Algebraic Structures and Their Applications Vol. 10 No. 1 (2023) pp 17-38.

Research Paper

ON GE-IDEALS OF BORDERED GE-ALGEBRAS

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Abstract. In this paper, the properties of GE-ideals of transitive bordered GE-algebra are studied and characterizations of GE-ideals are given. We have observed that the set of all GE-ideals of a transitive bordered GE-algebra forms a complete lattice. The notion of bordered GE-morphism is introduced and established fundamental bordered GE-morphism theorem. A congruence relation on a bordered GE-algebra with respect to GE-ideal is introduced and some bordered GE-morphism theorems are derived.

1. Introduction

BCK-algebras (see [7, 8]) were introduced by Y. Imai and K. Iséki in 1966 as the algebraic semantics for a non-classical logic possessing only implication. Since then, the generalized concepts of BCK-algebras have been studied by various scholars. Hilbert algebras were introduced by L. Henkin and T. Skolem in the fifties for investigations in intuitionistic and other

DOI: 10.22034/as.2022.2700

MSC(2010): Primary: 03G25, 06F35.

Keywords: Bordered GE-algebra, GE-ideal, Bordered GE-morphism, Dual kernel.

Received: 2 April 2022, Accepted: 27 May 2022.

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non-classical logics. A. Diego established that Hilbert algebras form a locally finite variety (see [5]). Later several researchers extended the theory on Hilbert algebras (see [3, 4, 6, 9, 10]). The notion of BE-algebra was introduced by H.S. Kim and Y.H. Kim as a generalization of a dual BCK-algebra (see [12]). A. Rezaei et al. discussed relations between Hilbert algebras and BE-algebras (see [13, 16]). In the study of algebraic structures, the generalization process is also an important topic. As a generalization of Hilbert algebras, R.K. Bandaru et al. introduced the notion of GE-algebras, and investigated several properties (see [1]). A. Rezaei et al. introduced the concept of prominent GE-filters in GE-algebras and discussed its properties (see [17]). R.K. Bandaru et al. introduced the concept of bordered GE-algebra and investigated its properties (see [2]). Later, M. A. Ozturk et al. introduced the concept of Strong GE-filters, GE-ideals of bordered GE-algebras and investigated its properties (see [14]). S. Z. Song et al. introduced the concept of Imploring GE-filters of GE-algebras and discussed its properties (see [18]). The isomorphism theorems play an important role in a general logical algebra, which were studied by several researches. Jun et al. derived isomorphism theorems by using Chinese Remainder Theorem in BCI-algebras (see [11]). J. K. Park et al. derived isomorphism theorems of IS-algebras (see [15]).

In this paper, we study the properties of GE-ideals of a transitive bordered GE-algebra and show that the set of all GE-ideals of a transitive bordered GE-algebra is a complete lattice. We introduce the notion of bordered GE-morphism and establish fundamental bordered GE-morphism theorem. We introduce a congruence relation on a bordered GE-algebra with respect to GE-ideal and derive some bordered GE-isomorphism theorems.

2. **Preliminaries**

Definition 2.1 ([1])**.** A *GE-algebra* is a non-empty set X with a constant 1 and a binary operation $\tilde{*}$ satisfying the following axioms:

(GE1) $\mu\tilde{*}\mu = 1$,

(GE2) $1\tilde{*}\mu = \mu$,

(GE3) $\mu\tilde{*}(\nu\tilde{*}\tau) = \mu\tilde{*}(\nu\tilde{*}(\mu\tilde{*}\tau))$

for all $\mu, \nu, \tau \in X$.

In a GE-algebra X, a binary relation "$\leq$" is defined by

$$(\forall \beta, \gamma \in X)\,(\beta \leq \gamma \;\Leftrightarrow\; \beta\tilde{*}\gamma = 1)\,. \tag{1}$$

In general, the dual BCK/BCI-algebra satisfies the transitivity, but GE-algebra does not. Therefore, it is necessary to define transitivity for the research of GE-algebra.

Definition 2.2 ([1])**.** A GE-algebra X is said to be

- *transitive* if it satisfies:

(2) $$(\forall \beta, \gamma, \alpha \in X)\,(\beta \widetilde{*} \gamma \leq (\alpha \widetilde{*} \beta) \widetilde{*} (\alpha \widetilde{*} \gamma))\,.$$

- *antisymmetric* if the binary relation "$\leq$" is antisymmetric.

Definition 2.3 ([2])**.** If a GE-algebra X has a special element, say 0, that satisfies $0 \leq \beta$ for all $\beta \in X$, we call X the *bordered GE-algebra*.

For every element β of a bordered GE-algebra X, we denote $\beta * 0$ by β^{ϱ}, and $(\beta^{\varrho})^{\varrho}$ is denoted by $\beta^{\varrho\varrho}$.

Definition 2.4 ([2])**.** If a bordered GE-algebra X satisfies the condition (2), we say that X is a *transitive bordered GE-algebra*.

Definition 2.5 ([2])**.** A bordered GE-algebra X is said to be *antisymmetric* if the binary operation "$\leq$" is antisymmetric.

Proposition 2.6 ([1])**.** *Every GE-algebra X satisfies the following items.*

(3) $$(\forall \mu \in X)\,(\mu \widetilde{*} 1 = 1)\,.$$

(4) $$(\forall \mu, \nu \in X)\,(\mu \widetilde{*} (\mu \widetilde{*} \nu) = \mu \widetilde{*} \nu)\,.$$

(5) $$(\forall \mu, \nu \in X)\,(\mu \leq \nu \widetilde{*} \mu)\,.$$

(6) $$(\forall \mu, \nu, \tau \in X)\,(\mu \widetilde{*} (\nu \widetilde{*} \tau) \leq \nu \widetilde{*} (\mu \widetilde{*} \tau))\,.$$

(7) $$(\forall \mu \in X)\,(1 \leq \mu \;\Rightarrow\; \mu = 1)\,.$$

If X is transitive, then

(8) $$(\forall \mu, \nu, \tau \in X)\,(\mu \leq \nu \;\Rightarrow\; \tau \widetilde{*} \mu \leq \tau \widetilde{*} \nu,\; \nu \widetilde{*} \tau \leq \mu \widetilde{*} \tau)\,.$$

