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Research Paper

A LENGTH FOR ARTINIAN MODULES

ALI REZA ALEHAFTTAN*

Abstract. In this paper we shall introduce a theory of length for Artinian modules over an arbitrary ring (with identity), assigning to each Artinian module A an ordinal number $len(A)$ which will briefly be called the length of A. We also demonstrate for some familiar properties of left Artinian ring be proved efficiently using length and arithmetic properties of ordinal numbers.

1. Introduction

Gulliksen in [7] introduced and studied a length and a dimension for Noetherian modules and these length and dimension were studied more and more deeply by Brookfield in [5] and also he showed some of their applications in Noetherian rings. These two papers and some works on Krull and Noetherian dimensions of modules such [1], [2], [3], [4], [9], [10] and [11] motivated us to introduce and study a length and a dimension (that, is called length dimension) for Artinian modules as the dual of length and dimension for Noetherian modules. These

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*Corresponding author

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length and dimension carry important information about Artinian module A and denoted by $len(A)$ and $l.dim(A)$ respectively. Being an ordinal, $len(A)$ can be expressed as a polynomial in ω with integral coefficients and ordinal exponents, ω denoting the first infinite ordinal. Length and length dimension are really measures of the size of the lattice of all submodules of an Artinian module ordered by inclusion. Suppose that A is an Artinian uniserial module, meaning that $L(A)$ is well ordered set with maximum element A and minimum element 0. We define the length of A to the ordinal number represented by $L(A) \setminus \{A\}$. Using this definition and the arithmetic of ordinal numbers we can then prove various properties of Artinian uniserial modules. For example, we notice that if B is an Artinian uniserial module and $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is exact, then $len(B) = len(C) + len(A)$. Consider the case $len(B) = len(C)$. Here $len(B) = len(B) + len(A)$, and since ordinal addition is cancellative on the left, we get $len(A) = 0$ and $A = 0$. Expressed differently, this says that a homomorphism $\lambda : A \longrightarrow A$ is injective if and only if $len(A) = len(\lambda(A))$.

This definition of length in this paper extends the above definition for Artinian uniserial module to all Artinian modules. It is natural because there is really only one possible way making this extension. In short for an Artinian module A, we define $len(B) = \varphi(0)$ where φ is the smallest strictly increasing function from $L(A)$ to ordinal numbers. The function φ can also be defined inductively as follows: First set $\varphi(A) = 0$. Suppose, for and ordinal α, we have already identified those submodules B of A such that $\varphi(B) \prec \alpha$. Then $\varphi(B) = \alpha$ if and only if B is maximal among those submodules of A on which has not yet been defined.

Once again ordinal arithmetic comes into play. We will show that if $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is an exact sequence of Artinian modules, then $len(C) + len(A) \preceq len(B) \preceq len(C) \oplus len(A)$. Here $\oplus$ is natural sum on ordinal numbers. The relationship between $len(A)$ and $l.dim(A)$ is simple one. If A is a nonzero Artinian module, then $len(A)$ can be writen uniquely in the long normal form $len(A) = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_n}$ where $\gamma_1 \succeq \gamma_2 \succeq \cdots \succeq \gamma_n$ are ordinal numbers. Then $l.dim(A) = \gamma_1$. In fact the possible values of $len(B)$ for a submodule B of A, are determined by $len(A)$. In particular, we have $l.dim(B) = \{-1, \gamma_1, \gamma_2, \ldots, \gamma_n\}$. Thus $len(A)$ contains a lot more information about A than $len(A)$.

Throughout this article, all rings are associative with $1 \neq 0$ and all modules are unital left $R-$modules. The notation $B \leq A$ (resp, $B < A$) means B is a submodule (resp, proper submodule) of A and The notation $\alpha \preceq \beta$ (resp, $\alpha \prec \beta$) means β is an ordinal number less (resp, strictly less) than α.

2. Preliminaries

We will use lowercase Greek letters for ordinal numbers. The smallest infinite ordinal is written by ω. We need the following results on ordinal numbers, for proofs and more details, see [8] and [13].

Lemma 2.1. *We have the following on ordinal numbers:*

(1) *Ordinal addition is associative but not commutative. For example $\omega + 1 \neq 1 + \omega = \omega$.*
(2) *Ordinal addition is cancelletive on the left: $\alpha + \beta = \alpha + \gamma \Longrightarrow \beta = \gamma$. Also $\alpha + \beta \preceq \alpha + \gamma \Longrightarrow \beta \preceq \gamma$.*
(3) *For the fix ordinal α, the map from ordinal numbers to itself given by $\beta \longmapsto \alpha + \beta$ is strictly increasing.*
(4) *If $\alpha \preceq \beta$, then $\beta - \alpha$ is the unique ordinal γ such that $\beta = \alpha + \gamma$, hence $\beta = \alpha + (\beta - \alpha)$. For any ordinal numbers α and β we have $\beta = (\alpha + \beta) - \alpha$.*
(5) *$\alpha n = \underbrace{\alpha + \alpha + \cdots + \alpha}_{n \text{ times}}$ when $n \in \mathbb{N}$. Note: $2\omega = \omega \neq \omega 2$.*

Proposition 2.2. *Any nonzero ordinal number α can be expressed uniquely in long normal form*

$$\alpha = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_n}$$

where $\gamma_1 \succeq \gamma_2 \succeq \cdots \succeq \gamma_n$. By collecting together terms which have identical exponents, this same frome can be written

$$\alpha = \omega^{\gamma_1} m_1 + \omega^{\gamma_2} m_2 + \cdots + \omega^{\gamma_n} m_n$$

where $\gamma_1 \succ \gamma_2 \succ \cdots \succ \gamma_n$ and $m_1, m_2, \ldots, m_n \in \mathbb{N}$. This will call the short normal form.

