{n-1} \overset{\mathfrak{p}_n}{\subset} N_n = M$ with $K = N_i$ for some i. We also say M is a regular divisor of M in M.

Let $\mathcal{D}_M(N)$ denote the set of all regular divisors of N in M.

Example 2.2. N is a prime submodule of M if and only if $\mathcal{D}_M(N) = \{N, M\}$. For, if N is a $\mathfrak{p}$-prime submodule of M, $\mathrm{Ass}(M/N) = \{\mathfrak{p}\}$ and M is the maximal $\mathfrak{p}$-prime extension of N. So $N \overset{\mathfrak{p}}{\subset} M$ is the only RPE filtration of M over N. In particular, an ideal $\mathfrak{a}$ is a prime ideal of R if and only if $\mathcal{D}_R(\mathfrak{a}) = \{\mathfrak{a}, R\}$.

Example 2.3. Let $R = k[x, y]$ and $\mathfrak{a} = (x^2y, xy^2)$. Since the primary decomposition of $\mathfrak{a}$ is $(x^2, y^2) \cap (x) \cap (y)$, $\mathrm{Ass}(R/\mathfrak{a}) = \{(x, y), (x), (y)\}$. Then $(\mathfrak{a} : (x, y)) = (xy)$ is the regular (x, y)-prime extension of $\mathfrak{a}$ in R. Now, $\mathrm{Ass}(R/(xy)) = \{(x), (y)\}$. Then $((xy) : (y)) = (x)$ and $((xy) : (x)) = (y)$ are the regular (y)-prime and (x)-prime extensions of (xy) respectively. So we have exactly two RPE filtrations of R over $\mathfrak{a}$,

$$\mathfrak{a} = (x^2y, xy^2) \overset{(x,y)}{\subset} (xy) \overset{(y)}{\subset} (x) \overset{(x)}{\subset} R,$$

$$\mathfrak{a} = (x^2y, xy^2) \overset{(x,y)}{\subset} (xy) \overset{(x)}{\subset} (y) \overset{(y)}{\subset} R.$$

Therefore, the set of all regular divisors of $\mathfrak{a}$ in R, $\mathcal{D}_R(\mathfrak{a}) = \{\mathfrak{a}, (xy), (x), (y), R\}$.

Now we show that the set of regular divisors of a submodule is finite.

Proposition 2.4. *A submodule N of M has a finite number of regular divisors in M.*

Proof. Since M is Noetherian, any RPE filtration is of finite length. While constructing an RPE filtration $N = N_0 \overset{\mathfrak{p}_1}{\subset} N_1 \subset \cdots \subset N_{n-1} \overset{\mathfrak{p}_n}{\subset} N_n = M$, N_1 must be one of the regular prime extensions of N in M; hence the number of choices for N_1 is the number of maximal elements in $\mathrm{Ass}(M/N)$ [Remark 1.2]. Similarly, for each i, the number of submodules N_i which can be regular prime extensions of N_{i-1} is the number of maximal elements in $\mathrm{Ass}(M/N_{i-1})$, and therefore is finite. So the number of RPE filtrations of M over N is finite, and hence the number of regular divisors of N in M is finite. □

Next we show if K is a regular divisor of N in M, then $\mathcal{P}_M(N)$ is a multiple of $\mathcal{P}_M(K)$ as a product of prime ideals.

Proposition 2.5. *If K is a regular divisor of N in M, then $\mathcal{P}_M(N) = \mathcal{P}_K(N)\mathcal{P}_M(K)$ and* $\mathrm{Ass}(M/K) \cup \mathrm{Ass}(K/N) = \mathrm{Ass}(M/N)$.

Proof. We have an RPE filtration $N = N_0 \overset{\mathfrak{p}_1}{\subset} N_1 \subset \cdots \overset{\mathfrak{p}_r}{\subset} N_r \overset{\mathfrak{p}_{r+1}}{\subset} \cdots \overset{\mathfrak{p}_n}{\subset} N_n = M$ with $K = N_r$. Then by Proposition 1.7, $N = N_0 \overset{\mathfrak{p}_1}{\subset} N_1 \subset \cdots \overset{\mathfrak{p}_r}{\subset} N_r = K$ and $K = N_r \overset{\mathfrak{p}_{r+1}}{\subset} \cdots \overset{\mathfrak{p}_n}{\subset} N_n = M$ are RPE filtrations. So $\mathcal{P}_M(N) = \mathfrak{p}_1 \cdots \mathfrak{p}_r \mathfrak{p}_{r+1} \cdots \mathfrak{p}_n = \mathcal{P}_K(N)\mathcal{P}_M(K)$. By Proposition 1.7, $\mathrm{Ass}(M/N) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$, $\mathrm{Ass}(M/K) = \{\mathfrak{p}_{r+1}, \ldots, \mathfrak{p}_n\}$, and $\mathrm{Ass}(K/N) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. This proves the Proposition. □

Remark 2.6. In particular, if $\mathcal{P}_M(N) = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_k}^{r_k}$, where $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ are distinct prime ideals and $r_1, \ldots, r_k$ positive integers, and K is a regular divisor of N in M, then $\mathcal{P}_K(N) = {\mathfrak{p}_1}^{s_1} \cdots {\mathfrak{p}_k}^{s_k}$, $\mathcal{P}_M(K) = {\mathfrak{p}_1}^{t_1} \cdots {\mathfrak{p}_k}^{t_k}$ with $0 \leq s_i, t_i \leq r_i$ and $s_i + t_i = r_i$ for $1 \leq i \leq k$.

The converse of Proposition 2.5 is not true. In Example 2.3, we have $\mathcal{P}_R(\mathfrak{a}) = (x, y)(x)(y)$. Let $\mathfrak{b} = (x, y)$. Then

$$\mathfrak{a} = (x^2y, xy^2) \overset{(y)}{\subset} (x^2, xy) \overset{(x)}{\subset} (x, y) = \mathfrak{b} \quad \text{and} \quad \mathfrak{b} = (x, y) \overset{(x,y)}{\subset} R.$$

are RPE filtrations. So $\mathcal{P}_{\mathfrak{b}}(\mathfrak{a})\mathcal{P}_R(\mathfrak{b}) = (y)(x)(x, y) = \mathcal{P}_R(\mathfrak{a})$ and $\mathrm{Ass}(\mathfrak{b}/\mathfrak{a}) \cup \mathrm{Ass}(R/\mathfrak{b}) = \{(y), (x), (x, y)\} = \mathrm{Ass}(R/\mathfrak{a})$, but $\mathfrak{b}$ is not a regular divisor of $\mathfrak{a}$ in R.

