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Research Paper

# SOME RESULTS OF $\alpha$-COSET GROUPS

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Abstract. We call $G$ to be an $\alpha$-coset group, if it contains a proper $\alpha$-invariant normal subgroup $N$ such that $Nx^{\alpha} = \{x^g \mid g \in G\}$, for some automorphism $\alpha$ of $G$ and any $x \in G \setminus N$. Clearly, if $\alpha$ is identity automorphism of $G$, one obtains the notion of con-cos groups, which was first introduced by Muktibodh in 2006.

In the present article, we discuss some properties of the new notion. Also, we introduce the concept of $\alpha$-Camina groups and give its connection with the groups of property $\mathcal{P}$, where $\mathcal{P}$ is the class of all finite groups such that their $\alpha$-centres are the same as $\alpha$-commutator subgroups of order $p$.

## 1. Introduction and Preliminaries

In 2006, A.S. Muktibodh in [6], defined the concept of conjugate coset (henceforth con-cos) group and showed that for such a group $G$, its centre has order at most 2. In 2010, L. Cangelmi

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and A.S. Muktibodh in [2] gave a relationship between con-cos groups and Camina groups, which was introduced by A.R. Camina in [1]. They showed that such groups are all Frobenius or extra special $p$-groups.

In [3], we introduced the concept of *right* and *left* $\alpha$-*commutator*, as follows:

**Definition 1.1.** For arbitrary elements $x$ and $y$ in a given group $G$ and $\alpha \in \mathrm{Aut}(G)$, we say $x$ and $y$ commute under the automorphism $\alpha$ whenever $yx = xy^{\alpha}$ or $y^{\varphi_x} = y^{\alpha}$, where $\varphi_x$ is the inner automorphism induced by $x$.

Moreover, $[x, y]_{\alpha} = x^{-1}y^{-1}xy^{\alpha}$ is called right $\alpha$-commutator of $x$ and $y$. Also, ${}_{\alpha}[x, y] = (x^{-1})^{\alpha}y^{-1}xy$ is called left $\alpha$-commutator of $x$ and $y$.

For $n \geqslant 3$, we may define inductively right and left $\alpha$-commutator of weight $n$ as follows:

$$[x_1, x_2, \cdots, x_n]_{\alpha} = [[x_1, x_2, \cdots, x_{n-1}]_{\alpha}, x_n]_{\alpha},$$

$${}_{\alpha}[x_1, x_2, \cdots, x_n] = {}_{\alpha}[{}_{\alpha}[x_1, x_2, \cdots, x_{n-1}], x_n],$$

for all $x_i \in G$ and $1 \leqslant i \leqslant n$. It is clear that, if $\alpha$ is the identity automorphism of $G$ or $x_i$'s are in $C_G(\alpha)$ then we have ordinary commutator $[x_1, x_2, \cdots, x_n]$, where

$$C_G(\alpha) = \{x \in G \mid [x, \alpha] = x^{-1}x^{\alpha} = x^{-1}\alpha(x) = 1\},$$

is the centralizer of $\alpha$ in $G$.

For a given group $G$ and automorphisms $\alpha$ and $\beta$ in $\mathrm{Aut}(G)$ we consider, $\alpha^{\beta} = \beta^{-1}\alpha\beta$. The following lemma is very useful in our further investigations.

**Lemma 1.2.** *Let $x, y$ and $z$ be elements of a group $G$ and $\alpha, \beta \in \mathrm{Aut}(G)$. Then the following identities hold:*
*(i)* $[x, x]_{\alpha} = [x, \alpha]$;
*(ii)* $[x, y]_{\alpha} = [x, y][y, \alpha]$;
*(iii)* $[x, y]_{\alpha}^{\alpha} = [x^{\alpha}, y^{\alpha}]_{\alpha}$;
*(iv)* $[x, y^{-1}]_{\alpha} = [x, y]_{\alpha}^{-(y^{\alpha})^{-1}}$;
*(v)* $[x, y]_{\alpha^{\beta}}^{\beta} = [x^{\beta}, y^{\beta}]_{\alpha}$;
*(vi)* $[xy, z]_{\alpha} = [x, z]_{\alpha}^{y}[y, z^{\alpha}]$;
*(vii)* $[x, yz]_{\alpha} = [x, z]_{\alpha}[x, y]_{\alpha}^{z^{\alpha}}$;
*(viii)* $[[x, y^{-1}]_{\alpha}, z]_{\alpha}^{y^{\alpha}} = [x, y^{-1}, z]^{y}[z^{y}, \alpha]$.