Lemma 2.3. *Suppose α, β and γ are ordinal numbers with $\alpha \succ 0$ and $m, n \in \mathbb{N} \cup \{0\}$.*

(1) *$\beta + \alpha \preceq \alpha$ if and only if $\beta + \alpha = \alpha$*
(2) *$\alpha = \omega^\gamma$ for some ordinal number γ if and only if $\beta + \alpha = \alpha$ for all $\beta \prec \alpha$.*
(3) *If $\beta + \omega^\gamma n \prec \omega^\gamma m$, then $\beta \prec \beta + \omega^\gamma (m - n)$*

Definition 2.4. Let α and β be nonzero ordinal numbers. With suitable re-labeling, the short normal forms for these ordinals can be written using the some strictly decreasing set of exponents $\gamma_1 \succ \gamma_2 \succ \cdots \succ \gamma_n$:

$$\alpha = \omega^{\gamma_1} m_1 + \omega^{\gamma_2} m_2 + \cdots + \omega^{\gamma_n} m_n \text{ and } \beta = \omega^{\gamma_1} t_1 + \omega^{\gamma_2} t_2 + \cdots + \omega^{\gamma_n} t_n$$

where $m_i, t_i \in \mathbb{N} \cup \{0\}$. Now we define

$$\alpha \oplus \beta = \omega^{\gamma_1}(m_1 + t_1) + \omega^{\gamma_2}(m_2 + t_2) + \cdots + \omega^{\gamma_n}(m_n + t_n)$$

Lemma 2.5. *Let $\alpha, \beta, \alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n$ be ordinal numbers.*

(1) $\alpha + \beta \preceq \alpha \oplus \beta$.

(2) $(\alpha_1 \oplus \beta_1) + (\alpha_2 \oplus \beta_2) + \cdots + (\alpha_n \oplus \beta_n) \preceq (\alpha_1 + \alpha_2 + \cdots + \alpha_n) \oplus (\beta_1 + \beta_2 + \cdots + \beta_n)$.

(3) $\alpha_1 + \beta_1 + \alpha_2 + \beta_2 + \cdots + \alpha_n + \beta_n \preceq (\alpha_1 + \alpha_2 + \cdots + \alpha_n) \oplus (\beta_1 + \beta_2 + \cdots + \beta_n)$.

Proposition 2.6. *Suppose* $\alpha \oplus \beta = \alpha + \beta = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_n}$. *Then* $\alpha = 0$ *or* $\beta = 0$ *or there is some* $1 \leqslant i \leqslant n-1$ *such that* $\alpha = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_i}$ *and* $\beta = \omega^{\gamma_{i+1}} + \omega^{\gamma_{i+2}} + \cdots + \omega^{\gamma_n}$.

We need the following results on modules, for proofs, see [6] and [15].

Definition 2.7. An essential (or large) submodule of a module A is any submodule E which has nonzero intersection with every nonzero submodule of A. We write $E \leq_e A$ to denote this situation.

Definition 2.8. A uniform module is a nonzero module A such that the intersection of any two nonzero submodules of A is nonzero or equivalently, every nonzero submodule of A is essential in A.

Definition 2.9. A module A has finite Goldie dimension if A has an essential submodule which is a finite direct sum of uniform submodules. Goldie dimension denoted by $G-dim(A)$.

Lemma 2.10. *Let A be a module and n a nonnegative integer. Then the following conditions are equivalent:*

(1) *$G - dim(A) = n$.*

(2) *A contains a direct sum of n nonzero submodules but no direct sum of $n+1$ nonzero submodules.*

(3) *For every ascending chain $A_0 \leq A_1 \leq A_2 \leq \ldots$ of submodules of A there is integer m such that $A_j \leq_e A_{j+1}$ for every $j \geqslant m$.*

(4) *For every descending chain $A_0 \geq A_1 \geq A_2 \geq \ldots$ of submodules of A there is integer k such that $A_{i+1} \leq_e A_i$ for every $i \geqslant k$.*

3. The Length of Artinian Modules

The following result is the counterpart of [5, Theorem 2.3].

Lemma 3.1. *Let $L(A)$ be lattice of submodules of Artinian $R-$module A and φ a function from $L(A)$ into ordinal numbers. Then the following are equivalent:*

(1) *φ is strictly increasing and for each strictly increasing function such ψ from $L(A)$ into ordinal numbers, $\varphi(B) \preceq \psi(B)$ for all $B \leq A$.*

(2) *For all $B \leq A$ and ordinal number α, $\varphi(B) = \alpha$ if and only if B is minimal with respect to property $\alpha \preceq \varphi(D)$.*

(3) *φ is strictly increasing and $\{\alpha \preceq \varphi(B) | \alpha$ is an ordinal number$\} \subseteq \varphi(\{C \in L(A) | C \leq B\})$.*