The next proposition shows that regular divisors extend the concept of divisors in integers.

Proposition 2.7. *Let R be a Dedekind domain and $\mathfrak{a}$ a non-zero ideal in R. Then an ideal $\mathfrak{b}$ is a regular divisor of $\mathfrak{a}$ in R if and only if $\mathfrak{b} \supseteq \mathfrak{a}$. In particular, for $d, n \in \mathbb{Z}$, $d\mathbb{Z}$ is a regular divisor of $n\mathbb{Z}$ in $\mathbb{Z}$ if and only if d is a divisor of n.*

Proof. If $\mathfrak{b}$ is a regular divisor of $\mathfrak{a}$ in R, then clearly, $\mathfrak{a} \subseteq \mathfrak{b}$. Next, we assume $\mathfrak{b} \supseteq \mathfrak{a}$. There exist distinct prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ in R and positive integers $r_1, \ldots, r_k$ such that $\mathfrak{a} = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_k}^{r_k}$. Since $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ are non-zero prime ideals, they are maximal ideals.

Note that $({\mathfrak{p}_i}^{r_i} : \mathfrak{p}_i) = {\mathfrak{p}_i}^{r_i - 1}$. For since R is Dedekind, $({\mathfrak{p}_i}^{r_i} : \mathfrak{p}_i) = {\mathfrak{q}_1}^{t_1} \cdots {\mathfrak{q}_m}^{t_m}$ for some distinct prime ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$ and positive integers $t_1, \ldots, t_m$. Then ${\mathfrak{p}_i}^{r_i - 1} \subseteq ({\mathfrak{p}_i}^{r_i} : \mathfrak{p}_i) \subseteq \mathfrak{q}_j$ for every $1 \leq j \leq m$. This implies that $\mathfrak{p}_i = \mathfrak{q}_j$ for $1 \leq j \leq m$. Therefore $({\mathfrak{p}_i}^{r_i} : \mathfrak{p}_i) = {\mathfrak{p}_i}^{t}$ for some integer t. That is, ${\mathfrak{p}_i}^{t}\mathfrak{p}_i \subseteq {\mathfrak{p}_i}^{r_i}$. So $t \geq r_i - 1$. Also, ${\mathfrak{p}_i}^{r_i - 1} \subseteq ({\mathfrak{p}_i}^{r_i} : \mathfrak{p}_i) = {\mathfrak{p}_i}^{t}$ implies that $r_i - 1 \geq t$. Therefore $t = r_i - 1$.

We claim that $(\mathfrak{a} : \mathfrak{p}_i) = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_i}^{r_i - 1} \cdots {\mathfrak{p}_k}^{r_k}$. Clearly ${\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_i}^{r_i - 1} \cdots {\mathfrak{p}_k}^{r_k} \subseteq (\mathfrak{a} : \mathfrak{p}_i)$. For $a \in (\mathfrak{a} : \mathfrak{p}_i)$, $a\mathfrak{p}_i \subseteq \mathfrak{a} = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_k}^{r_k} \subseteq {\mathfrak{p}_j}^{r_j}$ for $j = 1, \ldots, k$. For $j \neq i$, $a\mathfrak{p}_i \subseteq {\mathfrak{p}_j}^{r_j}$ implies $a \in {\mathfrak{p}_j}^{r_j}$ since ${\mathfrak{p}_j}^{r_j}$ is a primary ideal. Also, we have $a\mathfrak{p}_i \subseteq {\mathfrak{p}_i}^{r_i}$, that is, $a \in ({\mathfrak{p}_i}^{r_i} : \mathfrak{p}_i) = {\mathfrak{p}_i}^{r_i - 1}$. Therefore $a \in {\mathfrak{p}_1}^{r_1} \cap \cdots \cap {\mathfrak{p}_i}^{r_i - 1} \cap \cdots \cap {\mathfrak{p}_k}^{r_k} = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_i}^{r_i - 1} \cdots {\mathfrak{p}_k}^{r_k}$. Hence the claim.

Then since $\mathfrak{p}_i$ is a maximal element in $\mathrm{Ass}(R/\mathfrak{a})$, by Remark 1.2, $(\mathfrak{a} : \mathfrak{p}_i) = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_i}^{r_i - 1} \cdots {\mathfrak{p}_k}^{r_k}$ is the regular $\mathfrak{p}_i$-prime extension of $\mathfrak{a}$ in R. For an ideal $\mathfrak{b} \supseteq \mathfrak{a}$, $\mathfrak{b} = {\mathfrak{p}_1}^{s_1} \cdots {\mathfrak{p}_k}^{s_k}$, where $0 \leq s_i \leq r_i$. So we can have an RPE filtration

$$\mathfrak{a} = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_k}^{r_k} \overset{\mathfrak{p}_i}{\subset} {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_i}^{r_i - 1} \cdots {\mathfrak{p}_k}^{r_k} \subset \cdots \subset {\mathfrak{p}_1}^{s_1} \cdots {\mathfrak{p}_k}^{s_k} = \mathfrak{b} \subset \cdots \subset R.$$

Hence $\mathfrak{b}$ is a regular divisor of $\mathfrak{a}$ in R. □

If R is not Dedekind, then the above result is not true. In Example 2.3, the ideal (x^2, xy) contains $\mathfrak{a}$, but is not a regular divisor of $\mathfrak{a}$ in R.

3. Regular Prime Sequences

For a submodule N in M, for every RPE filtration there exists an ordered sequence of prime ideals. In this section, we characterize the regular divisors of N in M using these sequences.