*Proof.* All parts follow using the definition of right $\alpha$-commutator and the above notation. □

One can easily see that $[x, y]_{\alpha}^{-1} = {}_{\alpha}[y, x]$, hence we may state similar relations, as the above lemma, for left $\alpha$-commutator. Here we work with right $\alpha$-commutators in the rest of article.

Now, we recall that the following subgroup is called *$\alpha$-centre* of the group $G$

$$Z^{\alpha}(G) = \{y \in G \mid [x, y]_{\alpha} = 1, \forall x \in G\}.$$

Consider $y \in Z^{\alpha}(G)$, the above definition and Lemma 1.2 (i) imply that $[y, y]_{\alpha} = [y, \alpha] = 1$, and so $y \in C_G(\alpha)$. On the other hand, Lemma 1.2 (ii) implies

$$[x, y]_{\alpha} = [x, y][y, \alpha] = 1 \Rightarrow [x, y] = 1,$$

for all $x \in G$, which implies that $y \in Z(G)$. Therefore, $Z^{\alpha}(G) \subseteq Z(G) \cap C_G(\alpha)$, and hence $Z^{\alpha}(G) = Z(G) \cap C_G(\alpha)$. Note that the inclusion of $Z^{\alpha}(G)$ in $Z(G)$ restricted for many groups, as the following example shows.

One note that, if $y \in Z^{\alpha}(G)$ then for all $x \in G$, we have $[x, y]_{\alpha} = 1$ while $[y, x]_{\alpha} \neq 1$.

**Example 1.3.** Consider the direct product of Dihedral group $D_8 = \langle a, b \ : \ a^4 = b^2 = 1, a^b = a^{-1} \rangle$ of order 8 and the cyclic group $\mathbb{Z}_3 = \langle x \ : \ x^3 = 1 \rangle$.

Clearly, $\mathrm{Aut}(D_8 \times \mathbb{Z}_3) = \mathrm{Aut}(D_8) \times \mathrm{Aut}(\mathbb{Z}_3)$, take the automorphism $\theta = (\alpha, \beta)$ where $\alpha : a \mapsto a^3, b \mapsto a^2 b$ and $\beta : x \mapsto x^2$. Then one can easily seen that $Z^{\theta}(D_8 \times \mathbb{Z}_3) = \{1, a^2\} \subsetneqq Z(D_8 \times \mathbb{Z}_3) = \{1, a^2, x, x^2, a^2 x, a^2 x^2\}$.

Also, we define *$\alpha$-commutator* subgroup of $G$ as follows

$$K^{\alpha}(G) = \langle [x, y]_{\alpha} \mid x, y \in G \rangle.$$

Clearly, Lemma 1.2 (i) and (ii) imply that $G' \subseteq K^{\alpha}(G) \subseteq K(G)$, where $K(G)$ is the autocommutator subgroup of $G$ (see [4]). Note that, Lemma 1.2 (v) implies that $K^{\alpha}(G)$ is an $\alpha$-invariant subgroup of $G$.

The following example shows that $K^{\alpha}(G)$ may be much different from $KG)$, for a fixed automorphism $\alpha$ of a given group $G$.

**Example 1.4.** Consider the automorphism $\alpha : a \mapsto a^3, b \mapsto b$ of Dihedral group $D_8 = \langle a, b \ : \ a^4 = b^2 = 1, a^b = a^{-1} \rangle$ of order 8. Then one can calculate that $K^{\alpha}(D_8) = \{1, a^2\} \subsetneqq K(D_8) = \{1, a, a^2, a^3\}$.

