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Research Paper

ON CLOSEDNESS OF RIGHT(LEFT) NORMAL BANDS AND LEFT(RIGHT) QUASINORMAL BANDS

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ABSTRACT. It is well known that all subvarieties of the variety of all semigroups are not absolutely closed. So, it is worth to find subvarieties of the variety of all semigroups that are closed in itself or closed in the containing varieties of semigroups. We have gone through this open problem and able to determine that the varieties of right [left] normal bands and left [right] quasinormal bands are closed in the varieties of semigroups defined by the identities $axy = xa^n y$ $[axy = ay^n x]$, $axy = x^n ay$ $[axy = ayx^n]$ $(n > 1)$; and $axy = ax^n ay$ $[axy = ayx^n y]$ $(n > 1)$, $axy = a^n xa^r y$ $[axy = ay^r xy^n]$ $(n, r \in \mathbb{N})$, respectively.

1. Introduction and Preliminaries

It is well known that all subvarieties of the variety of all semigroups are not absolutely closed. So, it is worth to find subvarieties of the variety of all semigroups that are closed in itself or closed in the containing varieties of semigroups. As a first step in this direction,

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one attempts to find those varieties of semigroups that are closed in itself. Encouraged by the fact that Scheiblich [12] had shown that the variety of all normal bands was closed, Alam and Khan in [3, 4, 5] had shown that the varieties of all left [right] regular bands, left [right] quasi-normal bands and left [right] semi-normal bands were closed. In [2], Ahanger and Shah had proved a stronger fact that the variety of all left [right] regular bands was closed in the variety of all bands and, recently, Abbas and Ashraf [1] had shown that the variety of all left [right] normal bands was closed in some containing homotypical varieties (varieties admitting an identity containing same variables on both sides) of semigroups.

Let U be a subsemigroup of a semigroup S. Following Isbell [9], we say that U dominates an element d of S if for every semigroup P and for all homomorphisms $\gamma, \delta : S \longrightarrow P$ and $u\gamma = u\delta$ for every $u \in U$ implies $d\gamma = d\delta$. The set of all elements of S dominated by U is called *dominion* of U in S and we denote it by $Dom(U, S)$. It can be easily verified that $Dom(U, S)$ is a subsemigroup of S containing U. A subsemigroup U of a semigroup S is called closed in S if $Dom(U, S) = U$ whereas a semigroup is called absolutely closed if it is closed in every containing semigroup. Let $\mathcal{D}$ be a class of semigroups. A semigroup U is said to be $\mathcal{D}$-closed if $Dom(U, S) = U$ for all $S \in \mathcal{D}$ such that $U \subseteq S$. Let $\mathcal{A}$ and $\mathcal{D}$ be classes of semigroups such that $\mathcal{A}$ is a subclass of $\mathcal{D}$. We say that $\mathcal{A}$ is $\mathcal{D}$-closed if every member of $\mathcal{A}$ is $\mathcal{D}$-closed. A class $\mathcal{D}$ of semigroups is said to be closed if $Dom(U, S) = U$ for all $U, S \in \mathcal{D}$ with U as a subsemigroup of S. Let $\mathcal{B}$ and $\mathcal{C}$ be two categories of semigroups with $\mathcal{B}$ as a subcategory of $\mathcal{C}$. Then it can be easily verified that a semigroup U is $\mathcal{B}$-closed if it is $\mathcal{C}$-closed.

The following definitions and results are necessary for our investigations.

Result 1.1. *([9], Theorem 2.3) Let U be a subsemigroup of a semigroup S and let $d \in S$. Then $d \in Dom(U, S)$ if and only if $d \in U$ or there exists a series of factorizations of d as follows:*

$$d = a_0t_1 = y_1a_1t_1 = y_1a_2t_2 = y_2a_3t_2 = \cdots = y_ma_{2m-1}t_m = y_ma_{2m} \tag{1}$$

where $m \geq 1$, $a_i \in U$ $(i = 0, 1, \ldots, 2m)$, $y_i, t_i \in S$ $(i = 1, 2, \ldots, m)$, and

$$a_0 = y_1a_1, \qquad a_{2m-1}t_m = a_{2m},$$

$$a_{2i-1}t_i = a_{2i}t_{i+1}, \qquad y_ia_{2i} = y_{i+1}a_{2i+1} \qquad (1 \leq i \leq m-1).$$

Such a series of factorizations is called a zigzag in S over U with value d, length m and spine $a_0, a_1, \ldots, a_{2m}$.

Result 1.2. *([10, Result 3) Let U and S be semigroups with U as a subsemigroup of S. Take any $d \in S \backslash U$ such that $d \in Dom(U, S)$. If (1) is a zigzag of shortest possible length m over U with value d, then $t_j, y_j \in S \backslash U$ for all $j = 1, 2, \ldots, m$.*

Definition 1.3. A band S is said to be a left [right] normal band if S satisfies the identity $axy = ayx$ [$axy = xay$] for all $a, x, y \in S$.

