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Research Paper

SUMS OF UNITS IN BAER AND EXCHANGE RINGS

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Abstract. In this paper, we prove that every element in an exchange ring R with artinian primitive factors is n-tuplet-good iff 1_R is n-tuplet-good. Also, we show that for such rings the full matrix ring $M_n(R)$ (for $n \geq 2$) is n-tuplet-good. In [7], Fisher and Snider proved that every element of a strongly π-regular ring R with $\frac{1}{2} \in R$ is 2-good. We prove the same result under new condition and show that these rings are twin-good. We also consider the conditions under which endomorphism ring of a finitely generated projective module M over unit regular ring L is 2-tuplet-good. The main result of [14] states that regular self-injective rings are n-tuplet-good if such rings has no factor ring isomorphic to a field D with $|D| < n+2$. We generalized this result to regular Baer rings proving that every regular Baer ring R that has no factor ring isomorphic to a field of order less than $n+2$, is n-tuplet-good.

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1. Introduction

The study of rings whose all of its elements can be written as sum of invertible elements was started in 1953-1954 [22, 24]. Many authors have studied this rings and provided interesting results [1, 2, 19]. We refer the reader to [20] as a survey of this field. Henriksen in [10] introduced some classes of rings that elements can be written as the sums of exactly k invertible elements, he called such rings "(s,k)-*rings*". These rings were renamed "*k-good*" by Vámos in 2005 [21]. Following [17], a ring R is called *twin-good* if for every element a of R, there exists a unit $v \in U(R)$ such that $a \pm v$ are units in R. Although twin-good rings are 2-good, the converse is not true. For example $\mathbb{Z}_3$ is a 2-good ring but is not twin-good. In [1] we discussed this concept and we demonstrated every regular Baer ring that has no factor ring isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$ is twin-good. This definition led to a new concept that many authors were introduced in [14]. They said that for an integer $n \geq 1$, an element $a \in R$ is *n-tuplet-good* if for any set $\{u_1, u_2, \dots, u_n\}$ of central units, there exists a unit u in R such that $a + u_i u$ is a unit in R for each $i = 1, \dots, n$. A ring R is n-tuplet-good, if all of its elements are n-tuplet-good. For example, if D is a field with $|D| > n+1$, then D is n-tuplet-good. On the other hand, it is obvious that every 1-tuplet-good ring is specifically a 2-good ring. In [5, Theorem 3], Chen proved that each element of an exchange ring R with artinian primitive factor is 2-good iff 1_R is 2-good. In [4], he extend this result and proved that such rings are twin-good if and only if 1_R is twin-good.

A ring R is called an *exchange* ring if, for every right R-module A and any two decompositions $A = M \bigoplus N = \bigoplus_{i \in I} A_i$, where $M_R \cong R_R$ and I is a finite index set, there exist submodules $A_i^{'} \subseteq A_i$ such that $A = M \bigoplus (\bigoplus_{i \in I} A_i^{'})$. In the rest of this paper, we first derive some conditions under which exchange rings are n-tuplet-good. In other words, we prove that for an integer $n > 1$, any exchange ring R with artinian primitive factor is n-tuplet-good iff 1_R is n-tuplet-good. Furthermore, we show that if R is exchange, then $M_n(R)$ is n-tuplet-good for $n \geq 2$. It is renowned that every element of a strongly $\pi-$regular ring R is sum of two units if $\frac{1}{2} \in R$. Chen in [4] proved that even if 2 is not unit, this result is true. Here, we show that every strongly $\pi-$regular ring is twin-good, provided that 1_R is 2-tuplet-good. We also investigate some conditions under which the endomorphism ring of a finitely generated projective module over a unit regular ring is 2-tuplet-good. Following Kaplansky [11], a ring R is called *Baer* if for every non-empty subset S of R there exists an idempotent e such that $Ann_l(S) = Re$. The result of [14] show that under special condition every self-injective ring is n-tuplet-good. In this paper, we will generalize this result to Baer rings. We also investigate the needed requirements to prove that any regular Baer ring is n-tuplet-good. We conclude the paper by examining this concept for continuous modules and some classes of vector spaces.