Proof. (1) $\Longrightarrow$ (2): Suppose that $\varphi(B) = \alpha$, so $\alpha \preceq \varphi(B)$. Since φ is strictly increasing, for any $D < B$, we have $\varphi(D) \prec \varphi(B) = \alpha$. Hence B is minimal with respect to property $\alpha \preceq \varphi(D)$. Conversely; Suppose that B is minimal with respect to property $\alpha \preceq \varphi(D)$. So $\alpha \preceq \varphi(B)$ and we define λ from $L(A)$ into ordinal numbers by $\lambda(B) = \alpha$ and $\lambda(E) = \varphi(E)$ for all submodule E of A that $E \neq B$. Clearly ψ is strictly increasing and by hypothesis on φ we have $\varphi(B) \preceq \psi(B)$. Therefore $\varphi(B) = \alpha$.
(2) $\Longrightarrow$ (3): Suppose that $C < B$ and $\varphi(B) = \alpha$. Since B is minimal with respect to property $\alpha \preceq \varphi(D)$, we have $\varphi(C) \prec \varphi(B)$. Now let $\alpha \preceq \varphi(D)$. But A is Artinian module and hence $L(A)$ is Artinian lattice, so there exists $B \leq D$ which is minimal with respect to property $\alpha \preceq \varphi(E)$. Therefore $\alpha = \varphi(B)$ as desired.
(3) $\Longrightarrow$ (1): Let ψ be strictly increasing function from $L(A)$ into ordinal numbers. Suppose to the contrary that there exists $E \leq A$ such that $\psi(E) \prec \varphi(E)$. Let B be chosen so that $\psi(B)$ is minimum with respect to property. For any $C < B$ we have $\psi(C) \prec \psi(B)$, so $\varphi(C) \preceq \psi(C) \prec \psi(B) \prec \varphi(B)$. Thus we have an ordinal number $\alpha = \psi(B)$ such that $\alpha \prec \varphi(B)$ but there is no $C < B$ with $\varphi(C) = \alpha$. This is contradicts with $\{\alpha \preceq \varphi(B) | \alpha \textit{ is an ordinal number}\} \subseteq \varphi(\{C \in L(A) | C \leq B\})$. □

The following result is essentially the comment following Defnition 2.4, in [5].

Proposition 3.2. *Let A be an $R-$module and $L(A)$ be lattice of its submodules. If there exists a strictly increasing function from $L(A)$ to ordinal numbers, then A is Artinian and there exists a function φ from $L(A)$ into ordinal numbers.*

Proof. Any strictly increasing function from $L(A)$ to ordinal numbers, maps infinite decreasing sequence in $L(A)$ in ordinal numbers. Since no such sequence exist in ordinal numbers, there are no infinite strictly decreasing sequence in $L(A)$ either. Now we define

$$\varphi(B) = \min\{\psi(B) | \psi \textit{ is strictly increasing function from } L(A) \textit{ into ordinal numbers}\}$$

for all $B \leq A$. Since we are assuming that at least one strictly increasing function exists, φ is well define by this equation. If $B < C$ are submodules of A, then there is some strictly increasing function from $L(A)$ to ordinal numbers ψ such that $\psi(C) = \varphi(C)$, so $\varphi(B) \preceq \psi(B) \prec \psi(C) = \varphi(C)$. Thus φ is itself strictly increasing function. □

By Lemma 3.1 and Proposition 3.2 we have the following result.

Corollary 3.3. *Let A be an $R-$module and $L(A)$ be lattice of its submodules. If there exists a strictly increasing function from $L(A)$ to ordinal numbers then A is Artinian and there exists a function φ from $L(A)$ into ordinal numbers satisfying equivalent conditions in Lemma 3.1.*

Definition 3.4. The function introduced in the Corollary 3.3 will be called the length function of $R-$module A and we define the length of A by $len(A) = len(L(A)) = \varphi(A)$. In addition if $len(A) = \omega^{\gamma_1}n_1 + \omega^{\gamma_2}n_2 + \cdots + \omega^{\gamma_m}n_m$ we defin length dimension of A by $l.dim(A) = \gamma_1$ and the length rank of A by $l.rank(A) = n_1 + n_2 + \cdots + n_m$, the number of n_i will be called $\gamma_i - length$ of A, written $len_{\gamma_i}(A)$. By convention $l.dim(0) = -1$ and $l.rank(0) = 0$.

The following result is the counterpart of [5, Theorem 2.5] and shows that the length function of an Artinian module always exists.

Lemma 3.5. *Let A be an Artinian module and $B \leq A$. Then $L(B)$ has length function.*

Proof. Let φ define inductively as following

$$\varphi(0) = 0 \text{ and } \varphi(C) = \alpha \text{ if and only if } C \leq B \text{ and } C \text{ is minimal in } \{D \leq B | \varphi(D) \not< \alpha\}$$

Suppose $E \leq B$ is minimal among submodules of B for which $\varphi(E)$ undefined. Then for every $H < E$, $\varphi(H)$ is defined. Let $\alpha = \sup\{\varphi(H) + 1 | H < E\}$. This is well defined since any set of ordinal numbers has supremum. It is easy to check that E is minimal in $\{D \leq B | \varphi(D) \not< \alpha\}$ and so $\varphi(E) = \alpha$. This contradiction our assumption that $\varphi(E)$ is undefined. Consequently, φ is defined on all of $L(B)$, and by Corollary 3.3, is the length function of B. □

The following result is the counterpart of [5, Lemma 2.6].