Definition 1.4. A band S is said to be a left [right] quasinormal band if S satisfies the identity $axy = axay$ [$axy = ayxy$] for all $a, x, y \in S$.

A (semigroup)amalgam $\mathcal{U} = [\{S_i : i \in I\}; U \,; \{\varphi_i : i \in I\}]$ consists of a semigroup U (called the core of the amalgam), a family $\{S_i : i \in I\}$ of semigroups disjoint from each other and from U, and a family $\varphi_i : U \to S_i (i \in I)$ of monomorphisms. We shall simplify the notation to $\mathcal{U} = [S_i; U \,; \varphi_i]$ or to $\mathcal{U} = [S_i; U\,]$ when the context allows. We shall say that the amalgam $\mathcal{U}$ is embedded in a semigroup T if there exist a monomorphism $\lambda : U \to T$ and, for each $i \in I$, a monomorphism $\lambda_i : S_i \to T$ such that
(a) $\varphi_i \lambda_i = \lambda$ for each $i \in I$;
(b) $S_i\lambda_i \cap S_j\lambda_j = U\lambda$ for all $i, j \in I$ such that $i \neq j$.
A semigroup amalgam $\mathcal{U} = [\{S, S'\}; U; \{i\alpha \mid U\}]$ consisting of a semigroup S, a subsemigroup U of S, an isomorphic copy S' of S, where α: $S \to S'$ be an isomorphism and i is the inclusion mapping of U into S, is called a special semigroup amalgam. A class $\mathcal{C}$ of semigroups is said to have the special amalgamation property if every special semigroup amalgam in $\mathcal{C}$ is embeddable in $\mathcal{C}$.

Result 1.5. *([8, Theorem VII.2.3]). Let U be a subsemigroup of a semigroup S. Let S' be a semigroup disjoint from S and let α: $S \to S'$ be an isomorphism. Let $P = S *_U S'$ be the free product of the amalgam*

$$\mathcal{U} = [\{S, S'\};\ U\ ; \{i, \alpha \mid U\}],$$

where i is the inclusion mapping of U into S, and let μ, μ' be the natural monomorphisms from S, S' respectively into P . Then

$$(S\mu\ \cap\ S'\mu')\mu^{-1} = Dom(U, S).$$

From the above result, it follows that a special semigroup amalgam $[\{S,\, S'\};\, U\ ;\, \{i,\, \alpha \mid U\}]$ is embeddable in a semigroup if and only if $Dom(U, S) = U$. Therefore, the above amalgam with core U is embeddable in a semigroup if and only if U is closed in S.

The reader is referred to Petrich [11] for a complete description of all varieties of bands. The semigroup theoretic notations and conventions of Clifford and Preston [6] and Howie [8] will be used throughout without explicit mention.

2. Closedness of Right [Left] Normal Bands and Left [Right] Quasinormal Bands

In general, varieties of bands containing the variety of normal bands are not absolutely closed as Higgins [7, Chapter 4] had shown that variety of right [left] normal bands is not absolutely closed. Therefore, for the varieties of semigroups, it is worthwhile to find largest subvarieties of the variety of all semigroups in which the varieties of right [left] normal bands and left [right] quasinormal bands are closed.

To this end, we first note that Petrich [11, Theorem II.5.1] has classified an identity on bands in atmost three variables. Therefore, on the class of bands, varieties of semigroups defined by the identities $axy = xa^ny$ $[axy = ay^nx]$ and $axy = x^nay$ $[axy = ayx^n]$ are equivalent to right [left] normal bands; and varieties of semigroups defined by the identities $axy = ax^nay$ $[axy = ayx^ny]$ and $axy = a^nxa^ry$ $[axy = ay^rxy^n]$ are equivalent to left [right] quasinormal bands. In this section, we have shown that varieties of semigroups defined by the identities $axy = xa^ny$ $[axy = ay^nx]$, $axy = x^nay$ $[axy = ayx^n]$, $axy = ax^nay$ $[axy = ayx^ny]$ $(n > 1)$ and $axy = a^nxa^ry$ $[axy = ay^rxy^n]$ $(n, r \in \mathbb{N})$ are closed in itself and, as an application and consequence of these results, we conclude that the varieties of semigroups defined by the identities, $axy = xa^ny$ $[axy = ay^nx]$, $axy = x^nay$ $[axy = ayx^n]$ $(n > 1)$, $axy = ax^nay$ $[axy = ayx^ny]$ $(n > 1)$ and $axy = a^nxa^ry$ $[axy = ay^rxy^n]$ $(n, r \in \mathbb{N})$, have special amalgamation property, and right [left] normal bands and left [right] quasinormal bands are closed in the varieties of semigroups defined by the identities $axy = xa^ny$ $[axy = ay^nx]$, $axy = x^nay$ $[axy = ayx^n]$ $(n > 1)$ and $axy = ax^nay$ $[axy = ayx^ny]$ $(n > 1)$, $axy = a^nxa^ry$ $[axy = ay^rxy^n]$ $(n, r \in \mathbb{N})$, respectively. This is indeed a very positive and encouraging step towards a comprehensive solution of the above problem. However to find out largest varieties of semigroups in which the varieties of semigroups defined by the identities $axy = xa^ny$ $[axy = ay^nx]$, $axy = x^nay$ $[axy = ayx^n]$, $axy = ax^nay$ $[axy = ayx^ny]$ $(n > 1)$ and $axy = a^nxa^ry$ $[axy = ay^rxy^n]$ $(n, r \in \mathbb{N})$ are closed still remains open.

