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Research Paper

ON THE BASIC PROPERTIES OF THE COMPRESSED ANNIHILATOR GRAPH OF $\mathbb{Z}_n$

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ABSTRACT. For a commutative ring R, the *compressed annihilator graph* $AG_E(R)$ is defined by, taking the equivalence classes of zero divisors of R as the vertex set and two distinct vertices $[a]$ and $[b]$ are adjacent if and only if $ann(a) \cup ann(b) \subset ann(ab)$. In this paper, we discuss some of the basic properties such as degree of the vertices, Eulerian, regularity, domination number and planarity of $AG_E(\mathbb{Z}_n)$, where $\mathbb{Z}_n$ is the ring of integer modulo n.

1. Introduction

To solve many mathematical problems, the study on graphs from algebraic structures was initiated. Thereafter, a bulk of creations have been made related with algebraic graph theory such as, analyzing the basic invariants of the graph, investigating its topological properties, coloring of graphs, finding its spectral properties, investigating the interplay between the graphs and the algebraic structures and so on. To condense the size of the zero-divisor graph,

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Anderson and LaGrange [2] introduced the compressed zero divisor graph. Analogously, Sh. Payrovi and S. Babaei [8] established the study on the *compressed annihilator graph* of a ring R, which was designated by $AG_E(R)$ and is defined by, taking the classes of zero divisors of R determined by the annihilators as the vertices and two vertices a and b are connected by an edge if and only if $ann(a) \cup ann(b) \subset ann(ab)$. They studied some its basic properties and the role of the 2- obsorbing ideals in $AG_E(R)$. Impressed by their works, we are concerned on interpreting some of the basic invariants such as, degree of the vertices, Eulerian, regularity, domination number, domination polynomial. Added with that, we characterize some of its topological properties such as outer planar and planar of $AG_E(\mathbb{Z}_n)$, where $\mathbb{Z}_n$ is the ring of integer modulo n.

Throughout this article we take, $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$, where p_i's are distinct primes, k_i's are non negative integers, $s(k) = k_1 + k_2 + \dots + k_r$ and $p(k) = \prod_{i=1}^{r} k_i, k_i \geq 1$ for all i. For a deep view on graph theory one can see [5] and for the algebraic concepts one can mention [6].

2. Basic properties of $AG_E(\mathbb{Z}_n)$

Observation 2.1. Degree of the verices

Let us find out the degree of the vertices of $AG_E(\mathbb{Z}_n)$ where $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$.

The number of vertices in $AG_E(\mathbb{Z}_n)$ is $\prod_{i=1}^{r}(k_i + 1) - 2$.

Now we partition the vertex set into r disjoint sets namely, $A_1, A_2, \dots, A_r$, where

$A_1 = \{[p_i^{t_i}], 1 \leq i \leq r, 1 \leq t_i \leq k_i\}$

$A_2 = \{[p_i^{t_i} p_j^{t_j}], 1 \leq i, j \leq r, 1 \leq t_i \leq k_i, 1 \leq t_j \leq k_j\}$

Similarly we can define the remaining sets.

Let B_j be the sub collection of A_j, $1 \leq j \leq r-1$, where

$B_1 = \{[p_i^{k_i}], 1 \leq i \leq r\}$

$B_2 = \{[p_i^{k_i} p_j^{k_j}], 1 \leq i, j \leq r\}$

Similarly we can define the remaining sets. Now we are going to investigate the degree of the vertices. In each set, for a vertex x, we count the number of vertices that are not adjacent with x and we notate this by $\sigma(x)$.