Lemma 3.6. *Let A be an Artinian module with length function φ. Then we have the following:*

(1) *For all $B \leq A$, $len(L(B)) = \varphi(B)$.*
(2) *For all submodules $C \leq B$ of A, $len(L(C)) + len(\frac{B}{C}) \preceq len(L(B))$.*
(3) *For any ordinal number $\alpha \preceq len(A)$, there is some $B \leq A$ such that $\varphi(B) = \alpha$.*
(4) *If M is an Artinian module and $\Phi : L(A) \longrightarrow L(M)$ is a strictly increasing function, then $len(A) \preceq len(M)$.*

Proof. (1). It is easy to see that the restriction of φ to $L(B)$ is a strictly increasing and

$$\{\alpha \preceq \varphi|_{L(B)}(C) | \alpha \text{ is an ordinal number}\} \subseteq \varphi|_{L(B)}(\{D \in L(B) | D \leq C\})$$

So $\varphi|_{L(B)}$ is length function and in particular, $\varphi(B) = \varphi|_{L(B)}(B) = len(B)$.
(2). Let ψ be the length function of $L(B)$. Define τ from $L(\frac{B}{C})$ into ordinal numbers by $\tau(D) = \psi(D) - \psi(C)$. The fuction τ is a strictly increasing, so

$$len(L(\tfrac{B}{C})) \preceq \tau(B) = \psi(B) - \psi(C) = len(L(B)) - len(L(C))$$

Hence, $len(L(C)) + len(\frac{B}{C}) \preceq len(L(B))$. (3). It is follows immediately from following property of φ:

$$\{\alpha \preceq \varphi(B) | \alpha \text{ is an ordinal number}\} \subseteq \varphi(\{C \in L(A) | C \leq B\})$$

(4). Let ψ be the length function of M. Then the function $\psi\circ\varphi$ is a strictly increasing function from $L(A)$ into ordinal numbers, so from Lemma 3.1(1), $\varphi(B) \preceq \psi(\varphi(B))$ for all $B \leq A$. In particular, $len(A) = \varphi(A) \preceq \psi(\varphi(A)) \preceq \psi(M) = len(M)$. $\square$

It is easy to check that if $(P, \leqslant)$ and $(P', \leqslant')$ are two partially ordered set, then $(P \times P', \leqslant'')$ is a partially ordered set with order given by $(p_1, p_1') \leqslant'' (p_2, p_2') \Longleftrightarrow p_1 \leqslant p_2$ *and* $p_1' \leqslant' p_2'$. Also the above partially ordered is considerd for the direct product of the two sets involved.

Lemma 3.7. *If A is an Artinian module, then $f : L(A) \times L(A) \longrightarrow L(A) \times L(A)$ given by $f(B, C) = (B \cap C, B + C)$, is strictly increasing.*

Proof. Suppose that $(B_1, C_1) \leqslant (B_2, C_2)$ with $f(B_1, C_1) = f(B_2, C_2)$. Then $B_1 \leq B_2$, $C_1 \leq C_2$, $B_1 \cap C_1 = B_2 \cap C_2$ and $B_1 + C_1 = B_2 + C_2$. So

$$B_2 = (B_2+C_2)\cap B_2 = (B_1+C_1)\cap B_2 = B_1+(C_1\cap B_2) \subseteq B_1+(B_2\cap C_2) = B_1+(B_1\cap C_1) = B_1$$

Hence $B_1 = B_2$, and by symmetry $C_1 = C_2$. Thus $(B_1, C_1) = (B_2, C_2)$. Now Suppose that $(B_1, C_1) < (B_2, C_2)$. Since f is an increasing function, we have $f(B_1, C_1) \leqslant f(B_2, C_2)$. From above argument $f(B_1, C_1) = f(B_2, C_2)$ is imposible, and so we must have $f(B_1, C_1) < f(B_2, C_2)$. $\square$

The following result is the counterpart of [5, Theorem 3.2].

Proposition 3.8. *If A is an Artinian module and B, C are its submodules, then we have the following:*

(1) $len(B) + len(\frac{A}{B}) \preceq len(L(A)) \preceq len(B) \oplus len((\frac{A}{B}))$.
(2) $len(B \cap C) + len(B + C) \preceq len(B) + len(C) \preceq len(B \cap C) \oplus len(B + C)$.
(3) $len(\frac{A}{B\cap C}) + len(\frac{A}{B+C}) \preceq len(B) \oplus len(C) \preceq len(\frac{A}{B\cap C}) \oplus len(\frac{A}{B+C})$.

Proof. 1. The first inequality is directly from Lemma 3.6(2). To prove the second inequality, consider the restriction of the map in Lemma 3.7 to the domain $L(A) \times \{B\}$. This map is strictly increasing and its image is contained in $L(B) \times L(\frac{A}{B})$. from Lemma 3.6(4) we get

$$len(A) = len(L(A) \times \{B\}) \preceq len(L(B) \times L(\tfrac{A}{B})) = len(B) \oplus len(\tfrac{A}{B}).$$

2. To prove the first inequality we apply (1) to the lattices $L(B + C)$ and $L(C)$. This yields $len(B + C) \preceq len(B) \oplus len(\frac{B+C}{C})$ and $len(B \cap C) + len(\frac{C}{B\cap C}) \preceq len(C)$ respectively. From Lemma 3.7 we also have $len(\frac{B+C}{B}) = len(\frac{C}{B\cap C})$. Hence

$$len(B \cap C) + len(B + C) \preceq len(B \cap C) + (len(B) \oplus len(\tfrac{B+C}{B})) =$$
$$len(B \cap C) + (len(B) \oplus len(\tfrac{C}{B\cap C})) \preceq len(B) \oplus len(C)$$