Lemma 2.1. *Let U be a subsemigroup of semigroup S such that S satisfies an identity $axy = xa^ny$ $[axy = x^nay]$ $(n > 1)$ and let $d \in Dom(U, S)\backslash U$ has a zigzag of type (1) in S over U with value d of shortest possible length m. Then*

$$d = (\prod_{i=1}^{k} a_{(2i-1)}^{n-1})y_k a_{2k-1} t_k$$

for each $k = 1, 2, \ldots, m$.

Proof. Let $\mathcal{V}_1 = [axy = xa^ny]$ and $\mathcal{V}_2 = [axy = x^nay]$ $(n > 1)$ be the varieties of semigroups. First we show that, in both cases whether $S \in \mathcal{V}_1$ or $S \in \mathcal{V}_2$, S satisfies $xyz = y^{n-1}xyz$.

Case (i): When $S \in \mathcal{V}_1$, for any $x, y, z \in S$, we have

$$\begin{aligned}
xyz &= yx^nz \text{ (as } S \in \mathcal{V}_1) \\
&= x^ny^nz \text{ (as } S \in \mathcal{V}_1) \\
&= y^n(x^n)^nz \text{ (as } S \in \mathcal{V}_1) \\
&= y^{n-1}(y(x^n)^nz) \\
&= (y^{n-1}x^ny)z \text{ (as } S \in \mathcal{V}_1) \\
&= (x(y^{n-1}y)z) \text{ (as } S \in \mathcal{V}_1) \\
&= y^{n-1}(yx^nz) \text{ (as } S \in \mathcal{V}_1) \\
&= y^{n-1}xyz, \qquad (2)
\end{aligned}$$

as required.

Case (ii): When $S \in \mathcal{V}_2$, then for any $x, y, z \in S$, we have

$$\begin{aligned}
xyz &= y^nxz \text{ (as } S \in \mathcal{V}_2) \\
&= x^ny^nz \text{ (as } S \in \mathcal{V}_2) \\
&= x^n(y^{n-1}yz) \\
&= (x^n(y^ny^{n-1})z) \text{ (as } S \in \mathcal{V}_2) \\
&= (y^n(y^{n-1}x)z) \text{ (as } S \in \mathcal{V}_2) \\
&= y^{n-1}xyz \text{ (as } S \in \mathcal{V}_2).
\end{aligned}$$

Thus the claim is proved.

Now, we prove the lemma by using induction on k. Let U be a subsemigroup of semigroup S such that S belongs to either $\mathcal{V}_1$ or $\mathcal{V}_2$ and let $d \in Dom(U, S) \backslash U$ has a zigzag of type (1) in S over U with value d of shortest possible length m.

Now, for $k = 1$, we have

$$\begin{aligned}
d &= y_1a_1t_1 \text{ (by zigzag equations)} \\
&= a_1^{n-1}y_1a_1t_1 \text{ (by equation (2)).}
\end{aligned}$$

Thus the result holds for $k = 1$. Assume inductively that the result holds for $k = j < m$. Then we shall show that it also holds for $k = j + 1$. Now

$$\begin{aligned} d &= (\prod_{i=1}^{j} a_{(2i-1)}^{n-1}) y_j a_{2j-1} t_j \text{ (by inductive hypothesis)} \\ &= (\prod_{i=1}^{j} a_{(2i-1)}^{n-1}) y_{j+1} a_{2j+1} t_{j+1} \text{ (by zigzag equations)} \\ &= (\prod_{i=1}^{j} a_{(2i-1)}^{n-1}) a_{2j+1}^{n-1} y_{j+1} a_{2j+1} t_{j+1} \text{ (by equation (2))} \\ &= (\prod_{i=1}^{j+1} a_{(2i-1)}^{n-1}) y_{j+1} a_{2j+1} t_{j+1}, \end{aligned}$$

as required and, by induction, the lemma is established. □

Theorem 2.2. *The variety $\mathcal{V} = [axy = xa^n y]$ $(n > 1)$ of semigroups; i.e., the class of all semigroups satisfying the identity $axy = xa^n y$ is closed.*