Definition 3.1. An ordered sequence of prime ideals $(\mathfrak{p}_1, \ldots, \mathfrak{p}_n)$ is called a regular prime sequence of M with respect to N if there exists an RPE filtration

$$N = N_0 \overset{\mathfrak{p}_1}{\subset} N_1 \subset \cdots \subset N_{n-1} \overset{\mathfrak{p}_n}{\subset} N_n = M.$$

Proposition 3.2. *Let $(\mathfrak{p}_1, \ldots, \mathfrak{p}_n)$ be a regular prime sequence of M with respect to N. Then*

(i) $\mathfrak{p}_i \not\subset \mathfrak{p}_j$ *if $i < j$, that is, $\mathfrak{p}_i$ is a maximal element in $\{\mathfrak{p}_i, \mathfrak{p}_{i+1}, \ldots, \mathfrak{p}_n\}$.*

(ii) *Any other sequence $(\mathfrak{p}'_1, \ldots, \mathfrak{p}'_n)$ is a regular prime sequence of M with respect to N if and only if it is a permutation of $(\mathfrak{p}_1, \ldots, \mathfrak{p}_n)$ satisfying $\mathfrak{p}'_i \not\subset \mathfrak{p}'_j$ for $1 \leq i < j \leq n$.*

Proof. Let $N = N_0 \overset{\mathfrak{p}_1}{\subset} N_1 \subset \cdots \subset N_{n-1} \overset{\mathfrak{p}_n}{\subset} N_n = M$ be the RPE filtration with respect to $(\mathfrak{p}_1, \ldots, \mathfrak{p}_n)$. Then for every $1 \leq i \leq n$, N_i is a regular $\mathfrak{p}_i$-prime extension of N_{i-1}, and therefore $\mathfrak{p}_i$ is a maximal element in $\mathrm{Ass}(M/N_{i-1})$. By Lemma 1.3, $\mathrm{Ass}(M/N_{i-1}) = \{\mathfrak{p}_i, \mathfrak{p}_{i+1}, \ldots, \mathfrak{p}_n\}$. This proves (i).

Suppose $(\mathfrak{p}'_1, \ldots, \mathfrak{p}'_n)$ is a regular prime sequence of M with respect to N. Then by Lemma 1.4, $(\mathfrak{p}'_1, \ldots, \mathfrak{p}'_n)$ is a permutation of $(\mathfrak{p}_1, \ldots, \mathfrak{p}_n)$, and by (i), $\mathfrak{p}'_i \not\subset \mathfrak{p}'_j$ for $1 \leq i < j \leq n$. Conversely, if $(\mathfrak{p}'_1, \ldots, \mathfrak{p}'_n)$ is a permutation of $(\mathfrak{p}_1, \ldots, \mathfrak{p}_n)$ satisfying $\mathfrak{p}'_i \not\subset \mathfrak{p}'_j$ for $1 \leq i < j \leq n$, then $\mathfrak{p}'_1$ is maximal in $\{\mathfrak{p}'_1, \ldots, \mathfrak{p}'_n\} = \mathrm{Ass}(M/N)$. So there exists a regular $\mathfrak{p}'_1$-prime extension K_1 of $K_0 = N$ in M and $\mathrm{Ass}(M/K_1) = \{\mathfrak{p}'_2, \ldots, \mathfrak{p}'_n\}$ [Proposition 2.5]. Inductively, we assume that K_i is a regular $\mathfrak{p}'_i$-prime extension of K_{i-1} in M and $\mathrm{Ass}(M/K_i) = \{\mathfrak{p}'_{i+1}, \ldots, \mathfrak{p}'_n\}$. Since $\mathfrak{p}'_{i+1}$ is maximal in $\{\mathfrak{p}'_{i+1}, \ldots, \mathfrak{p}'_n\}$, by Lemma 1.1, there exists a regular $\mathfrak{p}'_{i+1}$-prime extension K_{i+1} of K_i in M. So we have an RPE filtration $N = K_0 \overset{\mathfrak{p}'_1}{\subset} K_1 \overset{\mathfrak{p}'_2}{\subset} K_2 \subset \cdots \overset{\mathfrak{p}'_n}{\subset} K_n = M$, and therefore $(\mathfrak{p}'_1, \ldots, \mathfrak{p}'_n)$ is a regular prime sequence of M with respect to N. □

Definition 3.3. A sequence of prime ideals $(\mathfrak{p}_1, \ldots, \mathfrak{p}_r)$ is called a part regular prime sequence of M with respect to N if there exist prime ideals $\mathfrak{p}_{r+1}, \ldots, \mathfrak{p}_n$ such that $(\mathfrak{p}_1, \ldots, \mathfrak{p}_r, \mathfrak{p}_{r+1}, \ldots, \mathfrak{p}_n)$ form a regular prime sequence of M with respect to N.

Proposition 3.4. *Let $(\mathfrak{p}_1, \ldots, \mathfrak{p}_r)$ be a part regular prime sequence of M with respect to N. Then*

(i) $\mathfrak{p}_i \not\subset \mathfrak{p}_j$ *for* $1 \leq i < j \leq r$.

(ii) *A permutation* $(\mathfrak{p}'_1, \ldots, \mathfrak{p}'_r)$ *of* $(\mathfrak{p}_1, \ldots, \mathfrak{p}_r)$ *is also a part regular prime sequence of M with respect to N if and only if* $\mathfrak{p}'_i \not\subset \mathfrak{p}'_j$ *for* $1 \leq i < j \leq r$.

(iii) *If* $\mathfrak{q} \in \mathrm{Ass}(M/N)$ *and* $\mathfrak{q} \supset \mathfrak{p}_i$ *for some* $\mathfrak{p}_i \in \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$, *then* $\mathfrak{q} \in \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$.

Proof. We have prime ideals $\mathfrak{p}_{r+1}, \ldots, \mathfrak{p}_n$ such that $(\mathfrak{p}_1, \ldots, \mathfrak{p}_r, \mathfrak{p}_{r+1}, \ldots, \mathfrak{p}_n)$ is a regular prime sequence of M with respect to N. Let $N \overset{\mathfrak{p}_1}{\subset} N_1 \subset \cdots \overset{\mathfrak{p}_r}{\subset} N_r \overset{\mathfrak{p}_{r+1}}{\subset} \cdots \overset{\mathfrak{p}_n}{\subset} N_n = M$ be the corresponding RPE filtration of M over N. By Proposition 1.7, $N \overset{\mathfrak{p}_1}{\subset} N_1 \subset \cdots \overset{\mathfrak{p}_r}{\subset} N_r$ is an RPE filtration of N_r over N, and hence $(\mathfrak{p}_1, \ldots, \mathfrak{p}_r)$ is a regular prime sequence of N_r with respect to N. So (i) and (ii) follow from Proposition 3.2.