In the present article, we consider finite groups and introduce the concepts of $\alpha$-coset and $\alpha$-Camina groups, which are the generalizations of the notions of con-cos and Camina groups. We then discuss some of their properties and also give their connections with the groups which have the property $Z^{\alpha}(G) = K^{\alpha}(G)$.

## 2. Main Results

In this section, we first introduce the notion of $\alpha$-coset groups, then we give their connections with groups which have the property $Z^{\alpha}(G) = K^{\alpha}(G)$.

**Definition 2.1.** A finite group $G$ is said to be an $\alpha$-coset group, if there exists a proper $\alpha$-invariant normal subgroup $N$ of $G$ such that

$$Nx^{\alpha} = cl(x) = \{x^g = g^{-1}xg \mid g \in G\},$$

for all $x \in G \setminus N$.

One observes that, if we consider the identity automorphism of a group $G$, then the above definition gives $Nx = cl(x)$, the conjugacy class of $x$ in $G$, which is the notion of con-cos groups and it was introduced by A.S. Muktibode in [6].

A.S. Muktibode showed that if $G$ is a con-cos group with a normal subgroup $N$ then $N = G'$.

The following theorem states that the $\alpha$-invariant normal subgroups of a given group $G$, which satisfy the condition of $\alpha$-coset groups are only the $\alpha$-commutator subgroups.

**Theorem 2.2.** *Let $G$ be a finite group and let $\alpha : G \to G$ be a group automorphism. If $G$ is an $\alpha$-coset group with respect to some proper $\alpha$-invariant normal subgroup $N$, then $N = K^{\alpha}(G)$. In particular, $G$ is an $\alpha$-coset group with respect to a unique proper $\alpha$-invariant normal subgroup.*

*Proof.* Let $G$ be an $\alpha$-coset group, then there exists an $\alpha$-invariant subgroup $N$ of $G$ such that $Nx^{\alpha} = \{g^{-1}xg \mid g \in G\}$, for all $x \in G \setminus N$. Then, for any element $n \in N$, there exists $g_0 \in G$ such that $nx^{\alpha} = g_0^{-1}xg_0$ and so $n = g_0^{-1}xg_0x^{-\alpha} = [g_0, x^{-1}]_{\alpha} \in K^{\alpha}(G)$. Hence $N \subseteq K^{\alpha}(G)$.

Conversely, by the definition of $\alpha$-coset group for any $g \in G \setminus N$, $ng^{\alpha} = x^{-1}gx$ for some $x \in G$, which implies that $[x, g^{-1}]_{\alpha} \in N$. On the other hand, if $g \in N$ then $[x, g]_{\alpha} = x^{-1}g^{-1}xg^{\alpha} = g^{-x}g^{\alpha} \in N$. Thus $K^{\alpha}(G)$ is contained in $N$ and so $N = K^{\alpha}(G)$. □

The following corollary gives a condition that the $\alpha$-commutator subgroup of a given group is equal to the set of $\alpha$-commutator elements.

**Corollary 2.3.** *If $G$ is an $\alpha$-coset group, then*

$$K^{\alpha}(G) = \{[x, g^{-1}]_{\alpha} \mid x \in G, g \in G \setminus K^{\alpha}(G)\}.$$

**Example 2.4.** The Dihedral group $D_8 = \langle a, b \mid a^4 = 1, b^2 = 1, a^b = a^{-1}\rangle$ is an $\alpha$-coset group, where $(\alpha : a \mapsto a^3; b \mapsto b)$.

Clearly, $N = \{1, a^2\}$ is the required $\alpha$-invariant normal subgroup. Since for any $x \in D_8 \setminus N = \{a, a^3, b, ab, a^2b, a^3b\}$, we have

$$Na^{\alpha} = N(a^3)^{\alpha} = cl(a) = \{a, a^3\};$$

$$Nb^{\alpha} = N(a^2b)^{\alpha} = cl(b) = \{b, a^2b\};$$

$$N(ab)^{\alpha} = N(a^3b)^{\alpha} = cl(ab) = \{ab, a^3b\}.$$

Moreover $N = K^{\alpha}(D_8)$.