Proof. Take any U, $S \in \mathcal{V}$ with U as a subsemigroup of S such that $d \in Dom(U, S)\backslash U$. Let d has a zigzag of type (1) in S over U of shortest possible length m. Now

$$\begin{aligned} d &= (\prod_{i=1}^{m} a_{(2i-1)}^{n-1}) y_m a_{2m-1} t_m \text{ (by Lemma 2.1)} \\ &= (\prod_{i=1}^{m-1} a_{(2i-1)}^{n-1}) a_{2m-1}^{n-2} (a_{2m-1}(y_m a_{2m-1}) t_m) \text{ (for } n = 2, \text{ we treat } a_{2m-1}^{n-2} a_{2m-1} \\ &\quad \text{as } a_{2m-1}) \\ &= (\prod_{i=1}^{m-1} a_{(2i-1)}^{n-1}) a_{2m-1}^{n-2} y_m (a_{2m-1} a_{2m-1}^{n} t_m) \text{ (as } S \in \mathcal{V}) \\ &= (\prod_{i=1}^{m-1} a_{(2i-1)}^{n-1}) a_{2m-1}^{n-2} (y_m a_{2m-1})(a_{2m-1} t_m) \text{ (as } S \in \mathcal{V}) \\ &= (\prod_{i=1}^{m-1} a_{(2i-1)}^{n-1}) a_{2m-1}^{n-2} y_{m-1} a_{2m-2} a_{2m} \text{ (by zigzag equations)} \\ &= (\prod_{i=1}^{m-2} a_{(2i-1)}^{n-1}) a_{2m-3}^{n-2} (a_{2m-3}(a_{2m-1}^{n-2} y_{m-1} a_{2m-2}) a_{2m}) \\ &= (\prod_{i=1}^{m-2} a_{(2i-1)}^{n-1}) a_{2m-3}^{n-2} a_{2m-1}^{n-2} y_{m-1} (a_{2m-2} a_{2m-3}^{n} a_{2m}) \text{ (as } S \in \mathcal{V}) \end{aligned}$$

$$= (\prod_{i=1}^{m-2} a_{(2i-1)}^{n-1})a_{2m-3}^{n-2}a_{2m-1}^{n-2}(y_{m-1}a_{2m-3})a_{2m-2}a_{2m} \text{ (as } U \in \mathcal{V})$$
$$= (\prod_{i=1}^{m-2} a_{(2i-1)}^{n-1})a_{2m-3}^{n-2}a_{2m-1}^{n-2}y_{m-2}a_{2m-4}a_{2m-2}a_{2m} \text{ (by zigzag equations)}$$
$$\vdots$$
$$= a_1^{n-1}a_3^{n-2}\cdots a_{2m-3}^{n-2}a_{2m-1}^{n-2}y_1a_2a_4\cdots a_{2m-2}a_{2m}$$
$$= a_1^{n-2}(a_1(a_3^{n-2}\cdots a_{2m-3}^{n-2}a_{2m-1}^{n-2}y_1a_2)(a_4\cdots a_{2m-2}a_{2m}))$$
$$= a_1^{n-2}a_3^{n-2}\cdots a_{2m-3}^{n-2}a_{2m-1}^{n-2}y_1(a_2a_1^na_4)a_6\cdots a_{2m-2}a_{2m} \text{ (as } S \in \mathcal{V})$$
$$= a_1^{n-2}a_3^{n-2}\cdots a_{2m-3}^{n-2}a_{2m-1}^{n-2}(y_1a_1)a_2a_4a_6\cdots a_{2m-2}a_{2m} \text{ (as } U \in \mathcal{V})$$
$$= a_1^{n-2}a_3^{n-2}\cdots a_{2m-3}^{n-2}a_{2m-1}^{n-2}a_0a_2a_4\cdots a_{2m-2}a_{2m} \text{ (by zigzag equations)}$$
$$\in U$$

$\Rightarrow Dom(U, S) = U$.

Thus the proof of the theorem is complete. □

The following corollary is an immediate consequence of Theorem 2.2:

Corollary 2.3. *The variety of all right normal bands is closed in the variety* $\mathcal{V} = [axy = xa^ny]$ $(n > 1)$ *of semigroups.*

Dually, we may prove the following results.

Theorem 2.4. *The variety* $\mathcal{V} = [axy = ay^nx]$ $(n > 1)$ *of semigroups; i.e., the class of all semigroups satisfying the identity* $axy = ay^nx$ *is closed.*

Corollary 2.5. *The variety of all left normal bands is closed in the variety* $\mathcal{V} = [axy = ay^nx]$ $(n > 1)$ *of semigroups.*

Theorem 2.6. *The variety* $\mathcal{V} = [axy = x^nay]$ $(n > 1)$ *of semigroups; i.e., the class of all semigroups satisfying the identity* $axy = x^nay$ *is closed.*

