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Research Paper

GENUS g GROUPS OF DIAGONAL TYPE

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Abstract. A transitive subgroup $G \leq S_n$ is called a genus g group if there exist non identity elements $x_1, ..., x_r \in G$ satisfying $G = \langle x_1, x_2, ..., x_r \rangle$, $\prod_{i=1}^{r} x_i = 1$ and $\sum_{i=1}^{r} ind\, x_i = 2(n + g - 1)$. The Hurwitz space $\mathcal{H}_{r,g}^{in}(G)$ is the space of genus g covers of the Riemann sphere $\mathbb{P}^1\mathbb{C}$ with r branch points and the monodromy group G. Isomorphisms of such covers are in one to one correspondence with genus g groups.

In this article, we show that G possesses genus one and two group if it is diagonal type and acts primitively on Ω. Furthermore, we study the connectedness of the Hurwitz space $\mathcal{H}_{r,g}^{in}(G)$ for genus 1 and 2.

1. Introduction

Let $F \colon X \to \mathbb{P}^1$ be a meromorphic function from a compact connected Riemann surface X of genus g into the Riemann sphere $\mathbb{P}^1$. For every meromorphic function there is a positive integer n such that all points have exactly n preimages. So every compact Riemann surface

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can be made into the branched covering of $\mathbb{P}^1$. The points p are called the branch points of F if $|F^{-1}(p)| < n$. It is well known that the set of branch points is finite and it will be denoted by $B = \{p_1, ..., p_r\}$. For $q \in \mathbb{P}^1 \setminus B$, the fundamental group $\pi_1(\mathbb{P}^1 \setminus B, q)$ is a free group which is generated by all homotopy classes of loops γ_i winding once around the point p_i. These loops of generators γ_i are subject to the single relation that $\gamma_1 \ldots \gamma_r = 1$ in $\pi_1(\mathbb{P}^1 \setminus B, q)$. The explicit and well known construction of Hurwitz shows that a Riemann surface X with n branching coverings of $\mathbb{P}^1$ is defined in the following way: consider the preimage $F^{-1}(q) = \{x_1, ..., x_n\}$, every loop in γ in $\mathbb{P}^1 \setminus B$ can be lifted to n paths $\widetilde{\gamma_1}, \cdots, \widetilde{\gamma_n}$ where $\widetilde{\gamma_i}$ is the unique path lift of γ and $\widetilde{\gamma_i}(0) = x_i$ for every i. The endpoints $\widetilde{\gamma_i}(1)$ also lie over q.

That is $\widetilde{\gamma_i}(1) = x_{\sigma(i)}$ in $F^{-1}(q)$ where σ is a permutation of the indices $\{1, ..., n\}$ and it depends only on γ. Thus it gives a group homomorphism $\varphi\colon \pi_1(\mathbb{P}^1 \setminus B, q) \to S_n$. The image of φ is called the monodromy group of F and denoted by $G = Mon(X, F)$. Since X is connected, then G is a transitive subgroup of S_n. Thus a group homomorphism is determined by choosing n permutations $x_i = \varphi(\gamma_i)$, $i = 1, ..., r$ and satisfying the relations

$$G = \langle x_1, x_2, ..., x_r \rangle \tag{1}$$

$$\prod_{i=1}^{r} x_i = 1,\ x_i \in G^{\#} = G \setminus \{1\},\ i = 1, ..., r \tag{2}$$

$$\sum_{i=1}^{r} ind\, x_i = 2(n + g - 1) \tag{3}$$

where $ind\, x = n - orb(x)$, $orb(x)$ is the number of orbits of the group generated by x on Ω where $|\Omega| = n$. Equation (3) is called the Riemann Hurwitz formula. A transitive subgroup $G \leq S_n$ is called a genus g group if there exist $x_1, ..., x_r \in G$ satisfying (1), (2) and (3) and then we call (G, Ω, S) a genus g system. If the action of G on Ω is primitive, we call G a primitive genus g group and (G, Ω, S) a primitive genus g system.