Let $b \in B_t$ for some $t, 1 \leq t \leq r-1$. Then $b = [p_{i_1}^{k_{i_1}} p_{i_2}^{k_{i_2}} \dots p_{i_t}^{k_{i_t}}]$ where $1 \leq i_j \leq r, 1 \leq j \leq t$. Then

$$\sigma(b) = \prod_{i=1, i \neq i_1, i_2, \dots, i_t}^{r} (k_i + 1) + \prod_{l=1}^{t} (k_{i_l} + 1) - 3 \tag{1}$$

Let $a_1 = p_j^s, s < k_j \in A_1/B_1$. Then

$$\sigma(a_1) = \prod_{i=1, i \neq j}^{r} (k_i + 1) \tag{2}$$

Now we move on to $A_t/B_t, 2 \leq t \leq r-1$
Let $a_2 = p_{i_1}^{l_1} p_{i_2}^{l_2} \dots p_{i_t}^{l_t} \in A_t/B_t$ for some t where $1 \leq i_j \leq r, 1 \leq l_t \leq k_t, 1 \leq j \leq t$. Suppose a_2 has m places, namely $j_1, j_2, \dots, j_m$, $1 \leq j_q \leq r$, $1 \leq q \leq m$ in which the power of p_i is k_i where $1 \leq m \leq t-1$, then

$$\sigma(a_2) = \prod_{i=1, i \neq j_1, j_2, \dots, j_m}^{r} (k_i + 1) + \prod_{l=1}^{m} (k_{j_l} + 1) - 3 \tag{3}$$

Suppose a_2 has all its powers $< k_i$, then

$$\sigma(a_2) = \prod_{i=1, i \neq i_1, i_2, \dots, i_t}^{r} (k_i + 1) \tag{4}$$

Now let us move on to A_r. Let $a_3 \in A_r$. Suppose a_3 has s places in which the power of p_i is k_i namely $j_1, j_2, \dots, j_s$ where $1 \leq s \leq r-1$, then

$$\sigma(a_3) = \prod_{i=1}^{s} (k_{j_i} + 1) \tag{5}$$

Suppose a_3 has all its powers less than k_i, then it will adjacent with all the other vertices.

Remark 2.2. *([8, Example 2.8]) Suppose p is a prime number and $n \geq 2$, then $AG_E(\mathbb{Z}_{p^{n+1}}) \cong K_n$*

Theorem 2.3. *$AG_E(\mathbb{Z}_n)$ is regular if and only if $2 \leq s(k) \leq 3$*

Proof. Suppose $s(k) = 2$ then $AG_E(\mathbb{Z}_n)$ is either K_1 or K_2.
Suppose $s(k) = 3$ then $AG_E(\mathbb{Z}_n)$ is either K_2, C_4 or Figure 1, which are regular.
Suppose $s(k) \geq 4$.
Case 1: $k_i = 1$ for all i.
Let A_i be the collection of vertices defined as in Observation 2.1. If r is odd then $|A_i| = |A_{r-i}|, 1 \leq i \leq \frac{r-1}{2}$ and the degree of each vertex in the above collection is $(2^i - 1)(2^{r-i} - 1)$. If r is even then $|A_i| = |A_{r-i}|, 1 \leq i \leq \frac{r-2}{2}$ and we have a midterm $A_{\frac{r}{2}}$ and the degree of each vertex in the above collection is $(2^i - 1)(2^{r-i} - 1)$. From this we can easily see that $AG_E(\mathbb{Z}_n)$ is non regular for $s(k) \geq 4$.
Case 2: $k_i \geq 2$ for all i. Then

$$deg[p_1] = k_1 \prod_{i=2}^{r} (k_i + 1) - 2$$

and

$$deg[p_1 p_2] = (k_1 + k_2 + k_1 k_2) \prod_{i=3}^{r} (k_i + 1) - 2$$

which are not same.

Case 3: $k_i = 1$ for some i , say k_1 and $k_j \geq 2 \ \forall \ i \neq 1$. Then

$$deg[p_1] = \prod_{i=2}^{r}(k_i + 1) - 2$$

and

$$deg[p_1p_2] = (2k_2 + 1)\prod_{i=3}^{r}(k_i + 1) - k_2$$

which are not same.

