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Research Paper

THE MINIMUM EDGE DOMINATING ENERGY OF THE CAYLEY GRAPHS ON SOME SYMMETRIC GROUPS

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Abstract. The minimum edge dominating energy of a graph G is defined as the sum of the absolute values of eigenvalues of the minimum edge dominating matrix of G. In this paper, for some finite symmetric groups Γ and subset S of Γ, the minimum edge dominating energy of the Cayley graph of the group Γ, denoted by $Cay(\Gamma, S)$, is investigated.

1. Introduction

Throughout the paper, we consider $G = (V, E)$ as a simple graph with the vertex set $V = \{v_1, \ldots, v_n\}$ and the edge set $E = \{e_1, \ldots, e_m\}$. The number of edges connected to vertex $v_i \in V$ is called the degree of v_i and denoted by d_i. For two vertices u and v in graph G, if $d_u = 0$ and $d_v = 1$, then u and v are called the isolated vertex and pendant vertex in the graph, respectively. Graph G is r-regular if the degree of all vertices is r.

For graph G, the adjacency matrix $A(G) = (a_{ij})$ of the order n is defined as $a_{ij} = 1$ if $v_iv_j \in E$

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and $a_{ij} = 0$ otherwise. The number of non-zero eigenvalues of G is called the rank of G and is denoted by $r = rank(A(G))$. The eigenvalues of graph G are the eigenvalues of the adjacency matrix $A(G)$ [17]. Assume that λ_i is the eigenvalue of a graph G with multiplicity m_i for $1 \leq i \leq t$. The spectrum of the graph G is as follows

$$Spec(G) = \begin{pmatrix} \lambda_1 & \dots & \lambda_t \\ m_1 & \dots & m_t \end{pmatrix}$$

One of the effective topological indices in graph theory that has many applications in molecular structures is graph energy. Ivan Gutman 1978 introduced the energy of a graph G which is the sum of the absolute eigenvalues of G [16]. The energy of line graph G is called the edge energy, denoted by $EE(G)$ [6]. The line graph of G, denoted by $L(G)$ is the graph with the vertex set $V(L(G)) = E(G)$ and two vertices of $L(G)$ are adjacent if and only if their corresponding edges are incident in G [17].
The dominating set of graph G is a subset D of V if every vertex of $V \setminus D$ is adjacent to some vertices in D [17]. The minimum dominating set of G is any dominating set with minimum cardinality. In [28], the minimum dominating matrix $A_D(G)$ is defined. Also, the authors introduced the minimum dominating energy of graph G, denoted by $E_D(G)$, which is the sum of the absolute values of eigenvalues of matrix $A_D(G)$. $F \subseteq E$ is the edge dominating set of graph G if every edge $e \in E \setminus F$ is adjacent to some edges in F. The edge domination number, denoted by $\gamma'(G)$ is an edge dominating set of G with the minimum cardinality [14]. Obviously, F is the edge dominating set of graph G if and only if F is the dominating set for its line graph. In [1], the minimum edge dominating energy of graph G, denoted by $EE_F(G)$ is defined as the sum of the absolute values of eigenvalues of matrix $A_F(G)$ in which the minimum edge dominating matrix $A_F(G)$ is as follows

$$A_F(G) = (a_{ij}) = \begin{cases} 1 & \text{if } e_i \, and \, e_j \, are \, adjacent, \\ 1 & \text{if } i = j \, and \, e_i \in F, \\ 0 & otherwise. \end{cases}$$

For more study of the minimum edge dominating energy of graphs, the reader is referred to [1, 25, 24, 26]. In [9], the minimum edge dominating energy of some Cayley graphs for the finite group S_n are investigated. Chokani et al. [10] are obtained the graph energy, Laplacian energy, signless Laplacian energy, edge energy and the minimum edge dominating energy of $\Gamma(R)$ for the commutative rings R.
Arthur Cayley in 1878, first defined the Cayley graph on the finite groups[7]. Let Γ be a finite group and $S \subseteq \Gamma \setminus \{1\}$ such that $S = S^{-1}$. The Cayley graph $G(V, E) = Cay(\Gamma, S)$ is an undirected and simple graph with the vertex set $V(G) = \Gamma$ and the edge set $E(G) =$

$\{(x,y)\,|\,xy^{-1} \in S\}$. A Cayley graph $Cay(\Gamma, S)$ is a connected graph if and only if S is the generating subset of Γ [4]. The Cayley graphs are the relation between group theory and graph theory that have many applications in the different sciences [29].