Since $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} = \mathrm{Ass}(M/N)$ [Lemma 1.3], $\mathfrak{q} = \mathfrak{p}_k$ for some $1 \leq k \leq n$. Then by Proposition 3.2 (i), $\mathfrak{p}_k \supset \mathfrak{p}_i$ implies $k < i \leq r$, and therefore $\mathfrak{q} \in \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. □

Definition 3.5. If $N = N_0 \overset{\mathfrak{p}_1}{\subset} N_1 \subset \cdots \overset{\mathfrak{p}_r}{\subset} N_r \subset \cdots \overset{\mathfrak{p}_n}{\subset} N_n = M$ is an RPE filtration of M over N, then we say N_r is the regular divisor of N in M defined by the part regular prime sequence $(\mathfrak{p}_1, \ldots, \mathfrak{p}_r)$.

Note. $N_r = (N : \mathfrak{p}_1 \cdots \mathfrak{p}_r)$ by the following lemma.

Lemma 3.6. [2, Lemma 3.1] *Let N be a proper submodule of an R-module M. If* $N = N_0 \overset{\mathfrak{p}_1}{\subset} N_1 \subset \cdots \overset{\mathfrak{p}_n}{\subset} N_n = M$ *is an RPE filtration of M over N, then* $N_i = \{x \in M \mid \mathfrak{p}_1 \cdots \mathfrak{p}_i x \subseteq N\} = (N : \mathfrak{p}_1 \cdots \mathfrak{p}_i)$ *for* $1 \leq i \leq n$.

Proposition 3.7. *If K is the regular divisor of N in M defined by a part regular prime sequence* $(\mathfrak{p}_1, \ldots, \mathfrak{p}_r)$, *then any permutation* $(\mathfrak{p}'_1, \ldots, \mathfrak{p}'_r)$ *of* $(\mathfrak{p}_1, \ldots, \mathfrak{p}_r)$ *satisfying* $\mathfrak{p}'_i \not\subset \mathfrak{p}'_j$ *for* $1 \leq i < j \leq r$ *also defines K.*

Proof. By Proposition 3.4 (ii), any permutation $(\mathfrak{p}'_1, \ldots, \mathfrak{p}'_r)$ of $(\mathfrak{p}_1, \ldots, \mathfrak{p}_r)$ satisfying $\mathfrak{p}'_i \not\subset \mathfrak{p}'_j$ for $1 \leq i < j \leq r$ is also a part regular prime sequence of M with respect to N. Then the regular divisor defined by $(\mathfrak{p}'_1, \ldots, \mathfrak{p}'_r)$ is $(N : \mathfrak{p}'_1 \cdots \mathfrak{p}'_r) = (N : \mathfrak{p}_1 \cdots \mathfrak{p}_r) = K$. □

Let $\mathcal{S}$ denote the set of all part regular prime sequences of M with respect to N. We define a relation $\sim$ on $\mathcal{S}$ as $(\mathfrak{p}_1, \ldots, \mathfrak{p}_r) \sim (\mathfrak{q}_1, \ldots, \mathfrak{q}_s)$, if $(\mathfrak{q}_1, \ldots, \mathfrak{q}_s)$ is a permutation of $(\mathfrak{p}_1, \ldots, \mathfrak{p}_r)$.

Clearly, $\sim$ is an equivalence relation. We denote the equivalence class containing the sequence $(\mathfrak{p}_1, \ldots, \mathfrak{p}_r)$ as $[\mathfrak{p}_1, \ldots, \mathfrak{p}_r]$.

Proposition 3.8. *Mapping an equivalence class $[\mathfrak{p}_1, \ldots, \mathfrak{p}_r]$ to the regular divisor defined by $(\mathfrak{p}_1, \ldots, \mathfrak{p}_r)$ is a one-to-one correspondence between the set of all equivalence classes in $\mathcal{S}$ under the relation $\sim$ defined above and $\mathcal{D}_M(N)$.*

Proof. By Proposition 3.7, every equivalence class $[\mathfrak{p}_1, \ldots, \mathfrak{p}_r]$ defines a unique regular divisor K of N in M. Suppose an element of $[\mathfrak{q}_1, \ldots, \mathfrak{q}_s]$ also defines K. Then we have two RPE filtrations of K over N

$$N = N_0 \overset{\mathfrak{p}_1}{\subset} N_1 \subset \cdots \overset{\mathfrak{p}_r}{\subset} N_r = K,$$

$$N = N'_0 \overset{\mathfrak{q}_1}{\subset} N'_1 \subset \cdots \overset{\mathfrak{q}_s}{\subset} N'_s = K.$$

By Lemma 1.4, $s = r$ and by Proposition 3.2 (ii), $(\mathfrak{q}_1, \ldots, \mathfrak{q}_s)$ is a permutation of $(\mathfrak{p}_1, \ldots, \mathfrak{p}_r)$, i.e., $(\mathfrak{q}_1, \ldots, \mathfrak{q}_s) \in [\mathfrak{p}_1, \ldots, \mathfrak{p}_r]$, and therefore $[\mathfrak{q}_1, \ldots, \mathfrak{q}_s] = [\mathfrak{p}_1, \ldots, \mathfrak{p}_r]$.

Let $K \in \mathcal{D}_M(N)$. Then there exists an RPE filtration $N = N_0 \overset{\mathfrak{p}_1}{\subset} N_1 \subset \cdots \overset{\mathfrak{p}_r}{\subset} N_r \subset \cdots \overset{\mathfrak{p}_n}{\subset} N_n = M$ with $N_r = K$ for some r. Then K is the regular divisor defined by $[\mathfrak{p}_1, \ldots, \mathfrak{p}_r]$. □

Let $\mathcal{P}_M(N) = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_k}^{r_k}$. We characterize the k-tuples $(s_1, \ldots, s_k)$ of integers such that there exists a regular divisor K of N in M with $\mathcal{P}_K(N) = {\mathfrak{p}_1}^{s_1} \cdots {\mathfrak{p}_k}^{s_k}$.