We have also used the fact that $\alpha+(\beta\oplus\gamma)\preceq(\alpha+\beta)\oplus\gamma$ which follows from Lemma 2.5(2). To prove the second inequality, consider the restriction of the map in Lemma 3.7 to the domain $L(B)\times L(C)$. This map is strictly increasing and its image is contained in $L(B\cap C)\times L(B+C)$, and so from Lemma 3.6(4) we get

$$len(B)\oplus len(C)\preceq len(B\cap C)\oplus len(B+C)$$

3. Proof is similar to that of (2). □

Corollary 3.9. *Let A be an Artinian module, $l.dime(A)=\gamma$ and $B\leq A$. Then we have the following:*

(1) $l.dim(A)=max\{l.dim(B), l.dim(\frac{A}{B})\}$.
(2) $len_\gamma(A)=len_\gamma(B)+len_\gamma(\frac{A}{B})$.

By part (2) of Lemma 2.3, part (3) of Lemma 3.6 and part (1) of Proposition 3.8 we have the following result.

Corollary 3.10. *Let A be a nonzero Artinian module. Then the following are equivalent:*

(1) $len(A)=\omega^\gamma$ *for some ordinal number* γ.
(2) $len(\frac{A}{B})=len(A)$ *for all* $B<A$.
(3) $l.dim(B)\prec l.dim(A)$ *for all* $B<A$.

By Proposition 3.8(1) part (3) and Corollary 3.9 we have the following result.

Corollary 3.11. *Let $0\longrightarrow B\longrightarrow A\longrightarrow C\longrightarrow 0$ be an exact sequence of Artinian modules and $l.dim(A)=\gamma$.*

(1) $len(C)+len(B)\preceq len(A)\preceq len(C)\oplus len(B)$
(2) $l.dim(A)=max\{l.dim(B), l.dim(C)\}$.
(3) $len_\gamma(A)=len_\gamma(B)+len_\gamma(C)$

The following result is the counterpart of [5, Corollary 4.2].

Proposition 3.12. *If A and B are Artinian modules and $f:A\longrightarrow B$ is a homomorphism, then f is injective if and only if $len(A)=len(f(A))$.*

Proof. We have short exact sequence $0\longrightarrow ker(f)\longrightarrow A\longrightarrow f(A)\longrightarrow 0$. So from Corollary 3.11(1), $len(f(A))+len(ker(f))\preceq len(A)$. If $len(A)=len(f(A))$, then we can cancel from this inequality to get $len(ker(f))=0$ and hence $ker(f)=0$. The converse implication is clear since if f is injective, then $A\simeq f(A)$. □

The following result is the counterpart of [5, Lemma 3.5].

Lemma 3.13. *Let A be an Artinian module. Suppose α and β are ordinal numbers such that $\alpha + \beta = \alpha \oplus \beta$. Then $len(A) = \alpha + \beta$ if and only if there is some submodule B of A such that $len(B) = \alpha$ and $len(\frac{A}{B}) = \beta$.*

Proof. Let $len(A) = \alpha + \beta$. From Lemma 3.6(3), there is some $B \subseteq A$ such that $len(B) = \alpha$. From Proposition 3.8(1), $\alpha + len(\frac{A}{B}) \preceq \alpha + \beta = \alpha \oplus \beta \preceq \alpha \oplus len(\frac{A}{B})$. Cancellation in first inequality gives $len(\frac{A}{B}) \preceq \beta$. Cancellation in second inequality gives $\beta \preceq len(\frac{A}{B})$. Hence $len(\frac{A}{B}) = \beta$. Conversely; This follows directly from Proposition 3.8(2). □

Definition 3.14. A nonzero Artinian module A is called $\gamma-$l.atomic if $len(A) = \omega^{\gamma}$. A is called l.atomic if it is a $\gamma-$l.atomic for some γ. An l.atomic series for an Artinian module A, is a sequence

$$0 = A_0 < A_1 < A_2 < \cdots < A_{n-1} < A_n = A$$

such that $\frac{A_i}{A_{i-1}}$ is γ_i-l.atomic for all i, and $\gamma_n \preceq \gamma_{n-1} \preceq \cdots \preceq \gamma_2 \preceq \gamma_1$.

The following result is the counterpart of [5, Lemma 3.8].

Lemma 3.15. *Let A ba an Artinian module. Then the following are equivalent:*

(1) *$len(A) = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_n}$.*
(2) *A has l.atomic series $0 = A_0 < A_1 < A_2 < \cdots < A_{n-1} < A_n = A$ with $\frac{A_i}{A_{i-1}}$ is γ_i-l.atomic for $i = 1, 2, \ldots, n$.*

Proof. If $len(A) = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_n}$, then from Lemma 2.6 and Lemma 3.13, there is $B \leq A$ such that $len(\frac{A}{B}) = \omega^{\gamma_n}$ and $len(B) = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_{n-1}}$. In paticular, $\frac{A}{B}$ is γ_n-l.atomic. A simple induction then shows that A has an l.atomic series. The convers is clear by definition. □

The following result is the counterpart of [5, Theorem 3.9].