Throughout this paper, all rings are assumed to be associative with identity. For a ring R, we write $U(R)$ for the group of units, and $|D|$ denotes the cardinality of a ring D.

2. Main results

Before proving the main results, we first give two lemmas that will be used later.

Lemma 2.1. *Let D be a division ring. Then for any $a \in D$ and every set of n elements $\{v_1, \dots, v_n\}$ in D^*, there exists a $v \in D^*$ such that $a + v_i v \in D^*$ iff 1 is n-tuplet-good.*

Proof. For any $a \in D$, if $a = 0$ then $a + v_i v \in D^*$. If $a \neq 0$, since 1 is n-tuplet-good there exists a $v \in D^*$ such that $1 + v_i v \in D^*$, thus $a + v_i a v \in D^*$. The converse is trivial.

Recall that a ring R is said to be an *elementary divisor ring* if for any $A \in M_n(R)$ there exist two units $P, Q \in GL_n(R)$ such that PAQ is a diagonal matrix.

Lemma 2.2. *Let D be a division ring and $n \geq 2$. Then $M_n(D)$ is n-tuplet-good.*

Proof. Since D is elementary divisor ring, if $A \in M_n(D)$ for $n \geq 2$ there exist invertible matrices U, V such that UAV is a diagonal matrix. By elementary row operations, $UAV = diag(I_m, 0)$ or $UAV = I_{n\times n}$. For every set $\{V_1, \dots, V_n\}$ of central units in $M_n(D)$, we know that there exists a $v_i \in Z(D)$ such that $V_i = v_i I$ for any $1 \leq i \leq n$. If $UAV = diag(I_m, 0)$ for the standard matrix units $e_{i,j}$, put $W = 0_{n,n} + e_{2,1} + e_{3,2} + \cdots + e_{n,n-1} + e_{1,n}$, where $0_{n,n}$ denotes the zero square matrix of order n. It is obvious that $W \in GL_n(D)$ and $UAV + V_i W \in GL_n(D)$. Therefore, in this case A is n-tuplet-good.

In the second case, let $UAV = I_{n\times n}$. Suppose that n is an even number, if for any $1 \leq i \leq n$, $v_i^n \neq -1$ put $W = 0_{n,n} + e_{2,1} + e_{3,2} + \cdots + e_{n-1,n-2} - e_{n,n-1} + e_{1,n} \in GL_n(D)$, thus $I_{n\times n} + V_i W \in GL_n(D)$. If n is an odd number for any $1 \leq i \leq n$, $v_i^n \neq -1$, put $W = 0_{n,n} + e_{2,1} + e_{3,2} + \cdots + e_{n-1,n-2} - e_{n,n-1} - e_{1,n} \in GL_n(D)$. Thus, in both cases, $I_{n\times n} + V_i W \in GL_n(D)$. Therefore, A is n-tuplet-good. To complete the proof, if $v_i^n \neq -1$ for some i and $n \geq 2$, we choice $w \in \{v_1, \dots, v_n\}$ such that $w \neq (v_i^n + 1)/v_i$, in these two cases put $W = 0_{n,n} - e_{2,1} - e_{3,2} + \cdots - e_{n-1,n-2} + e_{n,n-1} + e_{1,n} + w e_{n-1,n-1}$. Since $W \in GL_n(D)$ so $UAV + V_i W \in GL_n(D)$, the desired result is obtained.