A genus g group corresponds to the existence of an n sheeted branched covering of the Riemann sphere $\mathbb{P}^1$ by a Riemann surface X of genus g with r-branch points and monodromy group G. Guralnick and Thompson have observed that the conjecture reduce to consideration of genus g group where G acts primitively on Ω [5]. So the structure of G reduce into one of the five cases by their maximal subgroups whose structure has been described by Aschbacher and O'Nan-Scott Theorem [1].

Theorem 1.1. [1] *Suppose that G is a finite group and M is a maximal subgroup of G such that*

$$\bigcap_{g \in G} M^g = 1$$

Let S be a minimal normal subgroup of G, let L be a minimal normal subgroup of S, and let $\Delta = \{L = L_1, L_2, ..., L_m\}$ be the set of the G-conjugates of L. Then L is simple, $S = \langle L_1, ..., L_m \rangle$, $G = MS$ and furthermore either

(A): *L is of prime order p;*

or L is a non abelian simple group and one of the following holds:

(B): *$F^*(G) = S \times R$, where $S \cong R$ and $M \cap S = 1$;*

(C1): *$F^*(G) = S$ and $M \cap S = 1$;*

(C2): *$F^*(G) = S$ and $M \cap S \neq 1 = M \cap L$;*

(C3): *$F^*(G) = S$ and $M \cap S = M_1 \times M_2 \times \cdots \times M_m$, where $M_i = M \cap L_i$, $1 \leq i \leq m$.*

As far as we know (see [12, 8, 7]), there are four types of classification of genus g system as follows:

(1) Up to signature
(2) Up to ramification type
(3) Up to the braid action and diagonal conjugation by $Aut(G)$
(4) Up to the braid action and diagonal conjugation by $Inn(G)$.

The weakest classification is up to signature and the strongest one is up to the braid action and diagonal conjugation by $Inn(G)$, because it includes all 1,2 and 3.

In [12, 13, 5, 3], they have classified these cases (A),(B),(C1),(C2),(C3) up to signatures for genus zero. In [8, 9], they have produced a complete list of affine primitive genus 0, 1 and 2 groups up to the braid action and diagonal conjugation by $Inn(G)$. In [13], Shih shows that G cannot be a group of genus zero if it satisfies Theorem 1.1 (B).

In this paper, we consider the case (B) of Theorem 1.1 for genus g where $g = 1, 2$. The permutation representation of G on the coset space $\Omega = G/M$ is primitive. We show that G possesses genus 1 or 2 group. It can be seen in the following results.

Theorem 1.2. *Up to isomorphism, there exist one primitive genus one group satisfies Theorem 1.1 (B) and this group is represent on Ω by right multiplication. The corresponding primitive genus one group is enumerated in Table 5.*

Theorem 1.3. *Up to isomorphism, there exist two primitive genus two groups satisfy Theorem 1.1 (B) and these groups are represent on Ω by right multiplication. The corresponding primitive genus two groups are enumerated in Table 6.*

This work gets done by both the proof in group theory and calculations of GAP (Groups, Algorithms, Programming) software. So far the library of GAP contains all primitive actions whose degree are less than or equal to 4096. A calculation shows, there is exactly 4 and 8 braid orbits of primitive genus 1 and 2 groups of diagonal type respectively. The degree and the number of the branch points are given in Tables 1 and 2.

Table 1. Primitive Genus One Groups: Number of Components

Degree	Number of Group up to Isomorphsim	Number of Ramification Types	Number of connected components, $r = 3$	Number of connected components, total
168	1	2	4	4

Table 2. Primitive Genus Two Groups: Number of Components

Degree	Number of Group up to Isomorphsim	Number of Ramification Types	Number of connected components, $r = 3$	Number of connected components, $r = 4$	Number of connected components, total
60	1	1	0	1	1
168	1	1	7	-	7
Totals	2	2	7	1	8

Our paper is organized as follows. In section 2, we show that the existence the genus one or two groups of diagonal type up to signatures. In section 3, we give our algorithm and explain an example as appliction of it. In section 4, for the groups whose possesses genus one or two, we show the connectedness of the Hurwitz space.