The following result states that in a non-abelian $\alpha$-coset group, the $\alpha$-commutator subgroup contains the $\alpha$-centre of the group.

**Theorem 2.5.** *If $G$ is a non-abelian $\alpha$-coset group, then the $\alpha$-centre of $G$ is contained in $\alpha$-commutator subgroup of $G$.*

*Proof.* Let $x$ be non-trivial element in $Z^{\alpha}(G)$. Now, if $x \notin K^{\alpha}(G)$, then $K^{\alpha}(G)x^{\alpha} = \{x^g \mid g \in G\}$. Hence for every $k \in K^{\alpha}(G)$, we have $k = x^g(x^{-1})^{\alpha} = [g, x^{-1}]_{\alpha} = 1$, which implies $K^{\alpha}(G)$ is trivial. Therefore, $G' = \langle 1 \rangle$, which contradicts the assumption. □

Note that the above result does not hold for abelian groups, since for the $\alpha$-coset group $\mathbb{Z}_2$, we have $Z^{\alpha}(\mathbb{Z}_2) = \mathbb{Z}_2$, while $K^{\alpha}(\mathbb{Z}_2) = \langle 1 \rangle$.

It is clear that, the subgroup $K^{\alpha}(G)$ of a group $G$ does not form a normal subgroup for any $\alpha \in \mathrm{Aut}(G)$. In the following we give a condition, under which $K^{\alpha}(G)$ is a normal subgroup of $G$.

One may define the action of a group $G$ on $\mathrm{Aut}(G)$ given by $\alpha^g = \alpha^{\varphi_g} = \varphi_{g^{-1}} \circ \alpha \circ \varphi_g$ and the action of $\mathrm{Aut}(G)$ on $G$ given by $g^{\alpha} = \alpha(g)$, for all $g \in G$, $\alpha \in \mathrm{Aut}(G)$ and $\varphi_g \in \mathrm{Inn}(G)$, (see also [5]).

Also, if $x^{-1}\alpha(x) \in Z(G)$ for all $x \in G$, then we say that $\alpha$ is a *central automorphism* of $G$. The set of all such automorphisms is denoted by $\mathrm{Aut_Z}(G)$, and it is clear that, central automorphisms commute with inner automorphisms of the group $G$.

Now, we give a condition under which $K^{\alpha}(G)$ is a normal subgroup of the group $G$.

**Theorem 2.6.** *Let $\alpha$ be a central automorphism of a given group $G$, then $K^{\alpha}(G)$ is a normal subgroup of $G$.*

*Proof.* Assume that $\alpha$ is a central automorphism of $G$ and for any $x, y, g \in G$, Lemma 1.2 (ii) implies that

$$\begin{aligned} [x,y]_{\alpha}^{g} = [x,y]^{g}[y,\alpha]^{g} &= [x^g, y^g][y^g, \alpha^{\varphi_g}] \\ &= [x^g, y^g][y^g, \alpha] \\ &= [x^g, y^g]_{\alpha}, \end{aligned}$$

as $\mathrm{Aut_Z}(G) = C_{\mathrm{Aut}(G)}(\mathrm{Inn}(G))$. Hence, $K^{\alpha}(G) \trianglelefteq G$. □

Note that in Example 2.4, the automorphism $\alpha$ is central.

Now, we introduce the notion of $\alpha$-Camina groups and give their relationships with $\alpha$-coset groups.

**Definition 2.7.** Let $G$ be a finite group with non-trivial $\alpha$-invariant normal subgroup $N$. Then we call $(G, N)$ to be an $\alpha$-Camina pair, when $Nx^\alpha \subseteq \{x^g \mid g \in G\}$, for all $x \in G \setminus N$.