Lemma 2.7 ([1])**.** *In a GE-algebra X, the following facts are equivalent each other.*

(9) $$(\forall \beta, \gamma, \alpha \in X)\,(\beta \widetilde{*} \gamma \leq (\alpha \widetilde{*} \beta) \widetilde{*} (\alpha \widetilde{*} \gamma))\,.$$

(10) $$(\forall \beta, \gamma, \alpha \in X)\,(\beta \widetilde{*} \gamma \leq (\gamma \widetilde{*} \alpha) \widetilde{*} (\beta \widetilde{*} \alpha))\,.$$

Definition 2.8 ([1])**.** A subset K of a GE-algebra X is called a *GE-filter* of X if it satisfies:

(11) $$1 \in K,$$

(12) $$(\forall \beta, \gamma \in X)(\beta \widetilde{*} \gamma \in K,\; \beta \in K \;\Rightarrow\; \gamma \in K).$$

Lemma 2.9 ([1])**.** *In a GE-algebra X, every GE-filter K of X satisfies:*

(13) $$(\forall \beta, \gamma \in X)\,(\beta \leq \gamma,\; \beta \in K \;\Rightarrow\; \gamma \in K)\,.$$

Proposition 2.10 ([2]). *In a bordered GE-algebra X, the following assertions are valid.*

$$1^{\varrho} = 0, \; 0^{\varrho} = 1. \tag{14}$$

$$(\forall \beta \in X) \left(\beta \leq \beta^{\varrho\varrho} \right). \tag{15}$$

$$(\forall \beta, \gamma \in X) \left(\beta \widetilde{*} \gamma^{\varrho} \leq \gamma \widetilde{*} \beta^{\varrho} \right). \tag{16}$$

$$(\forall \beta, \gamma \in X) \left(\beta \leq \gamma^{\varrho} \; \Leftrightarrow \; \gamma \leq \beta^{\varrho} \right). \tag{17}$$

$$(\forall \beta, \gamma \in X) \left(\beta \widetilde{*} \gamma^{\varrho} = \beta \widetilde{*} (\gamma \widetilde{*} \beta^{\varrho}) \right). \tag{18}$$

If X is a transitive bordered GE-algebra, then

$$(\forall \beta, \gamma \in X) \left(\beta \leq \gamma \; \Rightarrow \gamma^{\varrho} \leq \beta^{\varrho} \right). \tag{19}$$

$$(\forall \beta, \gamma \in X) \left(\beta \widetilde{*} \gamma \leq \gamma^{\varrho} \widetilde{*} \beta^{\varrho} \right). \tag{20}$$

If X is an antisymmetric bordered GE-algebra, then

$$(\forall \beta, \gamma \in X) \left(\beta \widetilde{*} \gamma^{\varrho} = \gamma \widetilde{*} \beta^{\varrho} \right). \tag{21}$$

If X is a transitive and antisymmetric bordered GE-algebra, then

$$(\forall \beta \in X) \left(\beta^{\varrho\varrho\varrho} = \beta^{\varrho} \right). \tag{22}$$

Definition 2.11 ([2]). By a duplex bordered element in a bordered GE-algebra X, we mean an element β of X which satisfies $\beta^{\varrho\varrho} = \beta$.

The set of all duplex bordered elements of a bordered GE-algebra X is denoted by $0^2(X)$, and is called the duplex bordered set of X. It is clear that $0, 1 \in 0^2(X)$.

Definition 2.12 ([2]). A bordered GE-algebra X is said to be *duplex* if every element of X is a duplex bordered element, that is, $X = 0^2(X)$.

Definition 2.13 ([14]). Let X be a bordered GE-algebra. If a subset G of X meets the following conditions for all $\beta, \gamma \in X$, it is termed a *GE-ideal* of X:

(i) $0 \in G$,

(ii) $\beta \in G$ and $(\beta^{\varrho} \widetilde{*} \gamma^{\varrho})^{\varrho} \in G$ imply that $\gamma \in G$.

Proposition 2.14 ([14]). *Let G be a GE-ideal of X. Then we have*

(*i*) *For any $\beta, \gamma \in X, \beta \in G$ and $\gamma \leq \beta$ imply $\gamma \in G$.*

(*ii*) *For any $\beta, \gamma \in X, (\beta \widetilde{*} \gamma)^{\varrho} \in G, \; \gamma \in G \; \Rightarrow \; \beta \in G$.*

3. Characterizations of GE-ideals

In this section, we study properties of GE-ideals of a transitive bordered GE-algebra and derive characterization theorems of GE-ideals. Throughout this section, X means a transitive bordered GE-algebra $(X, \widetilde{*}, 1)$ unless otherwise mentioned.

Lemma 3.1. *For any $\beta, \gamma \in X$, we have*

(i) $\beta^{\varrho\varrho\varrho} \leq \beta^{\varrho}$,

(ii) $\beta\widetilde{*}\gamma^{\varrho} \leq \beta^{\varrho\varrho}\widetilde{*}\gamma^{\varrho}$,

(iii) $(\beta\widetilde{*}\gamma^{\varrho\varrho})^{\varrho\varrho} \leq \beta\widetilde{*}\gamma^{\varrho\varrho}$,

(iv) $(\beta^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho\varrho} \leq \beta^{\varrho}\widetilde{*}\gamma^{\varrho}$,

(v) $(\beta\widetilde{*}\gamma)^{\varrho\varrho} \leq \beta^{\varrho\varrho}\widetilde{*}\gamma^{\varrho\varrho}$.

Proof. (i). Let $\beta \in X$. Then, by (GE1), (6) and (20),

$$1 = (\beta\widetilde{*}0)\widetilde{*}(\beta\widetilde{*}0) \leq \beta\widetilde{*}((\beta\widetilde{*}0)\widetilde{*}0) = \beta\widetilde{*}\beta^{\varrho\varrho} \leq \beta^{\varrho\varrho\varrho}\widetilde{*}\beta^{\varrho}.$$

Hence $\beta^{\varrho\varrho\varrho}\widetilde{*}\beta^{\varrho} = 1$, which gives $\beta^{\varrho\varrho\varrho} \leq \beta^{\varrho}$.