In this paper, we investigate the minimum edge dominating energy of Cayley graphs of two symmetric groups namely, the dihedral groups D_{2n} and the symmetric group $\mathcal{Z}_n$.

In this paper, K_n and C_n are denoted for a complete graph and cycle of the order n, respectively. Two graphs G_1 and G_2 are called isomorphic, denoted by $G_1 \simeq G_2$ if there is a bijective correspondence between their vertices and edges.

2. Preliminaries

In this section, we recall some useful results which will be needed in the proofs of our main results.

Lemma 2.1. [1] *If C_n is a cycle of the order $n \geq 3$, then $EE_F(C_n) = E_D(C_n)$.*

Lemma 2.2. [25] *Let G be a graph of order n and size m whose vertices have degree d_i for $i = 1, \ldots, n$. If F is the minimum edge dominating set of G with cardinality k, then*

$$EE_F(G) \leq \sum_{i=1}^{n} d_i^2 - m.$$

Lemma 2.3. [25] *Let G be a graph of order n with m edges. If F is the minimum edge dominating set of G with cardinality k, then $EE_F(G) \leq 4m - 2n + k$.*

Lemma 2.4. [25] *Let G be a graph of order n with $m \geq n$ edges. If F is the minimum edge dominating set of G, then $EE_F(G) \geq 4(m - n + s) + 2p$, where p and s are the number of pendant and isolated vertices in G.*

Lemma 2.5. [25] *Let G be a regular graph of degree $r \geq 2$ with n vertices and $m = \frac{rn}{2}$ edges. If F is the minimum edge dominating set with cardinality k, then*

(i) *If $r = 2$, then $EE_F(G) \leq E(G) + k$,*
(ii) *If $r > 2$, then $EE_F(G) < E(G) + k + 2n(r-2)$.*

Lemma 2.6. [25] *Let G be a bipartite graph of order n with $m \geq 1$ edges and rank r. Then $EE_F(G) \geq 2\big(E(G) - r\big)$.*

Lemma 2.7. [8] *Let G be a graph of order n and γ' be the minimum edge dominion number of G. Then $\gamma' \leq \lfloor \frac{n}{2} \rfloor$.*

Lemma 2.8. [25] *Let G be a connected graph of order n. If v^+ is the number of the positive eigenvalues of the matrix $A(G)$, then $EE_F(G) \geq 2E(G) - 4v^+$.*

Lemma 2.9. [21] *Let G be a complete multipartite graph with minimum degree $\delta(G)$. Then $E(G) = 2\delta(G)$.*

Lemma 2.10. [13] *If M_1 and M_2 are $n \times n$ real matrices, then $E(M_1+M_2) = E(M_1)+E(M_2)$.*

Lemma 2.11. [15] *The graph energy of cycle C_n is given as follows*

$$E(C_n) = \begin{cases} 4cot\frac{\pi}{n} & if\ n \equiv 0(mod4), \\ 4csc\frac{\pi}{n} & if\ n \equiv 2(mod4), \\ 2cot\frac{\pi}{2n}cos\frac{\pi}{2n} & if\ n \equiv 1,3(mod4). \end{cases}$$

3. Main Results

We first consider the unitary Cayley graph X_n. Let $\mathcal{Z}_n = \{0, 1, \ldots, n-1\}$ be a additive cyclic group of integers modulo n and $U_n = \{a \in \mathcal{Z}_n : gcd(a, n) = 1\}$ be the multiplicative group of its units for $n > 1$. The unitary Cayley graph $X_n = Cay(\mathcal{Z}_n, U_n)$ is the Cayley graph of group $\mathcal{Z}_n$ on U_n with the vertex set $V(X_n) = \mathcal{Z}_n = \{0, 1, \ldots, n-1\}$ and $E(X_n) = \{(a, b) : a, b \in \mathcal{Z}_n, gcd(a-b, n) = 1\}$. The graph X_n is $|U_n|$-regular graph in which $|U_n| = \varphi(n)$ and φ is the Euler function. Note that graph X_n has n vertices and $\frac{n\varphi(n)}{2}$ edges [11].
We obtain the minimum edge dominating energy of the unitary Cayley graph X_n. We first state the following results of the unitary Cayley graph X_n.