Theorem 3.16. *Let A be an Artinian module with l.atomic series*

$$0 = A_0 < A_1 < A_2 < \cdots < A_{n-1} < A_n = A$$

with $\frac{A_i}{A_{i-1}}$ is γ_i-l.atomic for all $i = 1, 2, \ldots, n$. Let $B \subseteq A$ and set $B_i = B + A_i$ for $i = 1, 2, \ldots, n$. Then for $i = 1, 2, \ldots, n$, $len(\frac{B_{i-1}}{B_i})$ is either zero or ω^{γ_i}. Further, the sequence

$$0 = \frac{B_0}{B} < \frac{B_1}{B} < \frac{B_2}{B} < \cdots < \frac{B_{n-1}}{B} < \frac{B_n}{B} = \frac{A}{A}$$

after removal of duplicate entries, is an l.atomic series for $\frac{A}{B}$.

Proof. We have $\frac{A_i + B_{i-1}}{B_{i-1}} \simeq \frac{A_i \cap B_{i-1}}{A_i}$ for $i = 1, 2, \ldots, n$. Since

$$A_i + B_{i-1} = A_i + (A_{i-1} + B) = A_i + B = B_i$$

and

$$A_i \cap B_{i-1} = A_i \cap (A_{i-1} + B) = A_{i-1} + (A_i \cap B)$$

we get $\frac{B_{i-1}}{B_i} \simeq \frac{A_{i-1}+(A_i \cap B)}{A_i}$. We Also have $A_i \leq A_{i-1}+(A_i \cap B) \leq A_i$, and so $\frac{B_{i-1}}{B_i}$ is isomorphic to a final segment of $\frac{A_{i-1}}{A_i}$. Because $\frac{A_{i-1}}{A_i}$ is γ_i-l.atomic, Corollary 3.10(2) applies and either $B_{i-1} = B_i$ or $len(\frac{B_{i-1}}{B_i}) = \omega^{\gamma_i}$. The claime that $0 = \frac{B_0}{B} < \frac{B_1}{B} < \frac{B_2}{B} < \cdots < \frac{B_{n-1}}{B} < \frac{B_n}{B} = \frac{A}{A}$, after removal of duplicate entries, is an l.atomic series for $\frac{A}{B}$ is then clear. □

From this theorem we see that the factors in an l.atomic series for $\frac{A}{B}$ have lengths which are among the lengths of factors of l.atomic series of A. By Combining this with Lemma 3.13 we have the following result.

Corollary 3.17. *Let A be an Artinian module with $len(A) = \omega^{\gamma_1}n_1 + \omega^{\gamma_2}n_2 + \cdots + \omega^{\gamma_t}n_t$. Then for $B \leq A$, $len(\frac{A}{B}) = \omega^{\gamma_1}m_1 + \omega^{\gamma_2}m_2 + \cdots + \omega^{\gamma_t}m_t$, for some $m_i \in \mathbb{N}_0$ such that $m_i \preceq n_i$ for all i. In particular if $B \leq A$ we have the following:*

(1) *$l.rank(\frac{A}{B}) \preceq l.rank(A)$ with equality if and only if $len(\frac{A}{B}) = len(A)$.*
(2) *$l.dim(\frac{A}{B}) \in \{-1, \gamma_1, \gamma_2, \ldots, \gamma_t\}$.*

By Lemmas 2.5, 3.6(3) and 3.15 and Corollary 3.17 we have the following result.

Proposition 3.18. *Let A be an Artinian module.*

(1) *For every ordinal $\alpha \preceq len(A)$ there exists a submodule $B \leq A$ such that $len(\frac{A}{B}) = \alpha$.*
(2) *Suppose $len(A) = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_n}$ in long normal form. Then for ordinals $\alpha = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_i}$ and $\beta = \omega^{\gamma_{i+1}} + \omega^{\gamma_{i+2}} + \cdots + \omega^{\gamma_n}$, for some $0 \leqslant i \leqslant n-1$ there exists a submodule $B \leq A$ such that $len(\frac{A}{B}) = \alpha$ and $len(B) = \beta$.*
(3) *Suppose $len(A) = \omega^{\gamma_1}m_1 + \omega^{\gamma_2}m_2 + \cdots + \omega^{\gamma_n}m_n$ in short normal form. Then for any submodule $B \leq A$ we have $len(B) = \omega^{\gamma_1}t_1 + \omega^{\gamma_2}t_2 + \cdots + \omega^{\gamma_n}t_n$ for some $t_i \in \mathbb{N}_0$ such that $t_i \leqslant m_i$ for all i. In particular, $l.dim(B) \in \{-1, \gamma_1, \gamma_2, \ldots, \gamma_n\}$.*

Proposition 3.19. *Let A and B be an Artinian modules.*

(1) *If $B \leq A$, then $l.rank(B) \preceq l.rank(A)$ with equality if and only if $len(A) = len(B)$.*
(2) *$l.rank(A \oplus B) = l.rank(A) + l.rank(B)$.*
(3) *$G - dim(A) \leqslant l.rank(A)$.*

Proof. 1. Immediate from Corollary 3.17.
2. From Lemma 3.8(2), $len(A \oplus B) = len(A) \oplus len(B)$, so $l.rank(A \oplus B) = l.rank(A) + l.rank(B)$.
3. Any nonzero module has nonzero length rank, so if A contains a direct sum of $G - dim(A)$

nonzero submodules, the using (1) and (2), we must have $G - dim(A) \leqslant l.rank(A)$.