(ii). Let $\beta, \gamma \in X$. Then, by (16) and (20), $\beta\widetilde{*}\gamma^{\varrho} \leq \gamma\widetilde{*}\beta^{\varrho} \leq \beta^{\varrho\varrho}\widetilde{*}\gamma^{\varrho}$.

(iii). Let $\beta, \gamma \in X$. We can observe that $(\beta\widetilde{*}\gamma^{\varrho\varrho})^{\varrho} \leq (\beta\widetilde{*}\gamma^{\varrho\varrho})^{\varrho\varrho\varrho}$. By (8), we get $\gamma^{\varrho}\widetilde{*}(\beta\widetilde{*}\gamma^{\varrho\varrho})^{\varrho} \leq \gamma^{\varrho}\widetilde{*}(\beta\widetilde{*}\gamma^{\varrho\varrho})^{\varrho\varrho\varrho}$ and so $\beta\widetilde{*}(\gamma^{\varrho}\widetilde{*}(\beta\widetilde{*}\gamma^{\varrho\varrho})^{\varrho}) \leq \beta\widetilde{*}(\gamma^{\varrho}\widetilde{*}(\beta\widetilde{*}\gamma^{\varrho\varrho})^{\varrho\varrho\varrho})$. Hence, by (GE1),(6), (15) and (16), we get

$$\begin{aligned} 1 &= (\beta\widetilde{*}\gamma^{\varrho\varrho})\widetilde{*}(\beta\widetilde{*}\gamma^{\varrho\varrho}) \\ &\leq \beta\widetilde{*}((\beta\widetilde{*}\gamma^{\varrho\varrho})\widetilde{*}\gamma^{\varrho\varrho}) \\ &\leq \beta\widetilde{*}(\gamma^{\varrho}\widetilde{*}(\beta\widetilde{*}\gamma^{\varrho\varrho})^{\varrho}) \\ &\leq \beta\widetilde{*}(\gamma^{\varrho}\widetilde{*}(\beta\widetilde{*}\gamma^{\varrho\varrho})^{\varrho\varrho\varrho}) \\ &\leq \beta\widetilde{*}((\beta\widetilde{*}\gamma^{\varrho\varrho})^{\varrho\varrho}\widetilde{*}\gamma^{\varrho\varrho}) \\ &\leq (\beta\widetilde{*}\gamma^{\varrho\varrho})^{\varrho\varrho}\widetilde{*}(\beta\widetilde{*}\gamma^{\varrho\varrho}). \end{aligned}$$

Thus $(\beta\widetilde{*}\gamma^{\varrho\varrho})^{\varrho\varrho}\widetilde{*}(\beta\widetilde{*}\gamma^{\varrho\varrho}) = 1$. Therefore $(\beta\widetilde{*}\gamma^{\varrho\varrho})^{\varrho\varrho} \leq \beta\widetilde{*}\gamma^{\varrho\varrho}$.

(iv). By (16), we have $\beta^{\varrho}\widetilde{*}\gamma^{\varrho} \leq \gamma\widetilde{*}\beta^{\varrho\varrho}$. Hence, by (20), (iii) and (16), we get

$$(\beta^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho\varrho} \leq (\gamma\widetilde{*}\beta^{\varrho\varrho})^{\varrho\varrho} \leq \gamma\widetilde{*}\beta^{\varrho\varrho} \leq \beta^{\varrho}\widetilde{*}\gamma^{\varrho}.$$

(v). By (20), we get $\beta\widetilde{*}\gamma \leq \beta^{\varrho\varrho}\widetilde{*}\gamma^{\varrho\varrho}$. Hence $(\beta\widetilde{*}\gamma)^{\varrho\varrho} \leq (\beta^{\varrho\varrho}\widetilde{*}\gamma^{\varrho\varrho})^{\varrho\varrho}$. Also, by (iv), we can observe that $(\beta^{\varrho\varrho}\widetilde{*}\gamma^{\varrho\varrho})^{\varrho\varrho} \leq \beta^{\varrho\varrho}\widetilde{*}\gamma^{\varrho\varrho}$. Hence (v) follows, since X is transitive. □

Proposition 3.2. *Let G be a GE-ideal of X. Then we have*

(i) *For any $\beta, \gamma \in X, \beta^\varrho = \gamma^\varrho, \beta \in G$ imply $\gamma \in G$,*

(ii) *For any $\beta \in X, \beta \in G$ if and only if $\beta^{\varrho\varrho} \in G$.*

Proof. (i). Let $\beta, \gamma \in X$ be such that $\beta^\varrho = \gamma^\varrho$ and $\beta \in G$. Then $(\beta^\varrho \widetilde{*} \gamma^\varrho)^\varrho = 1^\varrho = 0 \in G$. Hence $\gamma \in G$ since G is a GE-ideal of X.
(ii). Let $\beta \in X$. Suppose $\beta \in G$. Then, by (GE1), (16) and (19),

$$1 = \beta^{\varrho\varrho} \widetilde{*} \beta^{\varrho\varrho} \leq \beta^\varrho \widetilde{*} \beta^{\varrho\varrho\varrho} \text{ which implies that } (\beta^\varrho \widetilde{*} \beta^{\varrho\varrho\varrho})^\varrho \leq 1^\varrho = 0 \in G.$$

By Proposition 2.14(i), we get $(\beta^\varrho \widetilde{*} \beta^{\varrho\varrho\varrho})^\varrho \in G$. Now $\beta \in G$ and G is a GE-ideal of X, we have $\beta^{\varrho\varrho} \in G$. Conversely, let $\beta^{\varrho\varrho} \in G$ for any $\beta \in X$. Since $\beta \leq \beta^{\varrho\varrho}$ and $\beta^{\varrho\varrho} \in G$, by Proposition 2.14(i), we get $\beta \in G$. □

Given a transitive bordered GE-algebra X, consider the next assertion:

$$(\forall \beta, \gamma \in X)(\beta^\varrho \widetilde{*} \gamma^\varrho \leq \gamma \widetilde{*} \beta). \tag{23}$$

Question 3.3. *Does every transitive bordered GE-algebra X satisfy the condition* (23)*?*

The answer to Question 3.3 is negative as seen in the following example.