Proof. Take any U, $S \in \mathcal{V}$ with U as a subsemigroup of S such that $d \in Dom(U, S)\backslash U$. Let d has a zigzag of type (1) in S over U of shortest possible length m. Now

$$
\begin{aligned}
d &= (\prod_{i=1}^{m} a_{(2i-1)}^{n-1}) y_m a_{2m-1} t_m \text{ (by Lemma 2.1)} \\
&= (\prod_{i=1}^{m-1} a_{(2i-1)}^{n-1})(a_{2m-1}^{n-2} a_{2m-1}(y_m a_{2m-1})) t_m \text{ (for } n=2\text{, we treat } a_{2m-1}^{n-2} a_{2m-1} \\
&\quad \text{as } a_{2m-1}) \\
&= (\prod_{i=1}^{m-1} a_{(2i-1)}^{n-1})(a_{2m-1}^{n}(a_{2m-1}^{n-2} y_m) a_{2m-1}) t_m \text{ (as } S \in \mathcal{V}) \\
&= (\prod_{i=1}^{m-1} a_{(2i-1)}^{n-1}) a_{2m-1}^{n-2}(y_m a_{2m-1})(a_{2m-1} t_m) \text{ (as } S \in \mathcal{V}) \\
&= (\prod_{i=1}^{m-1} a_{(2i-1)}^{n-1}) a_{2m-1}^{n-2} y_{m-1} a_{2m-2} a_{2m} \text{ (by zigzag equations)} \\
&= (\prod_{i=1}^{m-2} a_{(2i-1)}^{n-1})(a_{2m-3}^{n-2} a_{2m-3}(a_{2m-1}^{n-2} y_{m-1} a_{2m-2})) a_{2m} \\
&= (\prod_{i=1}^{m-2} a_{(2i-1)}^{n-1})(a_{2m-3}^{n}(a_{2m-3}^{n-2} a_{2m-1}^{n-2} y_{m-1}) a_{2m-2}) a_{2m} \text{ (as } S \in \mathcal{V}) \\
&= (\prod_{i=1}^{m-2} a_{(2i-1)}^{n-1}) a_{2m-3}^{n-2} a_{2m-1}^{n-2}(y_{m-1} a_{2m-3}) a_{2m-2} a_{2m} \text{ (as } S \in \mathcal{V}) \\
&= (\prod_{i=1}^{m-2} a_{(2i-1)}^{n-1}) a_{2m-3}^{n-2} a_{2m-1}^{n-2} y_{m-2} a_{2m-4} a_{2m-2} a_{2m} \text{ (by zigzag equations)} \\
&\vdots \\
&= a_1^{n-1} a_3^{n-2} \cdots a_{2m-3}^{n-2} a_{2m-1}^{n-2} y_1 a_2 a_4 \cdots a_{2m-2} a_{2m} \\
&= (a_1^{n-2} a_1(a_3^{n-2} \cdots a_{2m-3}^{n-2} a_{2m-1}^{n-2} y_1 a_2)) a_4 \cdots a_{2m-2} a_{2m} \\
&= (a_1^{n}(a_1^{n-2} a_3^{n-2} \cdots a_{2m-3}^{n-2} a_{2m-1}^{n-2} y_1) a_2) a_4 \cdots a_{2m-2} a_{2m} \text{ (as } S \in \mathcal{V}) \\
&= a_1^{n-2} a_3^{n-2} \cdots a_{2m-3}^{n-2} a_{2m-1}^{n-2}(y_1 a_1) a_2 a_4 \cdots a_{2m-2} a_{2m} \text{ (as } S \in \mathcal{V}) \\
&= a_1^{n-2} a_3^{n-2} \cdots a_{2m-3}^{n-2} a_{2m-1}^{n-2} a_0 a_2 a_4 \cdots a_{2m-2} a_{2m} \text{ (by zigzag equations)} \\
&\in U
\end{aligned}
$$

$\Rightarrow Dom(U, S) = U$.

Thus the proof of the theorem is complete. □

The following corollary is an immediate consequence of Theorem 2.6.

Corollary 2.7. *The variety of all right normal bands is closed in the variety* $\mathcal{V} = [axy = x^n ay]$ $(n > 1)$ *of semigroups.*

Dually, we may prove the following results.

Theorem 2.8. *The variety* $\mathcal{V} = [axy = ayx^n]$ $(n > 1)$ *of semigroups; i.e., the class of all semigroups satisfying the identity* $axy = ayx^n$ *is closed.*

Corollary 2.9. *The variety of all left normal bands is closed in the variety* $\mathcal{V} = [axy = ayx^n]$ $(n > 1)$ *of semigroups.*

Theorem 2.10. *The variety* $\mathcal{V} = [axy = ax^n ay]$ $(n > 1)$ *of semigroups; i.e., the class of all semigroups satisfying the identity* $axy = ax^n ay$ *is closed.*

Proof. Take any U, $S \in \mathcal{V}$ with U as a subsemigroup of S such that $d \in Dom(U,S)\backslash U$. Let d has zigzag of type (1) in S over U of shortest possible length m. In order to prove the theorem, we first prove the following lemma.