Lemma 3.1. [19] *For the unitary Cayley graph X_n, if $n = p^{\alpha}$, where $\alpha > 1$, then X_n is the complete p-partite graph $K_{p^{\alpha-1},\ldots,p^{\alpha-1}}$.*

Lemma 3.2. [11] *Let X_n be the unitary Cayley graph. If n is an even number, then X_n is a bipartite graph.*

Lemma 3.3. [18] *Let $n = p_1^{\alpha_1} \ldots p_t^{\alpha_t}$ where $p_1, \ldots, p_t$ are distinct primes and $\alpha_1, \ldots, \alpha_t$ are positive integers. If X_n is the unitary Cayley graph, then $E(X_n) = 2^t\varphi(n)$.*

Theorem 3.4. *Let X_n be the unitary Cayley graph. If F is the minimum edge dominating set of X_n with $|F| = k$, then*

$$EE_F(X_n) < \varphi(n)\big(2^t + 2n\big) + (k - 4n),$$

where $\varphi(n)$ and t are the Euler function and the number of distinct prime factors dividing n, respectively.

Proof. Assume that X_n is the unitary Cayley graph of order n and size $m = \frac{n\varphi(n)}{2}$ with the degree $\varphi(n)$ for all vertices. According to the parameters of Lemma 2.5 in the graph X_n, $m = \frac{nr}{2}$. Thus, we investigate $EE_F(X_n)$ for $n \geq 2$.
If $r = 2$, then $\varphi(n) = 2$ and obviously $n = 2$. Therefore, X_n is the 2-regular graph with

2 vertices and 2 edges. This is a contradiction with the definition of simple and undirected graphs. So, X_n is the r-regular graph for $r > 2$.
Suppose that $n = p_1^{\alpha_1} \dots p_t^{\alpha_t}$ where p_i's and α_i's are distinct primes and positive integers, respectively. Using Lemmas 3.10(ii) and 3.3, we get

$$\begin{aligned} EE_F(X_n) &< E(G) + k + 2n(r-2) \\ &= 2^t\varphi(n) + k + 2n\big(\varphi(n) - 2\big) \\ &= \varphi(n)\big(2^t + 2n\big) - 4n + k. \end{aligned}$$

Therefore, the result holds. □

Corollary 3.5. *Let X_n be the unitary Cayley graph where $n = p^\alpha$, for $\alpha > 1$ and p is a prime. If F is the minimum edge dominating set of X_n, then*

$$EE_F(X_n) < 2n\big(\varphi(p^\alpha) - \frac{1}{p} - \frac{3}{4}\big),$$

where $\varphi(.)$ is the Euler function.

Proof. Let X_n be the $\varphi(n)$-regular graph of order p^α and size $\frac{p^\alpha\varphi(p^\alpha)}{2}$. Let F is the minimum edge dominating set of X_n with $|F| = k$. Using Lemma 2.7, we have $k \le \lfloor \frac{n}{2} \rfloor$. The similar to the proof of Theorem 3.4, we get

$$\begin{aligned} EE_F(X_n) &< 2\varphi(p^\alpha) + k + 2(p^\alpha)\big(\varphi(p^\alpha) - 2\big) \\ &\le 2\big(p^\alpha - p^{\alpha-1}\big) + 2(p^\alpha)\big(p^\alpha - p^{\alpha-1} - 2\big) + \lfloor \frac{n}{2} \rfloor \\ &\le 2p^\alpha - 2p^{\alpha-1} + 2p^{2\alpha} - 2p^{2\alpha-1} - 4p^\alpha + \frac{n}{2} \\ &= 2p^\alpha\big(p^\alpha - p^{\alpha-1} - \frac{1}{p} - 1\big) + \frac{n}{2} \\ &= 2n\big(\varphi(p^\alpha) - \frac{1}{p} - \frac{3}{4}\big). \end{aligned}$$

Therefore, the result completes. □

Theorem 3.6. *Let X_n be the unitary Cayley graph where $n = p^\alpha$, for $\alpha \ge 1$ and p a prime. If F is the minimum edge dominating set of X_n, then*

$$EE_F(X_n) \le n\Big(\frac{4\varphi(n) - 3}{2}\Big).$$

Proof. According to the definition of the unitary graph, X_n is $\varphi(p^\alpha)$-regular graph of order n and size $m = \frac{n\varphi(n)}{2}$. For any vertex v_i in graph X_n, $deg(v_i) = d_i = \varphi(n)$. Let F be the

minimum edge dominating set of X_n.
By applying Lemmas 2.3 and 2.7, we get

$$\begin{aligned} EE_F(X_n) &\leq 4m - 2n + |F| \\ &\leq 4\Big(\frac{n\varphi(n)}{2}\Big) - 2n + \lfloor\frac{n}{2}\rfloor \\ &\leq 2\big(n\varphi(n)\big) - \frac{3n}{2} \\ &= \frac{4n\varphi(n) - 3n}{2}. \end{aligned}$$