□

4. Some Applications in Artinian Rings

The following result is the counterpart of [5, Lemma 5.3].

Lemma 4.1. *Let I and J be l.atomic left ideals in a left Artinian ring R.*

1. *If $IJ \neq 0$ and $len(I) \preceq len(J)$, then $len(I) = len(J)$ and there is some $x \in J$ such that $I \simeq Ix$, $len(I) = len(Ix) = len(Rx) = len(J)$ and $len(I \oplus ann(x)) = len(R)$.*
2. *If $I^2 \neq 0$, then there is some $\in I$ such $I \simeq Ix$, $len(I) = len(Ix) = len(Rx)$ and $len(I \oplus ann(x)) = len(R)$.*
3. *If I is nil, then $I^2 = 0$.*

Proof. 1. Let $x \in J$ be chosen so that $0 \neq Ix \leq J$. Since J is l.atomic we have $len(J) = len(Ix) = len(R)$. Now we define epimorhism $f : I \longrightarrow Ix$ by $f(r) = rx$. Since $Ix = f(I)$, we also have $len(Ix) \preceq len(I)$ and so $len(I) = len(Ix) = len(Rx) = len(J)$ and from Proposition 3.12 f is injective, so $I \simeq Ix$, $I \cap ann(x) = 0$ and $len(I \oplus ann(x))$ is a left ideal of R. From the exact sequence $0 \longrightarrow ann(x) \longrightarrow R \longrightarrow Rx \longrightarrow 0$ we get $len(R) \preceq len(Rx) \oplus len(ann(x)) = len(I \oplus ann(x))$.

2. It is clear from 1.

3. Suppose, contrary to the claim, that $I^2 \neq 0$. Then from (2), there is $x \in I$ such that $f : I \longrightarrow Ix$ defined by $f(r) = rx$ is an isomorphism. But this is impossible since $x^n = 0$ for some $n \in \mathbb{N}$, and hence $f^n = 0$. □

The following result is the counterpart of [5, Theorem 5.4].

Proposition 4.2. *Let R be a semiprime left Artinian ring, and I a left ideal such that $len(I) = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_n}$ in long normal form. Then there are $x_1, x_2, \ldots, x_n \in I$ such that*

1. *$len(Rx_1 \oplus Rx_2 \oplus \cdots \oplus Rx_n) = len(I)$.*
2. *$len(Rx_i) = \omega^{\gamma_i}$ for $i = 1, 2, \ldots, n$.*
3. *$xi_i x_j = 0$ whenever $i < j$ with $i, j = 1, 2, \ldots, n$.*

And if $x = x_1 + x_2 + \cdots + x_n$ we also have

4. *$f : I \longrightarrow Ix$ by $f(r) = rx$ is injective.*
5. *$I \simeq Ix$ and $len(I) = len(Rx) = len(Ix)$.*

(6) $len(I \oplus ann(x)) = len(R)$.

Proof. From Proposition 3.18(2), the left ideal I contains an l.atomic ideal I_n of length ω^{γ_n}. Since R is semiprime, $I_n^2 \neq 0$, and from Lemma 4.1(2) there is some $x_n \in I_n$ such that $len(I_n x_n) = len(I x_n) = len(R x_n) = \omega^{\gamma_n}$ and $ann(x_n) \cap R x_n = 0$. Let $J = ann(x_n) \cap I$. Then $J \oplus R x_n \leq I$, so that $len(J) \oplus \omega^{\gamma_n} \preceq len(I)$. From short exact sequence $0 \longrightarrow J \longrightarrow I \longrightarrow I x_n \longrightarrow 0$ we get $len(I) \preceq len(J) \oplus len(I x_n) = len(J) \oplus \omega^{\gamma_n}$. Thus we have $len(I) = len(J) \oplus \omega^{\gamma_n}$. Canceling ω^{γ_n} from this equation we get $len(J) = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_{n-1}}$, and so, $l.rank(J) = n - 1 \prec l.rank(I) = n$. By induction there are $x_1, x_2, \ldots, x_{n-1} \in I_n$ satisfying the above conditions with respect to J. We claim that $x_1, x_2, \ldots, x_n$ satisfy these conditions with respect to I. By induction we have $J \cap ann(x_1, x_2, \ldots, x_{n-1}) = 0$. We also have $R(x_1, x_2, \ldots, x_{n-1}) \cap R x_n \leq J \cap R x_n = 0$, from which it follows that $ann(x) = ann(x_1, x_2, \ldots, x_{n-1}) \cap ann(x_n)$. A simple calculation then yields $I \cap ann(x) = 0$. Claims 4, 5, 6 follows from this as the proof of Proposition 3.18(2). The remaining claims are easy to check. □

The following result is the counterpart of [5, Corollary 5.5].

Proposition 4.3. *Let R be a semiprime left Artinian ring, a I a left ideal and $r \in R$.*

(1) *$len(ann(r)) = len(R)$ if and only if $r = 0$.*
(2) *$ann(r) = 0$ if and only if $len(Rr) = len(R)$ if and only if r is regular.*
(3) *I is essential in R if and only if $len(I) = len(R)$ if and only if I contains a regular element.*
(4) *If I is nill, then $I = 0$.*
(5) *$G - dime(I) = l.rank(I)$.*

Proof. 1. Applying Proposition 4.2(6) to the left ideal Rr, we see that there is some $s \in R$ such that $Rr \cap ann(s) = 0$. But if $f : R \longrightarrow R$ is homomorphism by $f(x) = sx$, then $ann(sr) = f^{-1}(ann(r))$, so from Lemma 4.1(2), we have $len(ann(sr)) = len(R)$, and in particular, $ann(sr)$ is essential in R. Thus $Rr = 0$ and $r = 0$.