Lemma 2.11.

$$y_i a_{2i-1} t_i = y_i a_{2i-1}^n y_{i+1} a_{2i+1} t_{i+1} \ (1 \leq i \leq m-1).$$

Proof.

$$\begin{aligned}
y_i a_{2i-1} t_i &= y_i a_{2i-1}^n y_i t_i \text{ (as } S \in \mathcal{V}\text{)} \\
&= y_i a_{2i-1}^{n-2}(a_{2i-1}a_{2i-1}y_i)t_i \text{ (for } n=2\text{, we treat } a_{2i-1}^{n-2}a_{2i-1}a_{2i-1} \\
&\quad \text{as } a_{2i-1}a_{2i-1}) \\
&= y_i a_{2i-1}^{n-2} a_{2i-1} a_{2i-1}^n a_{2i-1} y_i t_i \text{ (as } S \in \mathcal{V}\text{)} \\
&= y_i (a_{2i-1}a_{2i-1})^n y_i t_i \\
&= y_i a_{2i-1}(a_{2i-1} t_i) \text{ (as } S \in \mathcal{V}\text{)} \\
&= (y_i a_{2i-1} a_{2i}) t_{i+1} \text{ (by zigzag equations)} \\
&= y_i a_{2i-1}^n y_i a_{2i} t_{i+1} \text{ (as } S \in \mathcal{V}\text{)} \\
&= y_i a_{2i-1}^n y_{i+1} a_{2i+1} t_{i+1} \text{ (by zigzag equations),}
\end{aligned}$$

as required. □

Now

$$
\begin{aligned}
d &= a_0t_1 \text{ (by zigzag equations)}\\
&= y_1a_1t_1 \text{ (by zigzag equations)}\\
&= y_1a_1^n(y_2a_3t_2) \text{ (by Lemma 2.11)}\\
&= y_1a_1^ny_2a_3^ny_3a_5t_3 \text{ (by Lemma 2.11)}\\
&\vdots\\
&= y_1a_1^ny_2a_3^n\cdots y_{m-2}a_{2m-5}^n(y_{m-1}a_{2m-3}t_{m-1})\\
&= y_1a_1^ny_2a_3^n\cdots y_{m-2}a_{2m-5}^ny_{m-1}a_{2m-3}^n(y_ma_{2m-1}t_m) \text{ (by Lemma 2.11)}\\
&= y_1a_1^ny_2a_3^n\cdots y_{m-2}a_{2m-5}^ny_{m-1}a_{2m-3}^ny_ma_{2m-1}^ny_mt_m \text{ (as } S\in\mathcal{V})\\
&= y_1a_1^ny_2a_3^n\cdots y_{m-2}a_{2m-5}^ny_{m-1}a_{2m-3}^ny_ma_{2m-1}^{n-2}(a_{2m-1}a_{2m-1}y_m)t_m\\
&\text{(for } n=2, \text{ we treat } a_{2m-1}^{n-2}a_{2m-1}a_{2m-1} \text{ as } a_{2m-1}a_{2m-1})\\
&= y_1a_1^ny_2a_3^n\cdots y_{m-2}a_{2m-5}^ny_{m-1}a_{2m-3}^ny_ma_{2m-1}^{n-2}a_{2m-1}a_{2m-1}^na_{2m-1}y_mt_m \text{ (as } S\in\mathcal{V})\\
&= y_1a_1^ny_2a_3^n\cdots y_{m-2}a_{2m-5}^ny_{m-1}a_{2m-3}^n(y_m(a_{2m-1}a_{2m-1})^ny_mt_m)\\
&= y_1a_1^ny_2a_3^n\cdots y_{m-2}a_{2m-5}^ny_{m-1}a_{2m-3}^n(y_ma_{2m-1})(a_{2m-1}t_m) \text{ (as } S\in\mathcal{V})\\
&= y_1a_1^ny_2a_3^n\cdots y_{m-2}a_{2m-5}^n(y_{m-1}a_{2m-3}^ny_{m-1}a_{2m-2})a_{2m} \text{ (by zigzag equations)}\\
&= y_1a_1^ny_2a_3^n\cdots y_{m-2}a_{2m-5}^ny_{m-1}a_{2m-3}a_{2m-2}a_{2m} \text{ (as } S\in\mathcal{V})\\
&\vdots\\
&= y_1a_1^n(y_2a_3)a_4a_6\cdots a_{2m-2}a_{2m}\\
&= (y_1a_1^ny_1a_2)a_4a_6\cdots a_{2m-2}a_{2m} \text{ (by zigzag equations)}\\
&= y_1a_1a_2a_4a_6\cdots a_{2m-2}a_{2m} \text{ (as } S\in\mathcal{V})\\
&= a_0a_2a_4a_6\cdots a_{2m-2}a_{2m} \text{ (by zigzag equations)}\\
&\in U
\end{aligned}
$$

$\Rightarrow Dom(U,S)=U$.

Thus the proof of the theorem is complete. □

The following corollaries are immediate consequences of the Theorem 2.10.

Corollary 2.12. *The variety of all left quasinormal bands is closed in the variety* $\mathcal{V} = [axy = ax^n ay]$ $(n > 1)$ *of semigroups.*

Corollary 2.13. *The variety of all left and right normal bands are closed in the variety* $\mathcal{V} = [axy = ax^n ay]$ $(n > 1)$ *of semigroups.*

Dually, we have the following results.