2. Classification up to signature

Let Ω be a finite set of size n. For $x \in G$, $S \subseteq G^{\#}$, define $U(x) = \frac{c(x)}{n}$, where $c(x)$ denotes the number of orbits on Ω. $N(x) = f(x)/n$, $M(x) = max\{N(g) : g \in \langle x \rangle^{\#}\}$, $f(x) = |Fix(x)|$ is the number of the set of fixed points of x on Ω and d is the order of x. Also, $U(S) = \sum_{x \in S} U(x)$ and $r = |S|$. The general form of Riemann Hurwitz formula is $\sum_{i=1}^{r} c(x_i) = (r-2)n + 2(1-g)$, that is $U(S) = \sum_{i=1}^{r} \frac{c(x_i)}{n} = (r-2) + \frac{2(1-g)}{n}$. The signature of the r-tuple $(x_1, ..., x_r)$ is the r-tuple $(d_1, ..., d_r)$ where $o(x_i) = d_i$. The following lemma can be found in [13].

Lemma 2.1. *Let x be a permutation of Ω and $d = o(x)$. Then*

(1) $c(x) = \frac{1}{d}\{\sum_{\frac{d}{s}} \varphi(\frac{d}{s}) f(x^d)\}$ *where φ is the Eular function.*
(2) $U(x) \leq \frac{1}{d}\{1 + M(x)(d-1)\}$.
(3) $c(x) \leq c(x^i), U(x) \leq U(x^i), f(x) \leq f(x^i)$.
(4) *For any $x \neq 1, M(x) \leq \frac{1}{10}$ and $U(x) \leq \frac{3}{5}$.*
(5) $U(x) \leq \frac{7}{20}, \frac{11}{30}$ *for $o(x) \geq 4, o(x) = 3$ respectively.*
(6) $U(x) \leq \frac{8}{15}$ *for $o(x) = 2$, unless $L = A_5$, $t = 1, x$ acts on L as an outer involution and in which case $U(x) \leq \frac{11}{20}$.*

The following result is an interesting tool to eliminate some signatures which cannot generate G.

Proposition 2.2. [6] *Assume that a group G acts transitively and faithfully on Ω and $|\Omega| = n$. Let $r \geq 2$, $G = \langle x_1, ..., x_r \rangle$, $\prod_{i=1}^{r} x_i = 1$ and $o(x_i) = d_i > 1$, $i = 1, ..., r$. Then one of the following holds:*

(1) $\sum_{i=1}^{r} \frac{d_i - 1}{d_i} \geq \frac{85}{42}$.

(2) $r = 4$, $d_i = 2$ *for* $i = \{1, 2, 3, 4\}$ *and* $G^{''} = 1$.

(3) $r = 3$ *and (up to permutation)*$(d_1, d_2, d_3) =$

- **(a):** $(3, 3, 3), (2, 3, 6)$ *or* $(2, 4, 4)$ *and* $G^{''} = 1$.
- **(b):** $(2, 2, d)$ *and G is dihedral.*
- **(c):** $(2, 3, 3)$ *and* $G \cong A_4$.
- **(d):** $(2, 3, 4)$ *and* $G \cong S_4$.
- **(e):** $(2, 3, 5)$ *and* $G \cong A_5$.

(4) $r = 2$ *and G is cyclic.*

For the remaining of this paper, we assume that G is a group of genus 1 or 2 and satisfies Theorem 1.1 (B). The next two results give the boundenss of the number of branch points which is 3 except for $L = A_5$ (in this case $r = 4$).

Lemma 2.3. *If G is a primitive permutation group of genus 1 or 2 of diagonal type, then* $r \leq 4$.

Proof. Recall that $r-2 < U(S)$. By Lemma 2.1(4), $r-2 < U(S) \leq r.max\{U(x) : x \in S\} \leq \frac{3}{5}r$. This implies that $r < 5$. Hence $r \leq 4$. □

Lemma 2.4. *If G is a primitive permutation group of genus 1 or 2 of diagonal type and* $L \neq A_5$, *then* $r = 3$.