Case 4: Atleast two k_j's are one, say $k_1 = k_2 = 1$ and $k_j \geq 2 \ \forall \ i \neq 1, 2$. Then

$$deg[p_1] = \prod_{i=2}^{r}(k_i + 1)$$

and

$$deg[p_1p_2] = 3[\prod_{i=3}^{r}(k_i + 1) - 1]$$

which are not same. The proof is complete. □

Theorem 2.4. *$AG_E(\mathbb{Z}_n)$ is Eulerian if and only if $k_i = 1$ for some i and k_j's are even for $i \neq j$ or all the k_j's are even.*

Proof. Assume that $AG_E(\mathbb{Z}_n)$ is Eulerian. Suppose k_i is odd for some $i, 2 \leq i \leq r$. When $t = 1$, equation (1) of Observation 2.1, will yield that the degree is odd.
Suppose exactly one k_i is odd, say k_1 with $k_1 \geq 3$ and all the other k_j's are even, then also equation (2) of Observation 2.1, will yield that the $deg[p_1]$ must be odd.
Conversely suppose k_i is even for all $i, 1 \leq i \leq r$, then by equations (1) – (5) of Observation 2.1, we obtain that the degree of each vertex is even. Also suppose $k_1 = 1$ and k_j is even for $j \neq 1$, then by equations (1) – (5) of Observation 2.1, we will receive that the degree of each vertex is even. The proof is complete. □

We notate the domination number of a graph G by $\gamma(G)$. The following theorem gives the domination number of $AG_E(\mathbb{Z}_n)$.

Theorem 2.5. *Let $n \geq 4$. Then*
(i)$\gamma(AG_E(\mathbb{Z}_n)) = 1$ if and only if $n = p_1p_2$ or $p_1^{k_1}p_2^{k_2}\dots p_r^{k_r}$ with $k_i \geq 2$ for all $1 \leq i \leq r$
(ii)$\gamma(AG_E(\mathbb{Z}_n)) = 2$ otherwise

Proof. (i) suppose $k_i \geq 2$ for all i, then the vertex $[p_1^{l_1} p_2^{l_2} \dots p_r^{l_r}]$ where $l_i < k_i, 1 \leq i \leq r$ will adjacent with all the other elements. Hence $\gamma(AG_E(\mathbb{Z}_n)) = 1$.
(ii) In this case, the vertices $[p_i^{k_i}]$ for some i and $[\prod_{j=1 i \neq j}^{r} p_i^{k_j}]$ will cover all the vertices. Hence $\gamma(AG_E(\mathbb{Z}_n)) \leq 2$. From Observation 2.1, we know that there exist no dominating vertex, except the vertices described in (i). Hence $\gamma(AG_E(\mathbb{Z}_n)) = 2$. □

Observation 2.6. Domination polynomial of $AG_E(\mathbb{Z}_n)$
The domination polynomial $D(G, x)$ of a graph G is described as follows [1]:

$$D(G, x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G, i) x^i \tag{6}$$

where $d(G, i) = |\mathbb{D}(G, i)|$, $\mathbb{D}(G, i)$ is the collection of dominating sets of size i of G.

Now let us discuss about the dominating polynomial of $AG_E(\mathbb{Z}_n)$. We know that, a dominating vertex of $AG_E(\mathbb{Z}_n)$ is a product of all the r primes and the power of each prime number must be less than k_i, $1 \leq i \leq r$. Hence we have
$d(AG_E(\mathbb{Z}_n), 1) = \prod_{i=1}^{r} (k_i - 1), k_i \geq 2, \forall\ i$.
Next we move on to $\mathbb{D}(AG_E(\mathbb{Z}_n), 2)$. From the structure on $AG_E(\mathbb{Z}_n)$ we have analyzed that a dominating set of size m has the elements that are the partitions of the r distinct prime numbers(including their powers) into m distinct parts. Since each combination has $p(k)$ possibilities we have,

$$d(AG_E(\mathbb{Z}_n), 2) = \begin{cases} p(k)\left(\binom{r}{1} + \binom{r}{2} + \dots \binom{r}{\frac{r}{2}}\right) & \text{if } r \text{ is even} \\ p(k)\left(\binom{r}{1} + \binom{r}{2} + \dots \binom{r}{\frac{r-1}{2}}\right) & \text{if } r \text{ is odd} \end{cases}$$