Theorem 2.3. *Suppose R is an exchange ring for which any primitive factor ring is artinian. Then R is n-tuplet-good if and only if 1_R is n-tuplet-good.*

Proof. We prove by using a similar technique as used in [14, Lemma 3]. Assume that R is not n-tuplet-good, then there exists some element a and some set $\{v_1, \dots, v_n\}$ of central units in R, such that for any unit element v of R, $a + v_i v \notin U(R)$. Let $\Phi = \{I \trianglelefteq R | \overline{a + v_i v} \notin U(R/I)$ for any $\overline{v} \in U(R/I)\}$. Since $\Phi \neq \varnothing$, for an arbitrary chain of ideals in Φ like $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$, set $J := \bigcup I_i$. We claim that $J \in \Phi$. If not, there is a $\overline{v} \in U(R/J)$ such that $\overline{a + v_i v} \in U(R/J)$.

So there are v', v_i', u in R such that $\overline{vv'} = \overline{v'v} = \overline{1}$, $\overline{a + v_i v}\overline{u} = \overline{u}\overline{a + v_i v} = \overline{1}$ and $\overline{v_i v_i'} = \overline{v_i' v_i} = \overline{1}$ in R/J for $i = 1, \ldots, n$. Thus, we can find $k, l, m, n, s, t \in \mathbb{N}$ such that $\overline{vv'} = \overline{1}$ in R/I_k, $\overline{v'v} = \overline{1}$ in R/I_l, $\overline{a + v_i v}\overline{u} = \overline{1}$ in R/I_m, $\overline{u}\overline{a + v_i v} = \overline{1}$ in R/I_n, $\overline{v_i v_i'} = \overline{1}$ in R/I_n and $\overline{v_i' v_i} = \overline{1}$ in R/I_s. Choose $p = max\{k, l, m, n, t, s\}$ thus $\overline{a + v_i v}, \overline{v}, \overline{v_i}, \overline{u} \in U(R/I_p)$, therefore $I_p \notin \Phi$ this is a contradiction, thus Φ is inductive. By Zorn's lemma, we can fix a maximal ideal M in Φ . Let $S = R/M$, then $S/J(S)$ is an indecomposable ring. As R is an exchange ring such that every primitive factor ring is artinian, thus is $S/J(S)$. Using [23, Lemma 3.7], $S/J(S)$ is a simple artinian ring, so there exists a division ring D such that $S/J(S) \cong M_m(D)$ for every $m \in \mathbb{N}$. For $m = 1$, by assumption, there exsists $\overline{w}, \overline{v_i} \in U(S/J(S))$ such that $\overline{1} + \overline{v_i}\ \overline{w} \in U(S/J(S))$. On the other hand $S/J(S)$ is division ring, so by Lemma 2.1 we can conclude that $\overline{a} + \overline{v_i}\ \overline{w'} \in U(S)$ for some $w' \in U(S)$, which is a contradiction. For $m \geq 2$, by using Lemma 2.2, again we get a contradiction. Thus R is n-tuplet-good.

As an immediate consequense of Theorem 2.3, we get the following result.

Corollary 2.4. *If R is an exchange ring of bounded index, then R is n-tuplet-good iff 1_R is n-tuplet-good.*

Another consequence of Theorem 2.3 is the following result. Note that the following corollary follows also from [14, Lemma 3].

Corollary 2.5. *If R is an exchange ring with all idempotents central, then R is n-tuplet-good iff 1_R is n-tuplet-good.*

Theorem 2.6. *If R is an exchange ring that every primitive factor ring is artinian, then $M_n(R)$ is n-tuplet-good for $n \geq 2$.*

Proof. Since R is an exchange ring that every primitive factor ring is artinian, so is $M_n(R)$. Similar to the proof of Lemma 2.2, we can show that I is n-tuplet-good in $M_n(R)$. Thus, the result follows by Theorem 2.3.

Recall that a ring R is a *strongly π-regular ring* provided that, for any $a \in R$, there exist $n \geq 1$ and $x \in R$ such that $a^n = a^{n+1}x$. Since every commutative strongly π-regular ring is an exchange ring with primitive factor artinian, so we get the following corollary.