Example 3.4. Consider a set $X := \{0, 1, 2, 3, 4, 5\}$ with the binary operation "$\widetilde{*}$", which is given by Table 1. Then $(X, \widetilde{*}, 1)$ is a transitive bordered GE-algebra. But X does not satisfy (23), since

Table 1. Cayley table for the binary operation "$\widetilde{*}$"

$\widetilde{*}$	0	1	2	3	4	5
0	1	1	1	1	1	1
1	0	1	2	3	4	5
2	0	1	1	3	5	5
3	0	1	2	1	4	4
4	0	1	2	3	1	1
5	0	1	2	3	1	1

$$((2 \widetilde{*} 0) \widetilde{*} (4 \widetilde{*} 0)) \widetilde{*} (4 \widetilde{*} 2) = (0 \widetilde{*} 0) \widetilde{*} 2 = 1 \widetilde{*} 2 = 2 \neq 1.$$

Theorem 3.5. *If X satisfies* (23)*, then G is a GE-ideal of X if and only if $0 \in G$ and $(\beta \widetilde{*} \gamma)^\varrho \in G$ implies that $\beta \in G$ for all $\gamma \in G$.*

Proof. Let X be a transitive bordered GE-algebra satisfying (23). Suppose G is a GE-ideal of X. Then $0 \in G$. Let $(\beta\widetilde{*}\gamma)^{\varrho} \in G$ and $\gamma \in G$. $\beta\widetilde{*}\gamma \leq \gamma^{\varrho}\widetilde{*}\beta^{\varrho}$ implies that $(\gamma^{\varrho}\widetilde{*}\beta^{\varrho})^{\varrho} \leq (\beta\widetilde{*}\gamma)^{\varrho}$ by (19). By Proposition 2.14(i), we have $(\gamma^{\varrho}\widetilde{*}\beta^{\varrho})^{\varrho} \in G$. Since G is a GE-ideal of X and $\gamma \in G$, we get $\beta \in G$. Conversely, assume, on the other hand, that the given conditions hold. Let $(\beta^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho} \in G$ and $\beta \in G$. Then $\beta^{\varrho}\widetilde{*}\gamma^{\varrho} \leq \gamma\widetilde{*}\beta$ implies that $(\gamma\widetilde{*}\beta)^{\varrho} \leq (\beta^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho}$ by (19). Therefore $((\gamma\widetilde{*}\beta)^{\varrho}\widetilde{*}(\beta^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho})^{\varrho} = 0 \in G$. Since $(\beta^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho} \in G$, we get $(\gamma\widetilde{*}\beta)^{\varrho} \in G$. Now $\beta \in G$ and $(\gamma\widetilde{*}\beta)^{\varrho}$ implies that $\gamma \in G$. □

Theorem 3.6. *Let G be a GE-ideal of X. Then $(\beta\widetilde{*}\gamma)^{\varrho} \in G, \gamma \in G \Rightarrow \beta \in G,\ \forall \beta, \gamma \in X$ if and only if $(\alpha\widetilde{*}\beta)^{\varrho} \leq \gamma \Rightarrow \alpha \in G, \forall \beta, \gamma \in G, \forall \alpha \in X$.*

Proof. Suppose $(\beta\widetilde{*}\gamma)^{\varrho} \in G, \gamma \in G \Rightarrow \beta \in G,\ \forall \beta, \gamma \in X$. Let $\beta, \gamma \in G$ and $\alpha \in X$ be such that $(\alpha\widetilde{*}\beta)^{\varrho} \leq \gamma$. Then $((\alpha\widetilde{*}\beta)^{\varrho}\widetilde{*}\gamma)^{\varrho} = 0 \in G$ and hence $(\alpha\widetilde{*}\beta)^{\varrho} \in G$. Therefore $\alpha \in G$ since $\beta \in G$. Conversely assume that the condition holds. Let $\beta, \gamma \in X$ be such that $(\beta\widetilde{*}\gamma)^{\varrho} \in G$ and $\gamma \in G$. Since $(\beta\widetilde{*}\gamma)^{\varrho} \leq (\beta\widetilde{*}\gamma)^{\varrho}$, it follows from the assumption that $\beta \in G$. □

Theorem 3.7. *Let $\emptyset \neq G \subseteq X$. Then G is a GE-ideal of X if and only if it satisfies the following property:*

$$\beta^{\varrho} \leq \gamma^{\varrho}\widetilde{*}\alpha^{\varrho} \text{ implies that } \alpha \in G$$

for all $\beta, \gamma \in G$ and $\alpha \in X$.

Proof. Assume that G is a GE-ideal of X. Let $\beta, \gamma \in G$ and $\alpha \in X$. Suppose $\beta^{\varrho} \leq \gamma^{\varrho}\widetilde{*}\alpha^{\varrho}$. Then $\beta^{\varrho} \leq \gamma^{\varrho}\widetilde{*}\alpha^{\varrho} \leq (\gamma^{\varrho}\widetilde{*}\alpha^{\varrho})^{\varrho\varrho}$ and hence $(\beta^{\varrho}\widetilde{*}(\gamma^{\varrho}\widetilde{*}\alpha^{\varrho})^{\varrho\varrho})^{\varrho} = 1^{\varrho} = 0 \in G$. Since $\beta \in G$ and G is a GE-ideal of X, we get $(\gamma^{\varrho}\widetilde{*}\alpha^{\varrho})^{\varrho} \in G$. Since $\gamma \in G$, we get $\alpha \in G$.

Conversely, assume, on the other hand, that the G satisfies the provided condition. Since $G \neq \emptyset$, choose $\beta \in G$. Clearly $\beta^{\varrho} \leq 1 = \beta^{\varrho}\widetilde{*}0^{\varrho}$. Then by the given condition, we get $0 \in G$. Let $\beta, \gamma \in X$ be such that $\beta \in G$ and $(\beta^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho} \in G$. By Lemma 3.1(iv), we get $(\beta^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho\varrho} \leq \beta^{\varrho}\widetilde{*}\gamma^{\varrho}$. Now, by (8), we get

$$(\beta^{\varrho}\widetilde{*}\gamma^{\varrho})\widetilde{*}\gamma^{\varrho} \leq (\beta^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho\varrho}\widetilde{*}\gamma^{\varrho}.$$