Next, let us discuss about $\mathbb{D}(AG_E(\mathbb{Z}_n), 3)$. First let us write down the partitions of r into 3 distinct parts. The possible partitions are

$$\begin{array}{llll} 1 & 1 & r-2 & \\ 1 & 2 & r-3 & \\ \vdots & \vdots & \vdots & \\ 1 & \frac{r}{2}-1 & \frac{r}{2} & \text{if } r \text{ is even} \\ 1 & \frac{r-1}{2} & \frac{r-1}{2} & \text{if } r \text{ is odd} \end{array} \tag{7}$$

	2	2	$r-4$	
	2	3	$r-5$	
	$\vdots$	$\vdots$	$\vdots$	
	2	$\frac{r-2}{2}$	$\frac{r-2}{2}$	if r is even
(8)	2	$\frac{r-1}{2}-1$	$\frac{r-2}{2}$	if r is odd

	3	3	$r-6$	
	3	4	$r-7$	
	$\vdots$	$\vdots$	$\vdots$	
	3	$\frac{r-2}{2}-1$	$\frac{r-2}{2}$	if r is even
(9)	3	$\frac{r-3}{2}$	$\frac{r-3}{2}$	if r is odd

Also the partition ends with any one of the following pattern:
If r is divisible by 3, then the partition ends with

$$(r_1) \qquad \frac{r}{3} \qquad \frac{r}{3} \qquad \frac{r}{3}$$

Suppose $r-1$ is divisible by 3, then the partition ends with

$$(r_2) \qquad \frac{r-1}{3} \qquad \frac{r-1}{3} \qquad \frac{r-1}{3}+1$$

Suppose $r-2$ is divisible by 3, then the partition ends with the following two patterns:

$$\frac{r-2}{3} \qquad \frac{r-2}{3} \qquad \frac{r-2}{3}+2$$

$$(r_3) \qquad \frac{r-2}{3} \qquad \frac{r-2}{3}+1 \qquad \frac{r-2}{3}+1$$

Let $d(i)$ denotes the number of domination sets of $AG_E(Z_n)$ of type equation (i). Now

$$d(7) = \begin{cases} p(k)\binom{r}{1}\sum_{i=1}^{\frac{r}{2}-1}\binom{r-1}{i} & \text{if } r \text{ is even} \\ p(k)\binom{r}{1}\sum_{i=1}^{\frac{r-1}{2}}\binom{r-1}{i} & \text{if } r \text{ is odd} \end{cases}$$

$$d(8) = \begin{cases} p(k)\binom{r}{2}\sum_{i=1}^{\frac{r-2}{2}}\binom{r-2}{i} & \text{if } r \text{ is even} \\ p(k)\binom{r}{2}\sum_{i=1}^{\frac{r-1}{2}-1}\binom{r-2}{i} & \text{if } r \text{ is odd} \end{cases}$$

$$d(9) = \begin{cases} p(k)\binom{r}{3}\sum_{i=1}^{\frac{r-2}{2}-1}\binom{r-3}{i} & \text{if } r \text{ is even} \\ p(k)\binom{r}{3}\sum_{i=1}^{\frac{r-3}{2}}\binom{r-3}{i} & \text{if } r \text{ is odd} \end{cases}$$

and

$$d(r_1) = p(k)\binom{r}{\frac{r}{3}}\binom{r-\frac{r}{3}}{\frac{r}{3}}$$
$$d(r_2) = p(k)\binom{r}{\frac{r-1}{3}}\binom{r-\frac{r-1}{3}}{\frac{r-1}{3}}$$
$$d(r_3) = p(k)\binom{r}{\frac{r-2}{3}}\left(\binom{r-\frac{r-2}{3}}{\frac{r-2}{3}} + \binom{r-\frac{r-2}{3}}{\frac{r-2}{3}+1}\right)$$

Now $d(AG_E(\mathbb{Z}_n), 3) = d(7) + d(8) + \ldots + d(r_i)$ for some i, $1 \leq i \leq 3$. Similarly we can calculate $d(AG_E(\mathbb{Z}_n), t)$, for $t \geq 4$.