Theorem 2.14. *The variety* $\mathcal{V} = [axy = ayx^n y]$ $(n > 1)$ *of semigroups; i.e., the class of all semigroups satisfying the identity* $axy = ayx^n y$ *is closed.*

Corollary 2.15. *The variety of all right quasinormal bands is closed in the variety* $\mathcal{V} = [axy = ayx^n y]$ $(n > 1)$ *of semigroups.*

Corollary 2.16. *The variety of all left and right normal bands are closed in the variety* $\mathcal{V} = [axy = ayx^n y]$ $(n > 1)$ *of semigroups.*

Theorem 2.17. *The variety* $\mathcal{V} = [axy = a^n x a^r y]$ $(n, r \in \mathbb{N})$ *of semigroups; i.e., the class of all semigroups satisfying the identity* $axy = a^n x a^r y$ *is closed.*

Proof. Take any U, $S \in \mathcal{V}$ with U as a subsemigroup of S such that $d \in Dom(U, S) \backslash U$. Let d has zigzag of type (1) in S over U of shortest possible length m. In order to prove the theorem, we first prove the following lemma.

Lemma 2.18.

$$y_i a_{2i-1} t_i = y_i^n a_{2i-1}^{n+r-1} y_i^{r-1} y_{i+1} a_{2i+1} t_{i+1} \ (1 \le i \le m-1).$$

Proof.

$$\begin{aligned}
y_i a_{2i-1} t_i &= y_i^n (a_{2i-1} y_i^r t_i) \text{ (as } S \in \mathcal{V}) \\
&= (y_i^n a_{2i-1}^n y_i^r a_{2i-1}^r) t_i \text{ (as } S \in \mathcal{V}) \\
&= y_i a_{2i-1}^n a_{2i-1}^r t_i \text{ (as } S \in \mathcal{V}) \\
&= y_i a_{2i-1}^n a_{2i-1}^{r-1} (a_{2i-1} t_i) \text{ (for } r = 1, \text{ we treat } a_{2i-1}^{r-1} a_{2i-1} \text{ as } a_{2i-1}) \\
&= (y_i a_{2i-1}^{n+r-1} a_{2i}) t_{i+1} \text{ (by zigzag equations)} \\
&= y_i^n a_{2i-1}^{n+r-1} y_i^r a_{2i} t_{i+1} \text{ (as } S \in \mathcal{V}) \\
&= y_i^n a_{2i-1}^{n+r-1} y_i^{r-1} (y_i a_{2i}) t_{i+1} \text{ (for } r = 1, \text{ we treat } y_i^{r-1} y_i \text{ as } y_i) \\
&= y_i^n a_{2i-1}^{n+r-1} y_i^{r-1} y_{i+1} a_{2i+1} t_{i+1} \text{ (by zigzag equations),}
\end{aligned}$$