By simplifying the above inequality the result holds. □

Theorem 3.7. *Let X_n be the unitary Cayley graph where $n = 2^\alpha$, for $\alpha \geq 1$. If F is the minimum edge dominating set of X_n, then*

$$4(\beta - 1) \leq EE_F(X_n) \leq \beta^2\big(2\beta - 1\big),$$

where $\beta = 2^{\alpha-1} = \frac{n}{2}$.

Proof. Assume that X_n is the unitary Cayley graph of order $n = 2^\alpha$ and size $2^{2\alpha-2}$ with the degree $2^{\alpha-1}$ for all vertices. By applying Lemma 2.2, we get

$$\begin{aligned} EE_F(X_n) &\leq \sum_{i=1}^{n} d_i^2 - m \\ &= n\big(\varphi(n)\big)^2 - \frac{n\varphi(n)}{2} \\ &= n\varphi(n)\big(\varphi(n) - \frac{1}{2}\big) \\ &= 2^\alpha\varphi(2^\alpha)\big(\varphi(2^\alpha) - \frac{1}{2}\big) \\ &= 2^{2\alpha-1}\big(2^{\alpha-1} - \frac{1}{2}\big). \end{aligned} \tag{1}$$

With considering $\beta = 2^{\alpha-1}$ in (1), we have

$$\begin{aligned} EE_F(X_n) &\leq 2^{2\alpha-1}\big(2^{\alpha-1} - \frac{1}{2}\big) \\ &= 2\beta^2\big(\beta - \frac{1}{2}\big) \\ &= \beta^2\big(2\beta - 1\big). \end{aligned}$$

Thus, the result for the upper bound holds.
Since $n = 2^\alpha$, using Lemmas 3.1 and 3.2, graph X_n is the complete bipartite graph $K_{2^{\alpha-1},2^{\alpha-1}}$.

Therefore, $rank(X_n) = 2$. Using Lemmas 2.6 and 3.3, we get

$$\begin{aligned} EE_F(X_n) &\geq 2\big(E(X_n) - rank(X_n)\big) \\ &= 2\big(2(2^{\alpha-1}) - 2\big) \\ &= 2\big(2^{\alpha} - 2\big) \\ &= 4(2^{\alpha-1} - 1), \end{aligned}$$

By putting $\beta = 2^{\alpha-1}$, the lower bound is obtained. □

Now, we consider the Dihedral group D_{2n} which is the finite group with $2n$ elements of symmetries including rotations and reflections of regular polygon [2]. We first obtain the minimum edge dominating energy of the Cayley graph of group D_{2n} with respect to the generating subset $S = \{b, ab, \dots, a^{n-1}b\}$. We recall the following result of the Cayley graph $Cay(D_{2n}, S)$.

Lemma 3.8. [12] *If D_{2n} is the dihedral group of the order $2n$, for $n \geq 3$ and $S = \{b, ad, \dots, a^{n-1}b\} \subseteq D_{2n}$, then $Cay(D_{2n}, S) = K_{n,n}$ where $K_{n,n}$ is the complete bipartite graph.*

Theorem 3.9. *Let D_{2n} be the dihedral group of the order $2n$, where $n \geq 3$ and $S = \{b, ad, \dots, a^{n-1}b\} \subseteq D_{2n}$ be the generating subset of D_{2n}. If F is the minimum edge dominating set of the Cayley graph $Cay(D_{2n}, S)$ with cardinality k, then*

$$EE_F\big(Cay(D_{2n}, S)\big) \leq 4n(n-1) + k.$$

Proof. Let G be the Cayley graph $Cay(D_{2n}, S)$ of order $2n$ and size n^2 for $n \geq 3$ where $S = \{b, ad, \dots, a^{n-1}b\} \subseteq D_{2n}$. Using Lemma 3.8, G is the complete bipartite graph $K_{n,n}$. Assume that F is the minimum edge dominating set of G such that $|F| = k$. Using Lemma 2.3, we get

$$\begin{aligned} EE_F(G) &\leq 4m - 2n + k \\ &= 4(n^2) - 2(2n) + k \\ &= 4n^2 - 4n + k \\ &= 4n(n-1) + k. \end{aligned}$$

□

In the following result, we obtain the lower and upper bounds for the minimum edge dominating energy of the Cayley graph $Cay(D_{2n}, S)$ where $S = \{b, ad, \ldots, a^{n-1}b\} \subseteq D_{2n}$ in terms of the number of elements of group D_{2n}.