Corollary 2.7. *Let R be a commutative strongly π-regular ring. Then R is n-tuplet-good iff 1_R is n-tuplet-good.*

Fisher and Snider [7] proved that every element in a strongly π-regular ring R is 2-good if $2 \in U(R)$. Now, by the next theorem, we conclude that every strongly π-regular ring R is twin-good if 1_R is 2-tuplet-good. Since every twin-good ring is 2-good, so we deduce the same result as that of Fisher and Snider under new conditions. For this purpose, we recall the following useful result [16, Proposition 1].

Lemma 2.8. *An element a in a ring R is strongly π-regular ring if and only if for every element $a \in R$ there exists an integer $m \geq 1$, an idempotent e and a unit u in R such that e, a, u commute with each other and $a^m = eu$.*

Theorem 2.9. *Let R be a strongly π-regular ring. If 1_R is n-tuplet-good, then for every $a \in R$ there exists an idempotent e such that ea is n-tuplet-good.*

Proof. Given any $a \in R$, by Lemma 2.8 there exist $m \in \mathbb{N}$, $e \in Id(R)$ and $v \in U(R)$ such that $a^m = ev$. Also, for any set $\{u_1, \ldots, u_n\}$, there exists a unit u and some units $w_1, \ldots, w_n$ such that, $1 + u_i u = w_i$. Certainly $\{eu_1e, \ldots, eu_ne\}$ are central units in eRe and $e + u_ieue = ew_ie$. On the other hand $(ea)^m = ev$ implies that ea is a unit in eRe and $ea + u_ieuea = ew_iea$. By adding $u_i(1-e)$ to both side, we conclude that $ea + u_i(euea + (1-e)) = ew_iea + u_i(1-e)$ where $euea + (1-e)$ and $ew_iea + u_i(1-e)$ are units in R, with inverse $(euea)^{-1} + (1-e)$ and $(ew_iea)^{-1} + (1-e)$ respectively, while $(euea)^{-1}$ and $(ew_iea)^{-1}$ denote the inverses of $euea$ and ew_iea in $U(eRe)$. So the proof is complete.

Corollary 2.10. *Let R be a strongly $\pi-$regular ring. Then R is twin-good if 1_R is 2-tuplet-good.*

Proof. By Theorem 2.9, for any $a \in R$ there exists $e \in Id(R)$ such that ae is 2-tuplet-good, thus ae is twin-good. Therefore there exist $u, v, w \in U(R)$ such that $ae = -u + v$ and $ae = u + w$. On the other hand, for idempotent $f = 1 - e$ we have $(fa)^n = 0$, so $t := f - fa \in U(fRf)$ and $t' := f + fa \in U(fRf)$. By the proof of Theorem 2.9, for $u_1 = 1$ and $u_2 = -1$, there exists $w, w_1, w_2 \in U(eRe)$ such that $ae + (w + (1-e)) = w_1 + (1-e)$ and $ae - (w + (1-e)) = w_2 - (1-e)$. Thus $a = (-w + f) + (w_1 - t)$ and $a = (w - f) + (w_2 + t')$ while $(-w + f), (w_1 - t)$ and $(w_2 + t')$ are units in R with inverses $(-w^{-1} + f), (w_1^{-1} - t^{-1})$ and $(w_2^{-1} + t'^{-1})$ respectively. Therefore R is twin-good.

In the following, we shall discuss some conditions under which some classes of finitely generated projective modules are n-tuplet-good. For this purpose, we need the following theorem on modules whose endomorphism rings have unit stable range one (See [3, Theorem 2.2.6]). Recall that an element a in a ring R is said to have *stable range one* (sr1, for short) if whenever $Ra + Rb = R$ (for any $b \in R$) there exists $r \in R$ such that $a + rb$ is a unit. If r can be chosen to be a unit, we say that a has *unit stable range one* (usr1, for short).