Since G is transitive, we have

$$\begin{aligned} 1 &= (\beta^{\varrho}\widetilde{*}\gamma^{\varrho})\widetilde{*}(\beta^{\varrho}\widetilde{*}\gamma^{\varrho}) \\ &\leq \beta^{\varrho}\widetilde{*}((\beta^{\varrho}\widetilde{*}\gamma^{\varrho})\widetilde{*}\gamma^{\varrho}) \\ &\leq \beta^{\varrho}\widetilde{*}((\beta^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho\varrho}\widetilde{*}\gamma^{\varrho}). \end{aligned}$$

Hence, we get $\beta^{\varrho} \leq (\beta^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho\varrho}\widetilde{*}\gamma^{\varrho}$. Since $\beta \in G$ and $(\beta^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho} \in G$, we get $\gamma \in G$ by the assumption. Therefore G is a GE-ideal of X. □

Theorem 3.8. *Let G be a non-empty subset of X. Then G is a GE-ideal of X if and only if it satisfies the following condition for all $\beta \in X$:*

$$\text{for all } \mu, \nu \in G,\ (\mu^{\varrho} \widetilde{*} (\nu^{\varrho} \widetilde{*} \beta^{\varrho})^{\varrho\varrho})^{\varrho} = 0 \text{ implies } \beta \in G$$

Proof. Assume that G is a GE-ideal of X. Let $\mu, \nu \in G$ and $\beta \in X$ be such that $(\mu^{\varrho} \widetilde{*} (\nu^{\varrho} \widetilde{*} \beta^{\varrho})^{\varrho\varrho})^{\varrho} = 0 \in G$. Since $\mu \in G$ and G is a GE-ideal of X, we get that $(\nu^{\varrho} \widetilde{*} \beta^{\varrho})^{\varrho} \in G$. Since $\nu \in G$, we get that $\beta \in G$.

Conversely, assume, on the other hand, that the G satisfies the provided condition. For any $\beta \in G$, we have

$$(\beta^{\varrho} \widetilde{*} (\beta^{\varrho} \widetilde{*} 0^{\varrho})^{\varrho\varrho})^{\varrho} = (\beta^{\varrho} \widetilde{*} (\beta^{\varrho} \widetilde{*} 1)^{\varrho\varrho})^{\varrho} = (\beta^{\varrho} \widetilde{*} 1^{\varrho\varrho})^{\varrho} = 1^{\varrho} = 0.$$

Hence, by assumption we get $0 \in G$. Let $\beta, \gamma \in X$. Suppose $\beta \in G$ and $(\beta^{\varrho} \widetilde{*} \gamma^{\varrho})^{\varrho} \in G$. We know that $(\beta^{\varrho} \widetilde{*} \gamma^{\varrho})^{\varrho\varrho} \leq (\beta^{\varrho} \widetilde{*} \gamma^{\varrho})^{\varrho\varrho}$. Therefore $(\beta^{\varrho} \widetilde{*} \gamma^{\varrho})^{\varrho\varrho} \widetilde{*} (\beta^{\varrho} \widetilde{*} \gamma^{\varrho})^{\varrho\varrho} = 1$ and hence $((\beta^{\varrho} \widetilde{*} \gamma^{\varrho})^{\varrho\varrho} \widetilde{*} (\beta^{\varrho} \widetilde{*} \gamma^{\varrho})^{\varrho\varrho})^{\varrho} = 0$. Since $\beta \in G$ and $(\beta^{\varrho} \widetilde{*} \gamma^{\varrho})^{\varrho} \in G$, we get $\gamma \in G$ by assumption. Therefore G is a GE-ideal of X. □

Theorem 3.9. *A non-empty subset G of X is a GE-ideal of X if and only if it satisfies the following properties:*

(*i*) *$\beta \in G$ and $\gamma \leq \beta$ implies that $\gamma \in G$,*

(*ii*) *$\beta^{\varrho\varrho} \in G$ implies that $\beta \in G$,*

(*iii*) *$\beta \in G$ implies $(\gamma^{\varrho} \widetilde{*} \beta^{\varrho})^{\varrho} \in G$,*

(*iv*) *$\mu, \nu \in G$ implies $((\mu^{\varrho} \widetilde{*} (\nu^{\varrho} \widetilde{*} \beta^{\varrho})) \widetilde{*} \beta^{\varrho})^{\varrho} \in G$*

for all $\beta, \gamma \in X$.

Proof. Assume that G is a GE-ideal of X. Then (i) and (ii) follows by Proposition 2.14(i) and Proposition 3.2(ii). Let $\beta \in G$ and $\gamma \in X$. Clearly $\gamma^{\varrho} \widetilde{*} \beta^{\varrho} \leq (\gamma^{\varrho} \widetilde{*} \beta^{\varrho})^{\varrho\varrho}$. Then, by (8), (19) and (6), we get that $(\beta^{\varrho} \widetilde{*} (\gamma^{\varrho} \widetilde{*} \beta^{\varrho})^{\varrho\varrho})^{\varrho} \leq (\beta^{\varrho} \widetilde{*} (\gamma^{\varrho} \widetilde{*} \beta^{\varrho}))^{\varrho} \leq (\gamma^{\varrho} \widetilde{*} (\beta^{\varrho} \widetilde{*} \beta^{\varrho}))^{\varrho} = (\gamma^{\varrho} \widetilde{*} 1)^{\varrho} = 1^{\varrho} = 0 \in G$. Hence $(\beta^{\varrho} \widetilde{*} (\gamma^{\varrho} \widetilde{*} \beta^{\varrho})^{\varrho\varrho})^{\varrho} \in G$ by (i). Since $\beta \in G$, we get $(\gamma^{\varrho} \widetilde{*} \beta^{\varrho})^{\varrho} \in G$. Thus (iii) follows. Let $\mu, \nu \in G$. Then, by (15), (8) and (19), we have

$$\nu^{\varrho} \widetilde{*} \beta^{\varrho} \leq (\nu^{\varrho} \widetilde{*} \beta^{\varrho})^{\varrho\varrho}$$

which implies that

$$\mu^{\varrho} \widetilde{*} (\nu^{\varrho} \widetilde{*} \beta^{\varrho}) \leq \mu^{\varrho} \widetilde{*} (\nu^{\varrho} \widetilde{*} \beta^{\varrho})^{\varrho\varrho}$$

so that

$$(A) \qquad [(\mu^{\varrho} \widetilde{*} (\nu^{\varrho} \widetilde{*} \beta^{\varrho})) \widetilde{*} \beta^{\varrho}]^{\varrho} \leq [(\mu^{\varrho} \widetilde{*} (\nu^{\varrho} \widetilde{*} \beta^{\varrho})^{\varrho\varrho}) \widetilde{*} \beta^{\varrho}]^{\varrho}$$