Theorem 3.10. *Let $Cay(D_{2n}, S)$ be the Cayley graph of the dihedral group D_{2n} on the subset $S = \{b, ad, \ldots, a^{n-1}b\} \subseteq D_{2n}$ where $n \geq 3$. If F is the minimum edge dominating set of the Cayley graph $Cay(D_{2n}, S)$, then*

$$4n(n-1) \leq EE_F\big(Cay(D_{2n}, S)\big) \leq n^2(2n-1).$$

Proof. By applying Lemma 3.8, the Cayley graph $Cay(D_{2n}, S)$ of group D_{2n} on the subset $S = \{b, ad, \ldots, a^{n-1}b\} \subseteq D_{2n}$ for $n \geq 3$ is the complete bipartite graph $K_{n,n}$. Thus, this graph is n-regular of order $2n$ and size n^2.

For the upper bound, we apply Lemma 2.2 and have

$$\begin{aligned} EE_F\big(Cay(D_{2n}, S)\big) &\leq \sum_{i=1}^{2n} (n)^2 - n^2 \\ &= 2n(n)^2 - n^2 \\ &= n^2(2n-1). \end{aligned} \tag{2}$$

On the other hand, using Lemma 2.4, we have

$$EE_F\big(Cay(D_{2n}, S)\big) \geq 4(n^2 - 2n + s) + 2p,$$

where p and s are the numbers of pendant and isolated vertices in $Cay(D_{2n}, S)$. By applying Lemma 3.8 and the structure of the Cayley graph $Cay(D_{2n}, S)$, $s = p = 0$. Therefore,

$$EE_F\big(Cay(D_{2n}, S)\big) \geq 4n(n-1). \tag{3}$$

From (3) and (4), the result completes. □

Now, we consider distance-regular Cayley graphs on dihedral groups as two classes trivial and non-trivial. The distance-regular graph is a connected graph such that the cardinality of the intersection of two spheres depends only on their radii and the distance between their centers. In the first, the minimum edge dominating energy of trivial distance-regular Cayley graphs is obtained in the results [3].

Theorem 3.11. *Let $Cay(D_n, S)$ is the Cayley graph of the dihedral group D_n of the even order n and the subset $S = D_n \setminus \{1\}$. If F is the minimum edge dominating set of graph $Cay(D_n, S)$, then the minimum edge dominating energy of $Cay(D_n, S)$ is as follows*

i) if $n = 2$, then $EE_F(Cay(D_2, S)) = 0$,

ii) if $n \geq 4$, then $2n - 3 \leq EE_F(Cay(D_n, S)) < \frac{1}{2}(4n^2 - 7n - 4)$.

Proof. Using Theorem 1.2 in [23] the trivial-regular Cayley graph $G = Cay(D_n, S)$ of diameter 1 is the complete graph. Thus its eigenvalues are $n-1$ with multiplicity 1 and -1 with multiplicity $n-1$. Therefore, $E(G) = 2(n-1)$.
For $n = 2$, it is easy to see that $Cay(D_2, S) \simeq K_2$. Clearly, $EE_F(Cay(D_2, S)) = 0$.
For $n \geq 4$, by applying Lemmas 2.5(ii) and 2.7, we get

$$\begin{aligned} EE_F(Cay(D_n, S)) &< E(Cay(D_n, S)) + |F| + 2n(r-2) \\ &\leq 2(n-1) + \lfloor \frac{n}{2} \rfloor + 2n(n-3) \\ &\leq (2n^2 - 4n - 2) + \frac{n}{2}. \end{aligned}$$

By simplifications of the above relation, the result completes for the upper bound.
For the lower bound, by setting $E(Cay(D_n, S)) = 2(n-1)$ and the number of the positive eigenvalues $v^+ = 1$ in Lemma 2.8, the result holds. □

Theorem 3.12. *Let $Cay(D_n, S)$ be the Cayley graph of the dihedral group D_n of the even order $n = tq$ and the subset $S = D_n \setminus L$ in which L is a subgroup of the order q. If F is the minimum edge dominating set of graph $Cay(D_n, S)$, then*

$$2n(\alpha - 2) \leq EE_F(Cay(D_n, S)) \leq 2n(\alpha - 1) - q,$$

where $\alpha = q(t-1)$.