Proof. By Proposition 2.2, we have $d_4 \geq 3$. By Lemma 2.1, we obtain $U(S) \leq 3.\frac{8}{15} + \frac{11}{20} < 2$, which is a contradiction. Thus $r = 3$. □

It can be very hard to determine whether a set of signatures can generate the entire group in group theory. We know that each signature corresponds to some tuples. So one can use computational tool (via double cosets) to determine a tuple length 3 generate the entire group or not. The program exists in [9].

Lemma 2.5. *The group A_5^2 possesses genus 2 system.*

Table 3. Indices of A_5^2

$d=\|x\|$	2	3	5	h
$f(x)$	4	3	5	0
$ind\ x$	28	38	44	$(h-1/h)n$

Table 4. Indices of $L_2(7)^2.2$

$d=\|x\|$	2	3	4	6	7	8	h
$f(x)$	6,8	3	4	3	4	7	0
$ind\ x$	81,80	110	122	137	143	138	$(h-1/h)n$

Proof. From Table 3, we obtain the following signatures (2,2,2,2),(3,3,3),(2,3,6),(3,3,5),(2,5,5),(2,3,10),(2,5,6),(2,5,15) and (2,2,2,3) for genus 1 and 2 system. The first three signatures cannot generate G, by Proposition 2.2. The signatures (2,3,10) and (2,5,15) cannot generate the group because $Aut(A_5)$ doesn't contain elements of order 10 and 15. We left with the following signatures (3,3,5),(2,5,5),(2,5,6) and (2,2,2,3). Finally, the direct computation shows that the signatures (3,3,5),(2,5,5),(2,5,6) cannot generate G that is they do not satisfy Equation (1), however (2,2,2,3) generates G for genus 2. This completes the proof. □

Lemma 2.6. *The group $L_2(7)^2.2$ possesses genus 1 system if $n = 168$.*

Proof. Recall that $\sum_{i=1}^{r} ind\, x_i = 2(n+g-1)$. If $g = 1$ and $n = 168$, then $r \leq 4$. From Table 4, we have the following signatures (2,2,2,2),(2,4,4),(2,3,6),(2,3,7) and (2,3,8). The first three signatures cannot generate G, by Proposition 2.2. The computation shows that the signature (2,3,8) generates G, but (2,3,7) cannot. This completes the proof. □

Lemma 2.7. *The group $L_2(7)^2.2^2$ possesses genus 2 system if $n = 168$.*

Proof. The proof is similar as Lemma 2.6. □

The remaining primitive permeation groups of diagonal type do not possesses genus one or two. Some of these groups are $L_2(13)^2, L_2(7)^2, L_2(17)^2, A_6^2, A_7^2, L_2(8)^2, L_2(19)^2, L_2(11)^2, A_5^3$, $L_2(16)^2, \ldots$.

3. Algorithm and Example

Tables 5 and 6 contain our results . To obtain these tables we need to do the following steps:

- We extract all primitive permutation group G by using the GAP function AllPrimitiveGroups(DegreeOperation,n).
- One can check which primitive group satisfy Theorem 1.1 (B) by using the GAP function
  ONanScottType.
- For the diagonal group G, compute the conjugacy class representatives and permutation indices on n points.
- For given n, g and G we use the GAP function RestrictedPartions to compute all possible ramification types satisfying the Riemann-Hurwitz formula.
- Compute the character table of G if possible and remove those types which have zero structure constant.
- We use the class names from the Atlas notion of finite groups.
- For the genearting tuples of length at least 4, we use MAPCLASS package to compute braid orbits see Example 3.1
- For the genearting tuples of length 3 determine braid orbits via double cosets [9].

The next example show that how to compute the ramification types and braid orbits for the group $Alt(5)^2$.