Proposition 3.9. *Let $\mathcal{P}_M(N) = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_k}^{r_k}$. There is a one-to-one correspondence between the regular divisors of N in M and k-tuples $(s_1, \ldots, s_k)$ of integers such that $0 \le s_i \le r_i$, and whenever $\mathfrak{p}_i \supset \mathfrak{p}_j$ with $s_j \ge 1$, then $s_i = r_i$.*

Proof. If K is a regular divisor of N in M, then there exists an RPE filtration $N = N_0 \subset N_1 \subset \cdots \subset N_r \subset \cdots \subset N_n = M$ with $K = N_r$. Then by Remark 2.6, $\mathcal{P}_K(N) = {\mathfrak{p}_1}^{s_1} \cdots {\mathfrak{p}_k}^{s_k}$, where $0 \le s_i \le r_i$. This implies that each $\mathfrak{p}_i$ occurs s_i times in any part regular prime sequence $(\mathfrak{q}_1, \ldots, \mathfrak{q}_r)$ which defines K [Lemma 1.4]. By Proposition 3.7, the equivalence class $[\mathfrak{q}_1, \ldots, \mathfrak{q}_r]$ is uniquely determined by ${\mathfrak{p}_1}^{s_1}, \ldots, {\mathfrak{p}_k}^{s_k}$. Let $\mathfrak{p}_i \supset \mathfrak{p}_j$. Then $s_j \ge 1$ implies $\mathfrak{p}_j \in \{\mathfrak{q}_1, \ldots, \mathfrak{q}_r\}$, i.e., $\mathfrak{p}_j = \mathfrak{q}_t$ for some $1 \le t \le r$. Let $(\mathfrak{q}_1, \ldots, \mathfrak{q}_r, \mathfrak{q}_{r+1}, \ldots, \mathfrak{q}_n)$ be a regular prime sequence of M with respect to N. Suppose $\mathfrak{q}_l = \mathfrak{p}_i$ for some l. Since $\mathfrak{q}_t = \mathfrak{p}_j \subset \mathfrak{p}_i = \mathfrak{q}_l$, by Proposition 3.4 (iii), $l < t \le r$. This implies that $\mathfrak{p}_i$ occurs r_i times in $(\mathfrak{q}_1, \ldots, \mathfrak{q}_r)$. Therefore $s_i = r_i$.

Suppose the k-tuple $(s_1, \ldots, s_k)$ satisfies the given condition. Without loss of generality, we assume that $\mathfrak{p}_i \not\subset \mathfrak{p}_j$ for $i < j$. We denote the sequence

$$(\underbrace{\mathfrak{p}_1, \ldots, \mathfrak{p}_1}_{s_1 \text{times}}, \underbrace{\mathfrak{p}_2, \ldots, \mathfrak{p}_2}_{s_2 \text{times}}, \ldots, \underbrace{\mathfrak{p}_k, \ldots, \mathfrak{p}_k}_{s_k \text{times}}, \underbrace{\mathfrak{p}_1, \ldots, \mathfrak{p}_1}_{r_1 - s_1 \text{times}}, \underbrace{\mathfrak{p}_2, \ldots, \mathfrak{p}_2}_{r_2 - s_2 \text{times}}, \ldots, \underbrace{\mathfrak{p}_k, \ldots, \mathfrak{p}_k}_{r_k - s_k \text{times}})$$

as σ. Note that if $\mathfrak{p}_i \supset \mathfrak{p}_j$ and $s_j \geq 1$ then $s_i = r_i$, and therefore $\mathfrak{p}_i$ cannot occur after $\mathfrak{p}_j$ in σ. This implies that σ is a regular prime sequence [Proposition 3.2 (ii)]. Therefore, the sequence

$$\delta = (\underbrace{\mathfrak{p}_1, \ldots, \mathfrak{p}_1}_{s_1 \text{times}}, \underbrace{\mathfrak{p}_2, \ldots, \mathfrak{p}_2}_{s_2 \text{times}}, \ldots, \underbrace{\mathfrak{p}_k, \ldots, \mathfrak{p}_k}_{s_k \text{times}})$$

is a part regular prime sequence. Then the regular divisor K defined by δ has $\mathcal{P}_K(N) = {\mathfrak{p}_1}^{s_1} \cdots {\mathfrak{p}_k}^{s_k}$, and corresponds to the k-tuple $(s_1, \ldots, s_k)$. □

4. Computation of the Number of Regular Divisors of a Submodule

First, we compute the number of regular divisors of an ideal in a Dedekind domain.

Proposition 4.1. *Let $\mathfrak{a}$ be an ideal of a Dedekind domain R. If $\mathfrak{a} = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_k}^{r_k}$ is the prime ideal factorization of $\mathfrak{a}$, then $|\mathcal{D}_R(\mathfrak{a})| = \prod_{i=1}^{k}(r_i + 1)$.*

Proof. By Proposition 2.7, the regular divisors of $\mathfrak{a}$ in R are the ideals of R containing $\mathfrak{a}$. Since R is a Dedekind domain, ${\mathfrak{p}_1}^{s_1} \cdots {\mathfrak{p}_k}^{s_k}$, $0 \leq s_i \leq r_i$, are precisely the ideals of R which contain $\mathfrak{a}$. So the number of regular divisors is $\prod_{i=1}^{k}(r_i + 1)$. □

Proposition 4.2. *If* $\text{Ass}(M/N)$ *has only isolated prime ideals and $\mathcal{P}_M(N) = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_k}^{r_k}$, then $|\mathcal{D}_M(N)| = \prod_{i=1}^{k}(r_i + 1)$.*

Proof. If every element of $\text{Ass}(M/N)$ is isolated, then any set of k integers $s_1, \ldots, s_k$ with $0 \leq s_i \leq r_i$ satisfies the condition given in Proposition 3.9. Hence the number of regular divisors of N in M is the number of k-tuples $(s_1, \ldots, s_k)$ with $0 \leq s_i \leq r_i$, and therefore $|\mathcal{D}_M(N)| = \prod_{i=1}^{k}(r_i + 1)$. □

Next, we find a method to compute the number of regular divisors of N in M for the general case. For that, we consider the following combinatorics problem.

Definition 4.3. Let $(P, \preceq)$ be a partially ordered set. For $a, b \in P$, we say a and b are comparable if $a \preceq b$ or $b \preceq a$. Otherwise, we say a and b are incomparable. We say a subset S of a partially ordered set P is independent if the elements of S are pairwise incomparable. An independent subset is said to be maximal if it is not a proper subset of any other independent subset.

Every independent subset of a partially ordered set P is contained in a maximal independent subset of P. Also, if A is a maximal independent subset of P, then $b \in P \setminus A$ implies b is comparable with some element of A, that is, there exists $a \in A$ such that $a \preceq b$ or $b \preceq a$.