Proof. According to Theorem 1.2 in [23], the trivial distance-regular Cayley graph $G = Cay(D_n, S)$ on the dihedral group D_n of the order $n = tq$ which contains $L \subseteq D_n$ of the order q and $D_n \setminus L$ is a complete multipartite graph $K_{t\times q} = K_{q,q,\ldots,q}$. Since graph G is a $(t-1)q$-regular graph, thus using Lemma 2.9, $E(G) = 2(t-1)q$. If F is the minimum edge dominating set of $Cay(D_n, S)$, obviously $|F| = q$. Therefore, using Lemma 2.5(ii) we have

$$\begin{aligned} EE_F(G) &< E(G) + |F| + 2n(r-2) \\ &= 2(t-1)q + q + 2n(n-q-2) \\ &= 2n - q + 2n^2 - 2nq - 4n \\ &= 2n(n-q-1) - q. \end{aligned}$$

With simplification and by substituting for $\alpha = q(t-1)$ in the above inequality, the upper bound completes.
The number of edges in graph G is equal to $\frac{n(n-q)}{2}$. Using Lemma 2.4, for the lower bound we

get

$$\begin{aligned} EE_F(G) &\geq 4(m-n+s)-2p \\ &= 4\Big(\frac{n(n-q)}{2}-n\Big) \\ &= 2(n^2-nq)-4n \\ &= 2n(n-q-2). \end{aligned}$$

With considering $n-q=q(t-1)=\alpha$, the result holds. □

Theorem 3.13. *Suppose that $Cay(D_n, S)$ is the Cayley graph of the dihedral group $D_n = \langle \rho, \tau | \rho^n, \tau^2, (\rho\tau)^2 \rangle$ of order $n = 2q$ and the subset $S = \{\rho^i\tau | 1 \leq i \leq m-1\}$. If F is the minimum edge dominating set of graph $Cay(D_n, S)$, then*

$$2n(\alpha-1) \leq EE_F(Cay(D_n, S)) \leq 2(n\alpha+1),$$

where $\alpha = q-3$.

Proof. Using Theorem 1.2 in [23], the trivial distance-regular Cayley graph $G = Cay(D_n, S)$ for $S = \{\rho^i\tau | 1 \leq i \leq m-1\}$ is the graph $K_{q,q} - qK_2$ such that $n = 2q$ and G is the $(q-1)$-regular graph. By applying Lemma 2.10 and since the spectrum of graph $K_{q,q}$ is $\pm q$ and 0 with multiplicity $2q-2$, we have

$$0 \leq E(G) \leq E(K_{q,q}) - qE(K_2) = 2q - 2q = 0$$

Therefore, we have $E(G) = 0$.
If F is the minimum edge dominating energy of G, then $|F| = 2$. Therefore, using Lemma 2.5(ii) we get

$$\begin{aligned} EE_F(G) &< E(G) + |F| + 2n(r-2) \\ &= 0 + 2 + 2n(q-3) \\ &= 2(nq-3n+1) \\ &= 2(2q^2-6q+1) \\ &= 2(2q(q-3)+1). \end{aligned}$$

With considering $n = 2q$ and $\alpha = q - 3$, the result holds.
On the other hand, using Lemma 2.4 for $q \geq 4$, we have

$$\begin{aligned} EE_F(G) &\geq 4(m - n + s) - 2p \\ &= 4(q^2 - n - n) \\ &= 4(q^2 - 4q) \\ &= 4q(q - 4). \end{aligned}$$