Now we show that $[(\mu^{\varrho}\widetilde{*}(\nu^{\varrho}\widetilde{*}\beta^{\varrho})^{\varrho\varrho})\widetilde{*}\beta^{\varrho}]^{\varrho} \in G$. By Lemma 3.1(iv), we have

$$(\nu^{\varrho}\widetilde{*}\beta^{\varrho})^{\varrho\varrho} \leq \nu^{\varrho}\widetilde{*}\beta^{\varrho}.$$

Then, by (15), (8) and (19), we have

$$\mu^{\varrho}\widetilde{*}(\nu^{\varrho}\widetilde{*}\beta^{\varrho})^{\varrho\varrho} \leq \mu^{\varrho}\widetilde{*}(\nu^{\varrho}\widetilde{*}\beta^{\varrho})$$

which implies that

$$[\mu^{\varrho}\widetilde{*}(\nu^{\varrho}\widetilde{*}\beta^{\varrho})]\widetilde{*}\beta^{\varrho} \leq (\mu^{\varrho}\widetilde{*}(\nu^{\varrho}\widetilde{*}\beta^{\varrho})^{\varrho\varrho})\widetilde{*}\beta^{\varrho} \leq [(\mu^{\varrho}\widetilde{*}(\nu^{\varrho}\widetilde{*}\beta^{\varrho})^{\varrho\varrho})\widetilde{*}\beta^{\varrho}]^{\varrho\varrho}.$$

So that

$$\nu^{\varrho}\widetilde{*}([\mu^{\varrho}\widetilde{*}(\nu^{\varrho}\widetilde{*}\beta^{\varrho})]\widetilde{*}\beta^{\varrho}) \leq \nu^{\varrho}\widetilde{*}[(\mu^{\varrho}\widetilde{*}(\nu^{\varrho}\widetilde{*}\beta^{\varrho})^{\varrho\varrho})\widetilde{*}\beta^{\varrho}]^{\varrho\varrho} \leq [\nu^{\varrho}\widetilde{*}[(\mu^{\varrho}\widetilde{*}(\nu^{\varrho}\widetilde{*}\beta^{\varrho})^{\varrho\varrho})\widetilde{*}\beta^{\varrho}]^{\varrho\varrho}]^{\varrho\varrho}$$

Therefore

$$1 = \mu^{\varrho}\widetilde{*}(\nu^{\varrho}\widetilde{*}([\mu^{\varrho}\widetilde{*}(\nu^{\varrho}\widetilde{*}\beta^{\varrho})]\widetilde{*}\beta^{\varrho})) \leq \mu^{\varrho}\widetilde{*}([\nu^{\varrho}\widetilde{*}[(\mu^{\varrho}\widetilde{*}(\nu^{\varrho}\widetilde{*}\beta^{\varrho})^{\varrho\varrho})\widetilde{*}\beta^{\varrho}]^{\varrho\varrho}]^{\varrho\varrho})$$

Hence

$$\mu^{\varrho}\widetilde{*}([\nu^{\varrho}\widetilde{*}[(\mu^{\varrho}\widetilde{*}(\nu^{\varrho}\widetilde{*}\beta^{\varrho})^{\varrho\varrho})\widetilde{*}\beta^{\varrho}]^{\varrho\varrho}]^{\varrho\varrho}) = 1$$

Thus

$$[\mu^{\varrho}\widetilde{*}([\nu^{\varrho}\widetilde{*}[(\mu^{\varrho}\widetilde{*}(\nu^{\varrho}\widetilde{*}\beta^{\varrho})^{\varrho\varrho})\widetilde{*}\beta^{\varrho}]^{\varrho\varrho}]^{\varrho\varrho})]^{\varrho} = 0 \in G$$

Since $\mu, \nu \in G$, and G is a GE-ideal of X, we get

$$[(\mu^{\varrho}\widetilde{*}(\nu^{\varrho}\widetilde{*}\beta^{\varrho})^{\varrho\varrho})\widetilde{*}\beta^{\varrho}]^{\varrho} \in G.$$

Since $[(\mu^{\varrho}\widetilde{*}(\nu^{\varrho}\widetilde{*}\beta^{\varrho})^{\varrho\varrho})\widetilde{*}\beta^{\varrho}]^{\varrho} \in G$ and G is a GE-ideal of X, we get, from (A),

$$[(\mu^{\varrho}\widetilde{*}(\nu^{\varrho}\widetilde{*}\beta^{\varrho}))\widetilde{*}\beta^{\varrho}]^{\varrho} \in G.$$

Hence (iv) follows.

Conversely, assume, on the other hand, that the G satisfies the provided conditions. Take $\beta = \gamma$ in (iii). Then we can observe that $0 \in G$. Let $\beta, \gamma \in X$. Suppose that $\beta \in G$ and $(\beta^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho} \in G$. Then, by Lemma 3.1(iv), we have

$$(\beta^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho\varrho} \leq \beta^{\varrho}\widetilde{*}\gamma^{\varrho}$$

which implies that

$$(\beta^{\varrho}\widetilde{*}\gamma^{\varrho})\widetilde{*}\gamma^{\varrho} \leq (\beta^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho\varrho}\widetilde{*}\gamma^{\varrho}$$

So that

$$1 = \beta^{\varrho}\widetilde{*}((\beta^{\varrho}\widetilde{*}\gamma^{\varrho})\widetilde{*}\gamma^{\varrho}) \leq \beta^{\varrho}\widetilde{*}((\beta^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho\varrho}\widetilde{*}\gamma^{\varrho}).$$

Therefore

$$(\beta^{\varrho}\widetilde{*}((\beta^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho\varrho}\widetilde{*}\gamma^{\varrho}))\widetilde{*}\gamma^{\varrho} \leq \gamma^{\varrho}$$