Example 4.4. Consider the ring $k[x_1, x_2, x_3, x_4]$. Let $\mathfrak{p}_1 = (x_1)$, $\mathfrak{p}_2 = (x_1, x_2)$, $\mathfrak{p}_3 = (x_1, x_2, x_3)$, $\mathfrak{p}_4 = (x_1, x_4)$, and P be the partially ordered set $\{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_4\}$ with $\mathfrak{p}_i \preceq \mathfrak{p}_j$ if $\mathfrak{p}_i \supseteq \mathfrak{p}_j$. Then the maximal independent subsets of P are $\{\mathfrak{p}_1\}$, $\{\mathfrak{p}_2, \mathfrak{p}_4\}$, and $\{\mathfrak{p}_3, \mathfrak{p}_4\}$.

Definition 4.5. Let $(P, \preceq)$ be a partially ordered set. A subset S of P is said to be a node if whenever $a \in S$ and $b \in P$ with $b \preceq a$, then $b \in S$.

Clearly $\emptyset$ and P are nodes of P. Also, any intersection of nodes is a node. For if $a \in \cap_i S_i$ where each S_i is a node, and $b \in P$ with $b \preceq a$, then $b \in S_i$ for each i, and therefore $b \in \cap_i S_i$. Hence $\cap_i S_i$ is a node.

Example 4.6. Let $P = \{a_1, \dots, a_n\}$ with $a_i \preceq a_{i+1}$ for every i, i.e., P is a totally ordered set. Then $\{\{a_1\}, \{a_2\}, \dots, \{a_n\}\}$ is the set of all maximal independent subsets of P; and $\emptyset$, $\{a_1\}$, $\{a_1, a_2\}$, ..., $\{a_1, \dots, a_n\}$ are the nodes of P.

Example 4.7. Let $P = \{a_1, \dots, a_n\}$, where the elements of P are pairwise incomparable. Then $\{a_1, \dots, a_n\}$ is the only maximal independent subset of P. Clearly, any subset of P is a node. So the power set of P, $\mathcal{P}(P)$, is the collection of all nodes of P, and therefore the total number of nodes is 2^n.

Example 4.8. In Example 4.4, $P = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_4\}$ with $\mathfrak{p}_2, \mathfrak{p}_4$ pairwise incomparable, $\mathfrak{p}_3, \mathfrak{p}_4$ pairwise incomparable, $\mathfrak{p}_3 \prec \mathfrak{p}_2 \prec \mathfrak{p}_1$, and $\mathfrak{p}_4 \prec \mathfrak{p}_1$. So the nodes of the partially ordered set $(P, \preceq)$ are $\emptyset$, $\{\mathfrak{p}_3\}$, $\{\mathfrak{p}_4\}$, $\{\mathfrak{p}_3, \mathfrak{p}_2\}$, $\{\mathfrak{p}_3, \mathfrak{p}_4\}$, $\{\mathfrak{p}_3, \mathfrak{p}_2, \mathfrak{p}_4\}$ and P.

Definition 4.9. Let $(P, \preceq)$ be a partially ordered set and $P' \subseteq P$. Then the set

$$\langle P' \rangle = \{a \in P \mid a \preceq b \text{ for some } b \in P'\}$$

is a node of P, and we call it the node generated by the subset P'.

Proposition 4.10. *Let P' be a subset of a partially ordered set $(P, \preceq)$. Then every node of P containing P' contains $\langle P' \rangle$. In particular, $\langle P' \rangle$ is the intersection of all nodes of P containing P'.*

Proof. For any node S with $P' \subseteq S$, if $a \in \langle P' \rangle$, then $a \preceq b$ for some $b \in P' \subseteq S$. This implies that $a \in S$, and hence $\langle P' \rangle \subseteq S$. So $\langle P' \rangle$ is contained in the intersection of all nodes of P containing P'. The equality holds since $\langle P' \rangle$ itself is a node of P containing P'. □

In Example 4.6, $\langle \{a_i\} \rangle = \{a_1, \dots, a_i\}$ for each i. In Example 4.7, for every i, $\langle \{a_i\} \rangle = \{a_i\}$. In Example 4.8, $\langle \{\mathfrak{p}_2\} \rangle = \{\mathfrak{p}_2, \mathfrak{p}_3\}$, and $\langle \{\mathfrak{p}_1\} \rangle = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_4\}$.

Example 4.11. We consider a job that requires the completion of a set of n tasks, say $T = \{t_1, \ldots, t_n\}$. Certain tasks may have some prerequisite tasks in T that must be completed before starting them. We define a relation $\preceq$ in T as $t_i \preceq t_j$ if either $i = j$, or to do the task t_j, the completion of task t_i is required. Then $(T, \preceq)$ is a partially ordered set, and each node of T is a stage of the job. That is, a subset S of T is a stage if whenever $t_j \in S$, any prerequisite task t_i for t_j also belongs to S. In other words, if $t_j \in S$ and $t_i \preceq t_j$, then $t_i \in S$. So the number of distinct nodes of T gives the number of all possible stages of the job.

We compute the number of distinct nodes of a finite partially ordered set P by using the maximal independent subsets of P.

Theorem 4.12. *Let $(P, \preceq)$ be a finite partially ordered set and $\mathcal{T} = \{A_1, \ldots, A_t\}$ be the collection of all maximal independent subsets of P. Then the number of nodes of P is equal to*

$$\sum_{1\leq i\leq t} 2^{|A_i|} - \sum_{1\leq i<j\leq t} 2^{|A_i\cap A_j|} + \sum_{1\leq i<j<k\leq t} 2^{|A_i\cap A_j\cap A_k|} - \cdots + (-1)^{t+1}\, 2^{|\bigcap_{i=1}^{t} A_i|}.$$

Proof. For a node S of P, let S_M denote the set of all maximal elements of S, i.e., $S_M = \{a \in S \mid a \preceq b \text{ for some } b \in S \text{ implies } a = b\}$. Then S_M is an independent subset of P, and therefore contained in A_i for some i. So $S_M \in \bigcup_{i=1}^{t} \mathcal{P}(A_i)$, where $\mathcal{P}(A_i)$ denotes the power set of A_i. By Proposition 4.10, $\langle S_M \rangle \subseteq S$. If $a \in S$, then by the definition of S_M, there exists $b \in S_M$ such that $a \preceq b$. This implies $a \in \langle S_M \rangle$. So $S = \langle S_M \rangle$.