By substituting $n = 2q$, the result holds. □

Theorem 3.14. *Suppose that $Cay(D_n, S)$ is the Cayley graph of the dihedral group $D_n = \langle \rho, \tau | \rho^n, \tau^2, (\rho\tau)^2 \rangle$ of the order n and the subset $S = \{\tau, \rho\tau\}$. Let F be the minimum edge dominating set of graph $Cay(D_n, S)$ and $N = 2n$.*
i) If n is even, then

$$8cot(\frac{\pi}{N}) - 4N \leq EE_F(G) \leq 4cot(\frac{\pi}{N}) + \frac{N}{2}.$$

ii) If n is odd, then

$$8ccs(\frac{\pi}{N}) - 4N \leq EE_F(G) \leq 4csc(\frac{\pi}{N}) + \frac{N}{2}.$$

Proof. Let $G = Cay(D_n, S)$ be the Cayley graph on the dihedral group and the subset $S = \{\tau, \rho\tau\}$. Assume that F is the minimum edge dominating set of graph G. By applying Theorem 1.2 in [23], the Cayley graph on the dihedral group D_n with the subset $S = \{\tau, \rho\tau\}$, is the cycle C_N if $N = 2n$. According to Lemma 2.11, we consider two following cases.
Case 1: If $N = 4m$, thus n is even. Since the domination number of C_N is $\lceil \frac{N}{3} \rceil$ and G is 2-regular graph then using Lemmas 2.5(i) and 2.11 we get

$$\begin{aligned} EE_F(G) = EE_F(C_N) &\leq E(G) + |F| \\ &= 4cot(\frac{\pi}{N}) + \lceil \frac{N}{3} \rceil \\ &\leq 4cot(\frac{\pi}{N}) + \frac{N}{2} \end{aligned}$$

Therefore, the upper bound holds.
For the lower bound, we use Lemma 2.8 and write

$$\begin{aligned} EE_F(G) = EE_F(C_N) &\geq 2E(C_N) - 4v^+ \\ &\geq 8cot(\frac{\pi}{N}) - 4N. \end{aligned}$$

Case 2: Since N is even, thus N must be $4m + 2$. Therefore, n is odd. The proof of this case is similar to Case 1. □

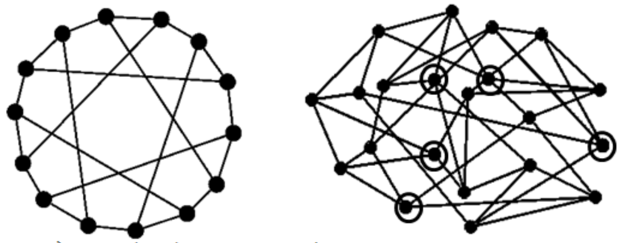

FIGURE 1. The Heawood graph and its line graph

Now, we investigate the smallest non-trivial distance-regular Cayley graphs on dihedral groups. The Heawood graph is a distance-regular Cayley graph on the dihedral group of the order 14 with 14 vertices and 21 edges (see Figure 1(a)). Note that the Heawood graph is the Cayley graph $Cay(D_{14}, S)$ of the dihedral group $D_{14} =< x, y \,|\, x^2 = y^7 = 1, y^x = y^{-1} >$ and $S = \{x, xy, xy^3\}$ is the subgroup of D_{14} [20].

Theorem 3.15. *Let $Cay(D_{14}, S)$ be the Cayley graph on the dihedral group D_{14} of order 14 and $S = \{x, xy, xy^3\}$. Then the minimum edge dominating energy of $Cay(D_{14}, S)$ is almost equal to* 36.3391.

Proof. Let F be the minimum edge dominating set of graph G. Since $G = Cay(D_{14}, S)$ is the Headwood graph, it is easy to obtain the line graph of G as Figure 1(b) that contains 21 vertices. Note that the minimum edge dominating set in G is the minimum dominating set in $L(G)$. Therefore, one can select the vertices marked by circles in figure 1(b) to dominate all vertices in $L(G)$. Thus, $\gamma(L(G)) = 5$.
By computing the minimum edge dominating matrix $A_F(G)$, we obtain the eigenvalues of $A_F(G)$ that are as follows

$$\{4.30716,\ 3.03253,\ 2.8662,\ 2.80177,\ 2.53979,\ 2.43958,\ 2.41421,$$
$$-2,\ -2,\ -2,\ -1.9215,\ -1.86805,\ -1.65544,\ -1.61655,\ -1.41044,$$
$$-0.414214,\ 0.347237,\ -0.22534,\ -0.210756,\ 0.199117,\ 0.069174\}.$$

Therefore,

$$EE_F(Cay(D_{14}, S)) = \sum_{i=1}^{21} |\lambda_i(A(G)| \simeq 36.3391.$$

□

Another kind of the smallest non-trivial distance-regular Cayley graph is the Shrikhande graph which is isomorphic to the Cayley graph of the generalized dihedral group $GD(< a > \times < b >)$ where $|a| = 1$ and $|b| = 4$ related to the subset $S = \{t, ta, tb, tab^3, b^{\pm 1}\}$ [22]. This graph is a 6-regular graph that contains 16 vertices and 48 edges. We obtain the upper and lower bound for the minimum edge dominating energy of the Shrikhande graph in the following theorem.