Example 3.1. `LoadPackage( "mapclass", false );`

```
gap> rts:=[];; N:=60;;
gap> a:=AllPrimitiveGroups(DegreeOperation,N);
[ Alt(5)^2, Alt(5)^2.2, Alt(5) wreath Sym(2), Alt(5) wreath Sym(2),
Alt(5)^2.2^2, PSL(2,59), PGL(2,59), A(60), S(60) ]
gap> g:=grps[1];;
gap> reps:= List( ConjugacyClasses( g ), Representative );;
gap> orders:= List( reps, Order );;
gap> Ind:= pi -> NrMovedPoints( pi ) - Sum( CycleStructurePerm( pi ), 0 );;
gap> ind:= List( reps, Ind );
[ 0, 30, 48, 48, 40, 30, 28, 54, 54, 50, 48, 54, 44, 48, 56, 48,
 54, 48, 44, 56, 40, 50, 56, 56, 38 ]
gap> cand:= RestrictedPartitions( 2*N-2, Set( ind{ [ 2 .. Length( ind ) ] } ) );;
gap> for l in cand do
UniteSet( rts, Set( Cartesian( List( l, x -> Positions( ind, x ))),SortedList)); od;
```

```
gap> Length(rts);
53
gap> cand:=rts[39];
 [ 7, 7, 7, 25 ]
gap> orbs:= GeneratingMCOrbits( g, 0, reps{ cand } : OutputStyle:= "silent" );;
gap> Length(orbs);
1
gap> tup:= orbs[1].TupleTable[1].tuple;;
```

The group g is primitive genus 2 group because it satisfies Equations (1),(2) and (3) respectively.

```
gap> g=Group(tup);
true
gap> Product(tup);
()
gap> Sum( List( tup, Ind ) );
122
```

4. Connectedness of $\mathcal{H}^{in}_{r,g}(G, C)$

The details of the relationship between the braid orbits on the Nielsen classes $\mathcal{N}(C)$ and the connected components of the hurwitz space $\mathcal{H}^{in}_{r,g}(G, C)$ can be found in section two in [11]. The multi set of non trivial conjugacy classes $C = \{C_1, ..., C_r\}$ in G is called the ramification type of the G-covers X. In general, to show that whether or not $\mathcal{H}_r(G, C)$ is connected is an open problem both computationally and theoretically for any finite group G. There are several well known results for some special groups in [8, 11]. For a given finite group and given type, there is a package which is called the MAPCLASS. It will be used to compute braid orbits. So we can show that the Hurwitz space $\mathcal{H}_r(G, C)$ is connected or not for given group which satisfy (B) of Theorem 1.1 for genus 1 and 2. To do this, one needs to find corresponding braid orbits which corresponds to the connected components $\mathcal{H}_r(G, C)$ of G-curves X such that $g(X/G) = 0$.

Table 5. Primitive Genus One Groups

degree	group	ramification type	Number of orbits	Length of orbits
168	$L_2(7)^2.2$	(2D,3C,8H)	2	1
		(2D,3C,8D)	2	1

Now, we present some results which shows the connectedness of the Hurwitz space for given groups.

Table 6. Primitive Genus Two Groups

Degree	group	ramification type	Number of orbits	Length of orbits
60	A_5^2	(2C,2C,2C,3C)	1	288
168	$L_2(7)^2.2^2$	(2B, 4D, 6E)	7	1

Proposition 4.1. *If G is a finite group satisfies Theorem 1.1 (B) and G is represent on Ω by right multiplication, $r \geq 4$ and $g = 2$ then $\mathcal{H}_{r,g}^{in}(G, C)$ is connected.*

Proof. Since we have just one braid orbit for all types C and the Nielsen classes $\mathcal{N}(C)$ are the disjoint union of braid orbits. From [14, Proposition 10.14], we obtain that the Hurwitz space $\mathcal{H}_{r,2}^{in}(G, C)$ is connected. □

Proposition 4.2. *If G is a finite group satisfies Theorem 1.1 (B) and G is represent on Ω by right multiplication, $r = 3$ and $g = 1, 2$ then $\mathcal{H}_{r,g}^{in}(G, C)$ is disconnected.*

Proof. Since we have at least two braid orbits for some type C and the Nielsen classes $\mathcal{N}(C)$ are the disjoint union of braid orbits. From [14, Proposition 10.14], we obtain that the Hurwitz space $\mathcal{H}_{r,g}^{in}(G, C)$ is disconnected. □

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