Hence

$$\gamma^{\varrho\varrho} \leq [(\beta^{\varrho}\widetilde{*}((\beta^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho\varrho}\widetilde{*}\gamma^{\varrho}))\widetilde{*}\gamma^{\varrho}]^{\varrho}.$$

Since $\beta \in G$ and $(\beta^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho} \in G$, by (iv), we obtain $[(\beta^{\varrho}\widetilde{*}((\beta^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho\varrho}\widetilde{*}\gamma^{\varrho}))\widetilde{*}\gamma^{\varrho}]^{\varrho} \in G$. Hence, by (i), $\gamma^{\varrho\varrho} \in G$. Therefore, by (ii), $\gamma \in G$. Thus G is a GE-ideal of X. □

4. Bordered GE-morphism theorems

Definition 4.1 ([17]). Let $(X, \widetilde{*}_X, 1_X)$ and $(Y, \widetilde{*}_Y, 1_Y)$ be GE-algebras. A mapping $\xi : X \to Y$ is called a *GE-morphism* if it satisfies:

$$(\forall \beta_1, \beta_2 \in X)(\xi(\beta_1 \widetilde{*}_X \beta_2) = \xi(\beta_1)\widetilde{*}_Y \xi(\beta_2)). \tag{24}$$

Note that every GE-morphism is order preversing (see [17]).

Definition 4.2. Let $(X, \widetilde{*}_X, 1_X)$ and $(Y, \widetilde{*}_Y, 1_Y)$ be bordered GE-algebras. A GE-morphism $\xi : X \to Y$ is called a *bordered GE-morphism* if it satisfies:

$$\xi(0_X) = 0_Y. \tag{25}$$

If a bordered GE-morphism $\xi : X \to Y$ is onto (resp., one-to-one), we say it is a *bordered GE-epimorphism* (resp., *bordered GE-isomorphism*).

Example 4.3. Consider two sets $X = \{0, 1, 2, 3, 4\}$ and $Y = \{0, 1, 2, 3, 4\}$ with binary operations "$\widetilde{*}_X$" and "$\widetilde{*}_Y$", respectively, which are given by the following Table 2. Then $(X, \widetilde{*}_X, 1_X)$

Table 2. Cayley tables for the binary operations "$\widetilde{*}_X$" and "$\widetilde{*}_Y$"

$\widetilde{*}_X$	0	1	2	3	4
0	1	1	1	1	1
1	0	1	2	3	4
2	0	1	1	1	0
3	4	1	1	1	4
4	1	1	1	3	1

$\widetilde{*}_Y$	0	1	2	3	4
0	1	1	1	1	1
1	0	1	2	3	4
2	0	1	1	0	4
3	1	1	1	1	4
4	3	1	1	3	1

and $(Y, \widetilde{*}_Y, 1_Y)$ are bordered GE-algebras. Let $\xi : X \to Y$ be a mapping defined by

$$\xi(\beta) = \begin{cases} 0 & \text{if } \beta \in \{0, 4\}, \\ 1 & \text{if } \beta \in \{1, 2, 3\}. \end{cases}$$

Then ξ is a bordered GE-morphism.

It is clear that every bordered GE-morphism is a GE-morphism, but the converse is not true in general as seen in the following example.

Example 4.4. Consider two sets $X = \{0, 1, 2, 3, 4\}$ and $Y = \{0, 1, 2, 3, 4\}$ with binary operations "$\widetilde{*}_X$" and "$\widetilde{*}_Y$", respectively, which are given by the following Table 3. Then $(X, \widetilde{*}_X, 1_X)$

TABLE 3. Cayley tables for the binary operations "$\widetilde{*}_X$" and "$\widetilde{*}_Y$"

$\widetilde{*}_X$	0	1	2	3	4
0	1	1	1	1	1
1	0	1	2	3	4
2	4	1	1	1	4
3	0	1	2	1	0
4	1	1	2	1	1

$\widetilde{*}_Y$	0	1	2	3	4
0	1	1	1	1	1
1	0	1	2	3	4
2	0	1	1	3	4
3	1	1	2	1	4
4	1	1	2	1	1

and $(Y, \widetilde{*}_Y, 1_Y)$ are bordered GE-algebras. Let $\xi : X \to Y$ be a mapping defined by

$$\xi(\beta) = 1 \text{ for all } \beta \in X.$$

Then ξ is a GE-morphism. But ξ is not bordered GE-morphism, since $\xi(0) = 1 \neq 0$.

For any bordered GE-morphism $\xi : X \to Y$, define the *dual kernel* of the bordered GE-morphism ξ as $Dker(\xi) = \{\beta \in X \mid \xi(\beta) = 0_Y\}$. It is easy to check that $Dker(\xi) = \{0_X\}$ whenever ξ is an injective bordered GE-morphism. If ξ is bordered, then

$$\xi(\beta^{\varrho}) = \xi(\beta\widetilde{*}_X 0_X) = \xi(\beta)\widetilde{*}_Y \xi(0_X) = \xi(\beta)\widetilde{*}_Y 0_Y = (\xi(\beta))^{\varrho}$$

for all $\beta \in X$.

Question 4.5. *Let X and Y be bordered GE-algebras.*

(i) *If $\xi : X \to Y$ is a GE-morphism, then is $(\xi(\beta))^{\varrho\varrho} = \xi(\beta)$ for all $\beta \in X$?*
(ii) *If $\xi : X \to Y$ is a bordered GE-morphism, then is $D\ker(\xi)$ a GE-ideal of X?*

The following example shows that the answer to Question 4.5 is negative.