Let $B \in \bigcup_{i=1}^{t} \mathcal{P}(A_i)$. Then $B \subseteq A_i$ for some maximal independent subset $A_i \in \mathcal{T}$. Then $\langle B \rangle$ is the node such that the set of all maximal elements of $\langle B \rangle$ is B, i.e., $\langle B \rangle_M = B$. Hence the correspondence $S \mapsto S_M$ induces a bijection between the set of all nodes of P and $\bigcup_{i=1}^{t} \mathcal{P}(A_i)$.
So the number of nodes of P is equal to $|\bigcup_{i=1}^{t} \mathcal{P}(A_i)|$
$= \sum_{1\leq i\leq t} |\mathcal{P}(A_i)| - \sum_{1\leq i<j\leq t} |\mathcal{P}(A_i) \cap \mathcal{P}(A_j)| + \sum_{1\leq i<j<k\leq t} |\mathcal{P}(A_i) \cap \mathcal{P}(A_j) \cap \mathcal{P}(A_k)| - \cdots + (-1)^{t+1} |\bigcap_{i=1}^{t} \mathcal{P}(A_i)|$
$= \sum_{1\leq i\leq t} 2^{|A_i|} - \sum_{1\leq i<j\leq t} 2^{|A_i\cap A_j|} + \sum_{1\leq i<j<k\leq t} 2^{|A_i\cap A_j\cap A_k|} - \cdots + (-1)^{t+1}\, 2^{|\bigcap_{i=1}^{t} A_i|}$
since $\cap_i \mathcal{P}(A_i) = \mathcal{P}(\cap_i A_i)$. □

Next, we compute the number of nodes of P containing given common elements.

Proposition 4.13. *Let P be a partially ordered set and P' a subset of P. Then the number of nodes of P which contain P' is equal to the number of nodes of the partially ordered set $P \setminus \langle P' \rangle$.*

Proof. Let S be any node of P containing P'. Then by Proposition 4.10, $\langle P'\rangle \subseteq S$. Let $a \in S \setminus \langle P'\rangle$ and $b \in P \setminus \langle P'\rangle$ with $b \preceq a$. Since S is a node, $b \in S$. Hence, $b \in S \setminus \langle P'\rangle$, and therefore $S \setminus \langle P'\rangle$ is a node of $P \setminus \langle P'\rangle$.

Conversely, let S' be a node of $P \setminus \langle P'\rangle$. We claim that $S' \cup \langle P'\rangle$ is a node of P containing P'. Let $a \in S' \cup \langle P'\rangle$ and $b \in P$ such that $b \preceq a$. If $b \notin \langle P'\rangle$, since $\langle P'\rangle$ is a node of P, $a \notin \langle P'\rangle$. Then $a \in S'$, and $b \in P \setminus \langle P'\rangle$ implies $b \in S'$. Therefore $b \in S' \cup \langle P'\rangle$, which implies that $S' \cup \langle P'\rangle$ is a node of P which contains P'. Since $S' \subseteq P \setminus \langle P'\rangle$, $(S' \cup \langle P'\rangle) \setminus \langle P'\rangle = S'$. Hence $S \mapsto S \setminus \langle P'\rangle$ is a one-to-one correspondence between the nodes of P which contain P' and the nodes of $P \setminus \langle P'\rangle$. This proves the proposition. □

We consider a partially ordered set consisting of products of prime ideals for the submodule N in M using $\mathcal{P}_M(N)$.

Notation. Let N be a submodule of M with $\mathcal{P}_M(N) = {\mathfrak{p}_1}^{r_1}\cdots{\mathfrak{p}_k}^{r_k}$, where $\mathfrak{p}_1,\dots,\mathfrak{p}_k$ are distinct prime ideals and $r_1,\dots,r_k$ positive integers. Let

$$\Sigma_{M/N} = \{{\mathfrak{p}_i}^s \mid 1 \le i \le k\,,\, 1 \le s \le r_i\}.$$

We define a partial order $\preceq$ on $\Sigma_{M/N}$ as ${\mathfrak{p}_i}^s \preceq {\mathfrak{p}_j}^t$ if $\mathfrak{p}_i \supset \mathfrak{p}_j$ or $\mathfrak{p}_i = \mathfrak{p}_j$ with $s \le t$.

Example 4.14. Let N be a submodule of M with $\mathcal{P}_M(N) = {\mathfrak{p}_1}^2\mathfrak{p}_2\mathfrak{p}_3{\mathfrak{p}_4}^2$, where the prime ideals $\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3,\mathfrak{p}_4$ are distinct, and $\mathfrak{p}_4 \subset \mathfrak{p}_3$ is the only inclusion. Then the set $\Sigma_{M/N} = \{\mathfrak{p}_1, {\mathfrak{p}_1}^2, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_4, {\mathfrak{p}_4}^2\}$ and the maximal independent subsets of $(\Sigma_{M/N}, \preceq)$ are $\{\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3\}$, $\{\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_4\}$, $\{\mathfrak{p}_1,\mathfrak{p}_2,{\mathfrak{p}_4}^2\}$, $\{{\mathfrak{p}_1}^2,\mathfrak{p}_2,\mathfrak{p}_3\}$, $\{{\mathfrak{p}_1}^2,\mathfrak{p}_2,\mathfrak{p}_4\}$, and $\{{\mathfrak{p}_1}^2,\mathfrak{p}_2,{\mathfrak{p}_4}^2\}$.

Next, we identify the regular divisors of N in M with the nodes of $\Sigma_{M/N}$, and with that, we compute the number of regular divisors of N in M.

Theorem 4.15. *Let N be a submodule of M with $\mathcal{P}_M(N) = {\mathfrak{p}_1}^{r_1}\cdots{\mathfrak{p}_k}^{r_k}$, where $\mathfrak{p}_1,\dots,\mathfrak{p}_k$ are distinct prime ideals and $r_1,\dots,r_k$ positive integers. Then the regular divisors of N in M are in one-to-one correspondence with the nodes of the partially ordered set $(\Sigma_{M/N}, \preceq)$, and therefore*

$$(4)\quad |\mathcal{D}_M(N)| = \sum_{1\le i\le t} 2^{|A_i|} - \sum_{1\le i<j\le t} 2^{|A_i\cap A_j|} + \sum_{1\le i<j<k\le t} 2^{|A_i\cap A_j\cap A_k|} - \cdots + (-1)^{t+1}\, 2^{|\bigcap_{i=1}^{t} A_i|},$$

where $\{A_1,\dots,A_t\}$ are the maximal independent subsets of the partially ordered set $(\Sigma_{M/N}, \preceq)$.