Theorem 2.11. *Let A be a right R-module and $E = End_R(A)$. Then E has unit stable range one iff for given right R-module decompositions $M = A_1 \bigoplus B_1 = A_2 \bigoplus B_2$ with $A_1 \cong A \cong A_2$, there exists some $C \subseteq M$ such that $M = A_1 \bigoplus C = A_2 \bigoplus C$.*

Theorem 2.12. *Let M be a finitely generated projective right module over a unit-regular ring L, such that $R = End_L(M)$ has unit stable range one. If $Ra + Rb = R$, then for all $u_1, u_2 \in U(R)$, there exists an unit $u \in R$ such that $u_i^{-1}a + ub \in U(R)$.*

Proof. First note that by [8, Corollary 4.7], R is unit-regular. So for any $a \in R$, $Ker(a)$ and $Im(a)$ are both summands of M. Obviously $Ker(u_i^{-1}a) = Ker(a)$ and $I_i := Im(u_ia) = u_iIm(a) \cong Im(a)$, for $i = 1, 2$. From regularity of u_ia we know that I_i is summand of M. Thus by Theorem 2.11 there exists a submodule $C \subseteq M$ such that $M = I_1 \bigoplus C = I_2 \bigoplus C$. On the other hand, $Ra + Rb = R$ implies that $Ker(a) \bigcap Ker(b) = 0$, thus $Ker(a) \cong b(Ker(a))$. Also by [6, Theorem 1], there exists an isomorphism $u \in R$ such that $Im(a) \bigoplus uKer(a) = M$. Now, by [6, Theorem 2], $I_1 \cong Im(a) \cong I_2$ implies that $C \cong uKer(a) \cong Ker(a) \cong b(Ker(a))$. We claim that $u_i^{-1}a + ub$ is unit in R for $i = 1, 2$. First, we show that $u_i^{-1}a + ub$ is epimorphism. For every $x \in Ker(a)$ we have $(u_i^{-1}a + ub)(x) = ub(x)$, thus $(u_i^{-1}a + ub)(Ker(a)) \cong C$. Also, as $ub(M) = ub(Ker(a))$, there exists $x \in ker(a)$ such that $ub(m) = ub(x)$ for any $m \in M$. Therefore, $(u_i^{-1}a + ub)(m - x) = u_i^{-1}a(m)$, whereas $m = (m - x) + x$. So $(u_i^{-1}a + ub)(m) = u_i^{-1}a(m) + (u_i^{-1}a + ub)(x)$. Thus $Im(u_i^{-1}a + ub) \cong I_i \bigoplus C \cong M$, by [8, Theorem 5.2] M is directly finite, so $Im(u_i^{-1}a + ub) = M$. Also, by [8, Theorem 5.4], we conclude that $u_i^{-1}a + ub$ is unit in R, as desired.

Recall that a ring R is called *unit-regular* if for each element $x \in R$ there exists a unit element $u \in R$ such that $xux = x$.

Corollary 2.13. *Let M be a finitely generated projective right module over a unit-regular ring K, and $R = End_K(M)$ has unit stable range one, then R is 2-tuplet-good if R has no factor ring isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.*

Proposition 2.14. *Let K be a regular ring whose primitive factor rings are artinian and M be a finitely generated projective right module over K and $R = End_K(M)$. Then R is 2-tuplet-good if R has no factor ring isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.*

Proof. By [8, Corollary 6.4 and Theorem 6.10], R is unit regular and by [3, Theorem 2.3.5], R has unit stable range one. So, by Corollary 2.12, R is 2-tuplet-good and the result follows.