Now $\mathbb{D}(AG_E(\mathbb{Z}_n), r)$ has the elements $\{[p_1^{l_1}], [p_2^{l_2}], \ldots, [p_r^{l_r}], 1 \leq i_i \leq k_i, 1 \leq i \leq r\}$. Hence $d(AG_E(\mathbb{Z}_n), r) = p(k)$.

Also adding any set of t vertices (other than the vertices in $\mathbb{D}(AG_E(\mathbb{Z}_n), r)$) with a set of $\mathbb{D}(AG_E(\mathbb{Z}_n), r)$ will produce a dominating set of size $r + t, 1 \leq t \leq n - r$. Hence

$$d(AG_E(\mathbb{Z}_n), r+1) = p(k)\binom{n-p(k)}{1}$$
$$d(AG_E(\mathbb{Z}_n), r+2) = p(k)\binom{n-p(k)}{2}$$
$$d(AG_E(\mathbb{Z}_n), n-2) = p(k)\binom{n-p(k)}{n-p(k)-2}$$

Also any set of $n-1$ vertices of $AG_E(\mathbb{Z}_n)$ is a dominating set. Hence $d(AG_E(\mathbb{Z}_n), n-1) = n$ and $d(AG_E(\mathbb{Z}_n), n) = 1$. Now substituting the values of $d(AG_E(\mathbb{Z}_n), i), 1 \leq i \leq n$ in equation (6), we get the domination polynomial of $AG_E(\mathbb{Z}_n)$.

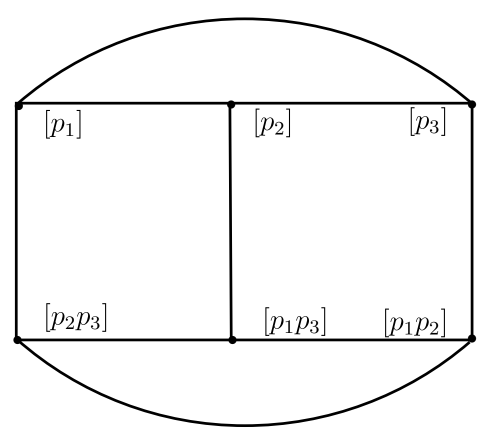

Figure 1: $AG_E(\mathbb{Z}_{p_1p_2p_3})$

3. Planarity of $AG_E(\mathbb{Z}_n)$

A graph is called outer planar if the graph can be drawn in the plane without any crossings so that all the vertices should be in the outer face of the embedding. A planar graph is the one which can be embedded on the plane in a manner that no two edges cannot intersect except at their starting point. In this section we characterize the values of n for which $AG_E(\mathbb{Z}_n)$ is outer planar or planar.

Theorem 3.1. [7, Theorem 1] A graph G is outer planar if and only if it contains no subgraph homeomorphic to $K_{2,3}$ or K_4.

Theorem 3.2. *$AG_E(\mathbb{Z}_n)$ is outer planar if and only if $n = p_1^{k_1}, 2 \leq k_1 \leq 4, p_1p_2$ or $p_1^2p_2$*