Proof. For a node S of $\Sigma_{M/N}$, we have the following.

(i) If ${\mathfrak{p}_i}^s \in S$ then ${\mathfrak{p}_i}^t \in S$ for $1 \le t \le s$.

(ii) If $\mathfrak{p}_i \preceq \mathfrak{p}_j$ and $\mathfrak{p}_j \in S$, then ${\mathfrak{p}_i}^s \in S$ for $1 \le s \le r_i$.

So, if for each $1 \leq i \leq k$, s_i is the largest integer such that ${\mathfrak{p}_i}^{s_i} \in S$, then $\langle\{{\mathfrak{p}_i}^{s_i} \mid 1 \leq i \leq k, s_i \neq 0\}\rangle = S$, and if $\mathfrak{p}_i \preceq \mathfrak{p}_j$ with $s_j \geq 1$, then by (ii), $s_i = r_i$. That is, the k-tuple $(s_1, \ldots, s_k)$ satisfies the condition

$$(*) \qquad 0 \leq s_i \leq r_i, \text{ and whenever } \mathfrak{p}_i \supset \mathfrak{p}_j \text{ with } s_j \geq 1, \text{ then } s_i = r_i.$$

Also, if $(s_1, \ldots, s_k)$ is a k-tuple of non-negative integers satisfying the condition $(*)$, then we have $\langle\{{\mathfrak{p}_i}^{s_i} \mid 1 \leq i \leq k, s_i \neq 0\}\rangle$, a node of $\Sigma_{M/N}$. So we have a one-to-one correspondence between the nodes of the partially ordered set $(\Sigma_{M/N}, \preceq)$ and the k-tuples $(s_1, \ldots, s_k)$ of integers satisfying the condition $(*)$. Hence, by Proposition 3.9, the regular divisors of N in M are in one-to-one correspondence with the nodes of the partially ordered set $(\Sigma_{M/N}, \preceq)$. Then (4) follows from Theorem 4.12. □

Using the above formula, the number of regular divisors of N in M in Example 4.14 is equal to 24.

Now we find the number of regular divisors of N which have a common factor in the generalized prime ideal factorization.

Definition 4.16. Let $\mathcal{P}_M(N) = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_k}^{r_k}$ and K be a regular divisor of N in M with $\mathcal{P}_K(N) = {\mathfrak{p}_1}^{s_1} \cdots {\mathfrak{p}_k}^{s_k}$. Then for $0 \leq t_i \leq s_i$, we say a prime ideal product ${\mathfrak{p}_1}^{t_1} \cdots {\mathfrak{p}_k}^{t_k}$ is a factor of the regular divisor K.

Corollary 4.17. *Let $\mathcal{P}_M(N) = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_k}^{r_k}$ and $t_1, \ldots, t_k$ be integers such that $0 \leq t_i \leq r_i$ for $i = 1, \ldots, k$. Then the number of regular divisors of N in M having ${\mathfrak{p}_1}^{t_1} \cdots {\mathfrak{p}_k}^{t_k}$ as a factor is equal to*

$$\sum_{1\leq i\leq s} 2^{|A'_i|} - \sum_{1\leq i<j\leq s} 2^{|A'_i \cap A'_j|} + \sum_{1\leq i<j<l\leq s} 2^{|A'_i \cap A'_j \cap A'_l|} - \cdots + (-1)^{s+1}\, 2^{|\bigcap_{i=1}^{s} A'_i|},$$

where $\{A'_i\}_{1\leq i\leq s}$ is the set of all maximal independent subsets of $\Sigma_{M/N} \setminus \big\langle\{{\mathfrak{p}_i}^{t_i} \mid 1 \leq i \leq k, t_i \neq 0\}\big\rangle$.

Proof. From Theorem 4.15, we have a one-to-one correspondence between the regular divisors of N in M and the nodes of $\Sigma_{M/N}$, which maps a regular divisor K of N in M with $\mathcal{P}_K(N) = {\mathfrak{p}_1}^{s_1} \cdots {\mathfrak{p}_k}^{s_k}$ to the node $\langle\{{\mathfrak{p}_i}^{s_i} \mid 1 \leq i \leq k, s_i \neq 0\}\rangle$ of $\Sigma_{M/N}$. So a prime ideal product ${\mathfrak{p}_1}^{t_1} \cdots {\mathfrak{p}_k}^{t_k}$ is a factor of the regular divisor K if and only if ${\mathfrak{p}_i}^{t_i} \in \langle\{{\mathfrak{p}_i}^{s_i} \mid 1 \leq i \leq k, s_i \neq 0\}\rangle$, for $1 \leq i \leq k$, $t_i \neq 0$. Therefore the number of regular divisors of N in M having ${\mathfrak{p}_1}^{t_1} \cdots {\mathfrak{p}_k}^{t_k}$ as a factor is exactly equal to the number of nodes of $\Sigma_{M/N}$ containing $\{{\mathfrak{p}_i}^{t_i} \mid 1 \leq i \leq k, t_i \neq 0\}$, which is equal to the number of nodes of the partially ordered set $\Sigma_{M/N} \setminus \big\langle\{{\mathfrak{p}_i}^{t_i} \mid 1 \leq i \leq k, t_i \neq 0\}\big\rangle$ by Proposition 4.13. □

Example 4.18. For N, M in Example 4.14, we compute the number of regular divisors of N in M having $\mathfrak{p}_1{}^2\mathfrak{p}_3$ as a factor. We have $\Sigma_{M/N} \setminus \langle\{\mathfrak{p}_1{}^2, \mathfrak{p}_3\}\rangle = \Sigma_{M/N} \setminus \{\mathfrak{p}_1, \mathfrak{p}_1{}^2, \mathfrak{p}_3\}$ $=\{\mathfrak{p}_2, \mathfrak{p}_4, \mathfrak{p}_4{}^2\}$, and the maximal independent subsets of this set are $\{\mathfrak{p}_2, \mathfrak{p}_4\}$ and $\{\mathfrak{p}_2, \mathfrak{p}_4{}^2\}$. Using the formula, the number of regular divisors of N in M having $\mathfrak{p}_1{}^2\mathfrak{p}_3$ as a factor is equal to 6.

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