Corollary 2.15. *Let R be a regular ring with primitive factor rings are artinian. Then R is 2-tuplet-good if R has no factor ring isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.*

Corollary 2.16. *Let R be a regular ring of bounded index. Then R is 2-tuplet-good if R has no factor ring isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.*

Recall that a ring R is called *Bear* if the left annihilator of every nonempty subset of R is generated by an idempotent. For $n = 1$ we proved that a regular Baer ring R is 2-good (i.e.,

R is 1-tuplet-good) if has no factor ring isomorphic to $\mathbb{Z}_2$ [1, Theorem 2.5]. Although any twin-good ring is not necessarily 2-tuplet-good [14, Example 6], it is proved (See [2, Theorem 3.7]) that a regular Baer ring is 2-tuplet-good if has no factor ring isomorphic to a field of order less than 4.

In the following, we prove that every element of a regular Baer ring R is n-tuplet-good if R has no factor ring isomorphic to a field of order less than $n+2$.

Theorem 2.17. *A regular ring R is a Bear ring if and only if R itself, regarded as a regular R-module, is a Baer semisimple module.*

Proof. See [9, Theorem 4].

Theorem 2.18. *A regular Baer ring R is n-tuplet-good if R has no factor ring isomorphic to a field of order less than $n+2$.*

Proof. By Theorem 2.17, R_R is a regular Baer semisimple module. Therefore, by using the theory stated in the proof of [1, Theorem 2.5], for each $1 \leq j \leq r$, there exist Baer simple $R-$modules $M_j^{n_j}$ such that $R \cong \prod_{j=1}^{r} End_R(M_j^{n_j}) \cong \prod_{j=1}^{r} M_{n_j}(End_R(M_j))$. On the other hand by [[9], Theorem 2], $End_R(M_j)$ is regular domain so, is a division ring for every $1 \leq j \leq r$. Let $D := End_R(M_j)$ so $R \cong \prod_{j=1}^{r} M_{n_j}(D_j)$. Since R has no factor ring isomorphic to a field of order less than $n+2$, by Lemma 2.1 $M_{n_j}(D_j)$ is n-tuplet-good for $1 \leq j \leq r$, thus R is n-tuplet-good.

In [14], the authors proved that every element of a right self-injective ring is n-tuplet-good if such rings has no factor ring isomorphic to a field of order less than $n+2$. Then, as a corollary, they concluded that the endomorphism ring of a vector space over a division ring D is n-tuplet-good, except when dimension of vector space is 1 and $|D| < n+2$. For every self-injective ring R, the factor ring $R/J(R)$ is a regular self-injective ring. Since every regular self-injective ring is Baer, we get the main result of [14] as a corollary of our Theorem 2.18.

Corollary 2.19. *A right self-injective ring R is n-tuplet-good if has no factor ring isomorphic to a field of order less than $n+2$.*

Recall that if V is a right vector space over a division ring D, then $End_D(V)$ is a regular Baer ring. Hence, the following corollary (see [14, Corollary 4]) is immediate.

Corollary 2.20. *Let V_D be a vector space over a division ring D. Then $End_D(V)$ is n-tuplet-good except when $dim(V_D) = 1$ and $|D| < n+2$.*

An R-module M is called *continuous* if every submodule of M is essential in a direct summand of M and every submodule of M isomorphic to direct summand of M is itself a direct summand of M.

We recall that the module M_R is n-tuplet-good if its endomorphism ring is n-tuplet-good. We conclude with the following result on continuous modules.

Proposition 2.21. *Let M be a continuous module. If $End_L(M)$ has no factor ring isomorphic to a field of order less than $n+2$, then M is n-tuplet-good.*

Proof. Let $R = End_L(M)$. If M is a continuous module, then by [[15], Theorem 3.11 and Proposition 3.5], $\overline{R} = R/J(R)$ is a regular right continuous ring, thus $\overline{R}_{\overline{R}}$ is an extending module with regular endomorphism ring. Therefore, $\overline{R}_{\overline{R}}$ is a Baer module, and subsequently $\overline{R}$ is a Baer ring (See [18, Proposition 4.12]). Thus, the result follows from Theorem 2.18.

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