as required. □

Now

$$
\begin{aligned}
d &= a_0t_1 \text{ (by zigzag equations)}\\
&= y_1a_1t_1 \text{ (by zigzag equations)}\\
&= y_1^na_1^{n+r-1}y_1^{r-1}(y_2a_3t_2) \text{ (by Lemma 2.18)}\\
&= y_1^na_1^{n+r-1}y_1^{r-1}y_2^na_3^{n+r-1}y_2^{r-1}y_3a_5t_3 \text{ (by Lemma 2.18)}\\
&\vdots\\
&= y_1^na_1^{n+r-1}y_1^{r-1}y_2^na_3^{n+r-1}y_2^{r-1}\cdots y_{m-1}^na_{2m-3}^{n+r-1}y_{m-1}^{r-1}(y_ma_{2m-1}t_m)\\
&= y_1^na_1^{n+r-1}y_1^{r-1}y_2^na_3^{n+r-1}y_2^{r-1}\cdots y_{m-1}^na_{2m-3}^{n+r-1}y_{m-1}^{r-1}y_m^n(a_{2m-1}y_m^rt_m) \text{ (as } S\in\mathcal{V})\\
&= y_1^na_1^{n+r-1}y_1^{r-1}y_2^na_3^{n+r-1}y_2^{r-1}\cdots y_{m-1}^na_{2m-3}^{n+r-1}y_{m-1}^{r-1}(y_m^na_{2m-1}^ny_m^ra_{2m-1}^r)t_m\\
&\text{(as } S\in\mathcal{V})\\
&= y_1^na_1^{n+r-1}y_1^{r-1}y_2^na_3^{n+r-1}y_2^{r-1}\cdots y_{m-1}^na_{2m-3}^{n+r-1}y_{m-1}^{r-1}y_ma_{2m-1}^na_{2m-1}^rt_m\\
&\text{(as } S\in\mathcal{V})\\
&= y_1^na_1^{n+r-1}y_1^{r-1}y_2^na_3^{n+r-1}y_2^{r-1}\cdots y_{m-1}^na_{2m-3}^{n+r-1}y_{m-1}^{r-1}(y_ma_{2m-1})a_{2m-1}^{n+r-2}(a_{2m-1}t_m)\\
&\text{(for } n,r=1, \text{ we treat } a_{2m-1}a_{2m-1}^{n+r-2}a_{2m-1} \text{ as } a_{2m-1}a_{2m-1})\\
&= y_1^na_1^{n+r-1}y_1^{r-1}y_2^na_3^{n+r-1}y_2^{r-1}\cdots y_{m-1}^na_{2m-3}^{n+r-1}y_{m-1}^{r-1}y_{m-1}a_{2m-2}a_{2m-1}^{n+r-2}a_{2m}\\
&\text{(by zigzag equations)}\\
&= y_1^na_1^{n+r-1}y_1^{r-1}y_2^na_3^{n+r-1}y_2^{r-1}\cdots y_{m-2}^na_{2m-5}^{n+r-1}y_{m-2}^{r-1}(y_{m-1}^na_{2m-3}^{n+r-1}y_{m-1}^ra_{2m-2})a_{2m-1}^{n+r-2}a_{2m}\\
&= y_1^na_1^{n+r-1}y_1^{r-1}y_2^na_3^{n+r-1}y_2^{r-1}\cdots y_{m-2}^na_{2m-5}^{n+r-1}y_{m-2}^{r-1}y_{m-1}a_{2m-3}^{n+r-1}a_{2m-2}a_{2m-1}^{n+r-2}a_{2m}\\
&\text{(as } S\in\mathcal{V})\\
&= y_1^na_1^{n+r-1}y_1^{r-1}y_2^na_3^{n+r-1}y_2^{r-1}\cdots y_{m-2}^na_{2m-5}^{n+r-1}y_{m-2}^{r-1}(y_{m-1}a_{2m-3})a_{2m-3}^{n+r-2}a_{2m-2}a_{2m-1}^{n+r-2}a_{2m}\\
&\text{(for } n,r=1, \text{ we treat } a_{2m-3}a_{2m-3}^{n+r-2} \text{ as } a_{2m-3})\\
&= y_1^na_1^{n+r-1}y_1^{r-1}y_2^na_3^{n+r-1}y_2^{r-1}\cdots y_{m-2}^na_{2m-5}^{n+r-1}y_{m-2}^{r-1}y_{m-2}a_{2m-4}a_{2m-3}^{n+r-2}a_{2m-2}a_{2m-1}^{n+r-2}a_{2m}\\
&\text{(by zigzag equations)}\\
&= y_1^na_1^{n+r-1}y_1^{r-1}y_2^na_3^{n+r-1}y_2^{r-1}\cdots y_{m-2}^na_{2m-5}^{n+r-1}y_{m-2}^{r}a_{2m-4}a_{2m-3}^{n+r-2}a_{2m-2}a_{2m-1}^{n+r-2}a_{2m}\\
&\vdots\\
&= (y_1^na_1^{n+r-1}y_1^ra_2)a_3^{n+r-2}a_4a_5^{n+r-2}a_6\cdots a_{2m-1}^{n+r-2}a_{2m}
\end{aligned}
$$

$= y_1 a_1^{n+r-1} a_2 a_3^{n+r-2} a_4 a_5^{n+r-2} a_6 \cdots\cdots a_{2m-1}^{n+r-2} a_{2m}$ (as $S \in \mathcal{V}$)

$= (y_1 a_1) a_1^{n+r-2} a_2 a_3^{n+r-2} a_4 a_5^{n+r-2} a_6 \cdots\cdots a_{2m-1}^{n+r-2} a_{2m}$

(for $n, r = 1$, we treat $a_1 a_1^{n+r-2}$ as a_1)

$= a_0 a_1^{n+r-2} a_2 a_3^{n+r-2} a_4 a_5^{n+r-2} a_6 \cdots\cdots a_{2m-1}^{n+r-2} a_{2m}$ (by zigzag equations)

$\in U$

$\Rightarrow Dom(U, S) = U$.

Thus the proof of the theorem is complete. □

The following corollaries are immediate consequences of the Theorem 2.17.

Corollary 2.19. *The variety of all left quasinormal bands is closed in the variety* $\mathcal{V} = [axy = a^n x a^r y]$ $(n \in \mathbb{N})$ *of semigroups.*

Corollary 2.20. *The variety of all left and right normal bands are closed in the variety* $\mathcal{V} = [axy = a^n x a^r y]$ $(n \in \mathbb{N})$ *of semigroups.*

Dually, we have the following results.

Theorem 2.21. *The variety* $\mathcal{V} = [axy = ay^r xy^n]$ $(n \in \mathbb{N})$ *of semigroups; i.e., the class of all semigroups satisfying the identity* $axy = ay^r xy^n$ *is closed.*

Corollary 2.22. *The variety of all right quasinormal bands is closed in the variety* $\mathcal{V} = [axy = ay^r xy^n]$ $(n \in \mathbb{N})$ *of semigroups.*

Corollary 2.23. *The variety of all left and right normal bands are closed in the variety* $\mathcal{V} = [axy = ay^r xy^n]$ $(n \in \mathbb{N})$ *of semigroups.*

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