It is clear that if $(G, N)$ is an $\alpha$-Camina pair with $\alpha$-invariant subgroup $K$ of $G$ which is contained in $N$, then $(G/K, N/K)$ is also $\bar{\alpha}$-Camina pair. The group $G$ is said to be $\alpha$-Camina group, if $(G, K^\alpha(G))$ is $\alpha$-Camina pair. Clearly, if one considers the identity automorphism of $G$, one obtains the concepts of Camina pairs and Camina groups. Clearly, every non-abelian $\alpha$-coset groups are $\alpha$-Camina groups and every $\alpha$-Camina groups are $\alpha$-coset groups.

**Lemma 2.8.** *If $(G, N)$ is an $\alpha$-Camina pair, then $Z^\alpha(G)$ is contained in $N$. In particular, if $G$ is an $\alpha$-Camina group then $K^\alpha(G)$ contains $Z^\alpha(G)$.*

*Proof.* Assume there exists an element $x \in Z^\alpha(G)$, which is not in $N$. Then by the assumption $Nx^\alpha \subseteq \{x^g \mid g \in G\}$. Hence for every $n \in N$, $nx^\alpha = x^g$ for some $g \in G$. Therefore $n = [g, x^{-1}]_\alpha = 1$, and hence the normal subgroup $N$ is trivial, which is a contradiction. □

In the following result, we state the condition under which a con-cos group $G$ is an $\alpha$-coset group.

**Theorem 2.9.** *Let $G$ be a con-cos group with a normal subgroup $N$. If $N$ is an $\alpha$-invariant subgroup of index 2, then $G$ is $\alpha$-Camina group.*

*Proof.* If $g \in N$ it is clear that $[x, g]_\alpha = g^{-x}g^\alpha \in N$. If $g \notin N$, then $g^\alpha \notin N$. Since $N$ is $\alpha$-invariant subgroup of index 2 in $G$, then one can easily see that $Ng = Ng^\alpha$ which implies that $K^\alpha(G) \subseteq K(G) \subseteq N$. Hence by the assumption $K^\alpha(G)x^\alpha \subseteq Nx^\alpha = x^G$, for any $x \in G \backslash K^\alpha(G)$ and so $G$ is $\alpha$-Camina group. □

Now, as in [6] we introduce the concept of $n$-$\alpha$-Camina group.

**Definition 2.10.** A finite group $G$ is said to be $n$-$\alpha$-Camina, if the following conditions hold:
($i$) $K^\alpha(G)x^\alpha \subseteq \{x^g \mid g \in G\}$, for all $x \in G \setminus K^\alpha(G)$;
($ii$) $K^\alpha(G) = \{1\} \bigcup_{i+1}^{n-1} cl(x_i)$, for some $x_1, \cdots, x_{n-1} \in G$.

Clearly, each $n$-$\alpha$-coset group is $\alpha$-coset but the converse is not true in general. For a counter example, one can easily check that the dihedral group of order 8, $D_8$, is $\alpha$-coset, which is not 2-$\alpha$-coset group. It is also easy to see that $\mathbb{Z}_p$ ($p$ is a prime), $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $S_3$ are 2-$\alpha$-coset groups.

**Theorem 2.11.** *Let $G$ be an $\alpha$-coset group with $Z^\alpha(G) = K^\alpha(G)$ and $|Z^\alpha(G)| = p$, where $p$ is a prime number. Then $G$ is a $p$-$\alpha$-Camina group.*

*Proof.* Theorem 2.2 guarantees the first condition of Definition 2.10. Now, if $x \in Z^{\alpha}(G)$, the assumption implies

$$Z^{\alpha}(G) = K^{\alpha}(G) = \{1, x, x^2, \cdots, x^{p-1}\} \cong \mathbb{Z}_p.$$

Clearly $\alpha(x^i) = x^i$, for all $i = 1, 2, \cdots, p-1$. So $K^{\alpha}(G) = \{1\} \cup \{x\} \cup \cdots \cup \{x^{p-1}\}$, which gives the second condition of $p$-$\alpha$-Camina group. □

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