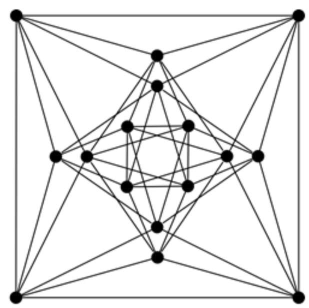

FIGURE 2. The Shrikhande graph

Theorem 3.16. *Let $Cay(GD, S)$ be the Cayley graph on the generalized dihedral group and $S = \{t, ta, tb, tab^3, b^{\pm 1}\}$. If F is the minimum edge dominating set in $Cay(GD, S)$, then* $128 \leq EE_F(Cay(GD, S)) \leq 178$.

Proof. Assume that $G = Cay(GD, S)$ is the Shrikhande graph. According to the structure of the Shrikhande graph in Figure 2, this graph is 6-regular with $n = 16$ vertices and $m = 48$ edges. An upper bound for the minimum domination number of G is obtained in [27] as follows

$$\gamma(G) \leq \frac{n\big(1 + ln(\delta + 1)\big)}{\delta + 1}. \tag{4}$$

where δ is the minimum degree of G.
Using the inequality (4), we have $\gamma(L(G)) \leq \frac{48(1+ln(11))}{11} \simeq 14.83$.
Since the spectrum of Shrikhande graph G is $Spec(G) = \begin{pmatrix} 6 & 2 & -2 \\ 1 & 6 & 9 \end{pmatrix}$, thus the number of the positive eigenvalues of G is $v^+ = 7$.
Also, the graph energy of G obtains as $E(G) = \sum_{i=1}^{16} |\lambda_i(G)| = 36$. Therefore, using Lemma 2.5(ii), we get

$$\begin{aligned} EE_F(G) &< E(G) + |F| + 2n(r - 2) \\ &\leq 36 + 14 + 2 \times 16(6 - 2) \\ &= 178. \end{aligned}$$

For the lower bound, we use Lemma 2.4 and have

$$\begin{aligned} EE_F(G) &\geq 4(m-n+s)+2p \\ &= 4(48-16) \\ &= 128. \end{aligned}$$

Therefore, the result completes. □

Finally, we consider the generalized Petersen graph $P(n,1)$, also called a Prism graph, with $2n$ vertices and $3n$ edges. The graph $P(n,1)$ is a trivalent Cayley graph $Cay(D_{2n},S)$ with the generating subset $S=\{x,x^{-1},y\}$ [5]. Thus, the line graph of $Cay(D_{2n},S)$ is the 4-regular graph with $3n$ vertices and $6n$ edges.

Theorem 3.17. *Let $Cay(D_{2n},S)$ be the trivalent Cayley graph on the dihedral group D_{2n} and the generating subset $S=\{x,x^{-1},y\}$. If F is the minimum edge dominating set in $Cay(D_{2n},S)$, then $12n \leq EE_F(Cay(D_{2n},S)) \leq 19n$.*

Proof. Assume that $G=Cay(D_{2n},S)$ is the Cayley graph on the dihedral group D_{2n} and the subset $S=\{x,x^{-1},y\}$. Let F be the minimum edge dominating set of G. Similar to the proof of Theorem 3.16, we get the domination number of line graph of G as follows

$$\gamma(L(G)) \leq \frac{3n(1+ln(5))}{5} \simeq 1.565n$$

Therefore using Lemma 2.3, we get

$$\begin{aligned} EE_F(G) &\leq 4m-2n+|F| \\ &\leq 4m-2n+n \\ &= 4(6n)-2(3n)-n = 19n. \end{aligned}$$

For the lower bound, by applying Lemma 2.4 we get

$$\begin{aligned} EE_F(G) &\geq 4(m-n+s)+2p \\ &= 4(6n-3n) \\ &= 12n. \end{aligned}$$

□

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