Proof. We know that, $AG_E(\mathbb{Z}_{p_1p_2}) \cong K_2$ and $AG_E(\mathbb{Z}_{p_1^2p_2}) \cong C_4$. Then the proof is clear by Remark 2.2 and Theorem 3.1.
Suppose $AG_E(\mathbb{Z}_n)$ is outer planar. Let $r \geq 3$ with $k_i \geq 1 \; \forall \; 1 \leq i \leq r$. Then the collection of vertices $\{[p_1], [p_2], [p_3], [p_1p_2], [p_1p_3], [p_2p_3]\}$ will form a subdivision of $K_{2,3}$, which is a contradiction. Hence $r \leq 2$.
Suppose $k_1 \geq 3$. Then the $\{[p_1], [p_2], [p_1^2], [p_1^3], [p_1p_2], [p_1^2p_2]\}$ will form a subdivision of $K_{2,3}$, which is a contradiction. Hence $k_1, k_2 \leq 2$. Suppose $k_1 = k_2 = 2$, then also the following collection $\{[p_2], [p_1^2], [p_2^2], [p_1p_2], [p_1p_2^2]\}$ will yield a $K_{2,3}$. Hence $n = p_1p_2$ or $p_1^2p_2$ □

Theorem 3.3. [4, Kuratowski] *A graph G is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.*

For a non-negative integer n, let S_n denote the sphere with n handles. The *genus* of a graph G, denoted by $g(G)$, is the minimum integer n such that G can be embedded in S_n.

Theorem 3.4. ([4, Theorem 4.4.4]) Let G be a connected graph with $n \geq 3$ vertices and q edges. Then $g(G) \geq \lceil \frac{q}{6} - \frac{n}{2} + 1 \rceil$

Theorem 3.5. *$AG_E(\mathbb{Z}_n)$ is planar if and only if $n = p_1^5$ or $p_1p_2p_3$*

Proof. Suppose $r \geq 4$. Then the collection of vertices $\{[p_1], [p_2], [p_3], [p_4], [p_1p_3], [p_2p_4]\}$ will form a $k_{3,3}$, which is a contradiction. Hence $r \leq 3$.
Case 1: $r = 3$
Suppose any one of k_i, say $k_1 \geq 2$. Then the subgraph $AG_E(\mathbb{Z}_{p_1^2p_2p_3})$ of $AG_E(\mathbb{Z}_n)$ contains 10 vertices and 29 edges. Then by Theorem 3.4, $g(AG_E(\mathbb{Z}_{p_1^2p_2p_3})) \geq 1$, which is a conflict. Hence $n = p_1p_2p_3$.
Case 2: $r = 2$
Suppose $k_1 \geq 3$, then the collection $\{[p_1], [p_2], [p_1p_2], [p_1^2], [p_1^3], [p_1^2p_2]\}$ will produce a $K_{3,3}$,

a contradiction. Hence $k_1 \leq 2$. Suppose $k_1 = k_2 = 2$ then also the set of vertices $\{[p_1], [p_2], [p_1^2], [p_2^2], [p_1^2 p_2], [p_1 p_2^2]\}$ will generate a $K_{3,3}$, which is a mismatch.

Case 3: $r = 1$

The proof is straightforward by Theorem 3.3 and Remark 2.2. Also the converse is clear by Figure 1. □

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References

[1] S. Alikhani and Y. H. Peng, *Introduction to domination polynomial of a graph*, Ars Combin., **114** (2014) 257-266.

[2] D. F. Anderson and J. D. LaGrange, *Commutative Boolean Monoids, reduced rings and the compressed zero-divisor graphs*, J. Pure and Appl. Algebra, **216** (2012) 1626-1636

[3] I. Beck, *Coloring of commutative rings*, J. Algebra, **116** (1988) 208-226.

[4] B. Mohar and C. Thomassen, *Graphs on Surfaces*, The Johns Hopkins University Press, Baltimore and London, 1956.

[5] J. A. Bondy and U. S. R. Murty, *Graph Theory and its Applications*, American Elsevier, New York, 1976.

[6] D. S. Dummit and R. M. Foote, *Abstract Algebra*, Second edition, Wiley Student edition, New Jersey, 2008.

[7] M. M. Syslo, *Characterizations of Outer planar graphs*, Discrete Math., **26** No. 1 (1979) 47-53.

[8] S. Payrovi and S. Babaei, *The compressed annihilator graph of a commutative ring*, Indian J. Pure Appl. Math., **49** No. 1 (2018) 177-186

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