2. Suppose $len(Rr) = len(R)$. Then by Proposition 3.12, the homomorphism $f : R \longrightarrow R$ by $f(x) = rx$ is injective and hence $ann(r) = 0$. Further, if $rs = 0$ for some $s \in R$, then $Rr \leq ann(s)$, and so $len(ann(s)) = len(R)$ and then, by (1), $s = 0$. Thus r is regular. The remainder claims are easy.

3. If I is essential, then from Proposition 4.2(6), I contains an element x such that $ann(x) = 0$. from 92), x is regular. The remainder claims are easy.

4. From Lemma 4.1(4), there is some $x \in I$ such that $f : I \longrightarrow Ix$ by $f(a) = ax$ is a monomorphism. Since $x^n = 0$ for some $n \in \mathbb{N}$, we have $f^n = 0$ and hence $I = 0$.

5. From Lemma 4.1(1), I contains a direct sum of $l.rank(I)$ nonzero submodules, and so $l.ran(I) \leqslant G - dime(I)$. The opposite inequalty is Proposition 3.19(3). $\square$

The following result is the counterpart of [5, Theorem 5.6].

Proposition 4.4. *If R is a left Artinian prime ring, then $len({}_RR) = \omega^\gamma n$ where $\gamma = l.dim({}_RR)$ and $n = G - dim({}_RR)$. Further, for an Artinian module A and $m \in \mathbb{N}$ we have*

(1) *$\omega^\gamma n \preceq len(A)$ if and only if A has a submodule isomorphic to a direct sum of m l.atomic left ideals.*

(2) *$(len({}_RR))m \preceq len(A)$ if and only if A has a submodule isomorphic to R^m.*

Proof. First we notice that for any tow l.atomic left ideals I and J of R we have $IJ \neq 0 \neq JI$ and so from Lemma 4.1(1), $len(I) = len(J)$, I has a submodule isomorphic to J, and vice versa. If $len({}_RR) = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_n}$ in long normal form, then from Proposition 4.2(1), there are l.atomic left ideals of length $\omega^{\gamma_1}, \omega^{\gamma_2}, \ldots, \omega^{\gamma_n}$. From above we must have $\gamma_1 = \gamma_2 = \cdots = \gamma_n$ and so we can write $len({}_RR) = \omega^\gamma n$ as required. This means in particular, that any l.atomic left ideal of R has length ω^γ.

(1). Proof by induction on m, the case $m = 0$ being trivial. Suppose $0 < m$ and $\omega^\gamma m \preceq len(A)$. Then by Proposition 3.18(1) there is some submodule $B \leq A$ uch that $len(\frac{A}{B}) = \omega^\gamma$. using Corollary 3.11(1), we have $\omega^\gamma m \preceq len(A) \preceq len(\frac{A}{B}) \oplus len(B) = \omega^\gamma \oplus len(B)$, so by cacellation $\omega^\gamma \preceq len(B)$. By induction, B contains a submodule isomorphic to a direct sum of $m-1$ l.atomic left ideals. Let $a \in A \setminus B$. Then $len(\frac{Ra+B}{B}) = \omega^\gamma$. from the exact sequence $0 \longrightarrow ann(a+B) \longrightarrow R \longrightarrow \frac{Ra+B}{B} \longrightarrow 0$ and llary 3.11(1) we get $len(\frac{Ra+B}{B}) + len(ann(a+B)) \preceq len(R)$, that is , $\omega^\gamma + len(ann(a+B)) \preceq \omega^\gamma(n-1) \prec len(R)$. From Proposition 4.3(3), $ann(a+B)$ is not essential in R, and there is an l.atomic left ideal I of R such that $I \cap ann(a+B) = 0$. The map $f : I \longrightarrow \frac{Ia+B}{B}$ by $f(y) = y(a+B)$ is then an isomorphism, so for any $u \in I$, $ua \in B$ implies that $u = 0$. Thus $Ia \cap B = 0$, and $Ia \simeq \frac{Ia+B}{B} \simeq I$ is l.atomic. Since $Ia \cap B = 0$, A contains a direct sum of m l.atomic modules.

(2). In view of (1), to prove (2), it suffices to show that any direct sum on n l.atomic ideals, contains a submodule isomorphic to R. Since for any tow l.atomic left ideals I and J of R I has a submodule isomorphic to J, and vice versa, it suffices to show this for any particular direct sum of n l.atomic left ideals. Now from Proposition 4.2(1), there are l.atomic left ideals $I_1, I_2, \ldots, I_n$ such that $len(I_1 \oplus I_2 \oplus \cdots \oplus I_n) = len(R)$, and $x \in I_1 \oplus I_2 \oplus \cdots \oplus I_n$ such that $Rx \simeq R$. Thus this in particular direct sum contains a submodule isomorphic to R as required.

$\square$

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Ali Reza Alehafttan

Department of Mathematics,

Jundi-Shapur University of Technology, Dezful, Iran.

A.R.Alehafttan@jsu.ac.ir