Example 4.6. Consider two sets $X = \{0, 1, 2, 3, 4\}$ and $Y = \{0, 1, 2, 3, 4\}$ with binary operations "$\widetilde{*}_X$" and "$\widetilde{*}_Y$", respectively, which are given by the following Table 4. Then $(X, \widetilde{*}_X, 1_X)$

TABLE 4. Cayley tables for the binary operations "$\widetilde{*}_X$" and "$\widetilde{*}_Y$"

$\widetilde{*}_X$	0	1	2	3	4
0	1	1	1	1	1
1	0	1	2	3	4
2	0	1	1	3	4
3	1	1	1	1	4
4	0	1	2	3	1

$\widetilde{*}_Y$	0	1	2	3	4
0	1	1	1	1	1
1	0	1	2	3	4
2	0	1	1	3	4
3	1	1	1	1	4
4	3	1	1	3	1

and $(Y, \widetilde{*}_Y, 1_Y)$ are bordered GE-algebras. Let $\xi : X \to Y$ be a mapping defined by

$$\xi(\beta) = \begin{cases} 0 & \text{if } \beta = 0, \\ 1 & \text{if } \beta \in \{1, 4\}, \\ 2 & \text{if } \beta = 2, \\ 3 & \text{if } \beta = 3. \end{cases}$$

Then ξ is a bordered GE-morphism and hence a GE-morphism. But Question 4.5(i) and Question 4.5(ii) does not hold since

$$(\xi(2)\widetilde{*}_X 0)\widetilde{*}_X 0 = (2\widetilde{*}_X 0)\widetilde{*}_X 0 = 0\widetilde{*}_X 0 = 1 \neq 2 = \xi(2).$$

Also, $D\ker(\xi) = \{0_X\}$ and it is not a GE-ideal of X since

$$((0\widetilde{*}_X 0)\widetilde{*}_X(3\widetilde{*}_X 0))\widetilde{*}_X 0 = (1\widetilde{*}_X 1)\widetilde{*}_X 0 = 1\widetilde{*}_X 0 = 0 \in D\ker(\xi) \text{ but } 3 \notin D\ker(\xi).$$

We provide conditions to ensure that the answer to Question 4.5(ii) is positive.

Theorem 4.7. *Let X and Y be bordered GE-algebras. If $\xi : X \to Y$ is a bordered GE-morphism satisfying*

$$(\forall \beta \in X)((\xi(\beta))^{\varrho\varrho} = \xi(\beta)),$$

then the dual kernel, $D\ker(\xi)$ is a GE-ideal of X.

Proof. Clearly $0_X \in Dker(\xi)$. Let $\beta, \gamma \in X$ be such that $\beta \in Dker(\xi)$ and $(\beta^\varrho \widetilde{*}_X \gamma^\varrho)^\varrho \in Dker(\xi)$. Then $\xi(\beta) = 0_Y$ and

$$\begin{aligned} 0_Y &= \xi((\beta^\varrho \widetilde{*}_X \gamma^\varrho)^\varrho) = (\xi(\beta^\varrho \widetilde{*}_X \gamma^\varrho))^\varrho = (\xi(\beta^\varrho)\widetilde{*}_Y \xi(\gamma^\varrho))^\varrho \\ &= ((\xi(\beta))^\varrho \widetilde{*}_Y (\xi(\gamma))^\varrho)^\varrho = ((0_Y)^\varrho \widetilde{*}_Y (\xi(\gamma))^\varrho)^\varrho \\ &= ((1\widetilde{*}_Y (\xi(\gamma))^\varrho)^\varrho = (\xi(\gamma))^{\varrho,\varrho} = \xi(\gamma), \end{aligned}$$

and so $\gamma \in Dker(\xi)$. Therefore $Dker(\xi)$ is a GE-ideal of X. □

Corollary 4.8. *Let $\xi : X \to Y$ be a bordered GE-morphism of bordered GE-algebras X and Y. If Y is duplex, then the dual kernel, $D\ker(\xi)$, is a GE-ideal of X.*

Proposition 4.9. *Let X and Y be two bordered GE-algebras and $\xi : X \to Y$ a bordered GE-morphism. Then $f^{-1}(G)$ is a GE-ideal of X for any GE-ideal G of Y.*

Proof. Let $\xi : X \to Y$ be a bordered GE-morphism. Suppose G is a GE-ideal of Y. Let $\beta, \gamma \in X$ be such that $\beta \in \xi^{-1}(G)$ and $(\beta^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho} \in \xi^{-1}(G)$. Then $\xi(\beta) \in G$ and $(\xi(\beta)^{\varrho}\widetilde{*}\xi(\gamma)^{\varrho})^{\varrho} = \xi((\beta^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho}) \in G$. Since $\xi(\beta) \in G$ and G is a GE-ideal, we get $\xi(\gamma) \in G$. Hence $\gamma \in \xi^{-1}(G)$. Thus $\xi^{-1}(G)$ is a GE-ideal of X. □

Let K be a GE-filter of a transitive GE-algebra X. Consider the set

$$R_K := \{(\beta, \gamma) \in X \times X \mid \beta\widetilde{*}\gamma \in K,\ \gamma\widetilde{*}\beta \in K\}. \tag{26}$$

It is routine to verify that R_K is a congruence relation on X. For each $\delta \in X$, let $[\delta]$ denote the set of elements of X to which δ is related under R_K, that is,

$$[\delta] = \{\beta \in X \mid (\delta, \beta) \in R_K\}.$$

We call $[\delta]$ the *equivalence class* of δ in X under R_K. The collection of all such equivalence classes is denoted by X/R_K, that is,

$$X/R_K = \{[\delta] \mid \delta \in X\},$$

which is called the *quotient set* of X by R_K. Then $(X/R_K, \widetilde{*}_K, [1])$ is a GE-algebra where $\widetilde{*}_K$ is defined as follow:

$$(\forall [\beta], [\gamma] \in X/R_K)([\beta]\widetilde{*}_K[\gamma] = [\beta\widetilde{*}\gamma]).$$

If X is bordered, then X/R_K is also a bordered GE-algebra with the special element $[0_X]$.

Proposition 4.10. *For any GE-filter K of a transitive bordered GE-algebra X, the congruence class $[0]_K$ is a GE-ideal of X.*

Proof. Let K be a GE-filter of X. Since X is transitive, we have R_K is a congruence relation on X. Clearly $0 \in [0]_K$. Let $\beta \in [0]_K$ and $(\beta^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho} \in [0]_K$. Hence $\beta^{\varrho} = \beta\widetilde{*}_X 0 \in K$ and $(\beta^{\varrho}\widetilde{*}_X\gamma^{\varrho})^{\varrho\varrho} = (\beta^{\varrho}\widetilde{*}_X\gamma^{\varrho})^{\varrho}\widetilde{*}0 \in K$. Since $(\beta^{\varrho}\widetilde{*}_X\gamma^{\varrho})^{\varrho\varrho} \leq \beta^{\varrho}\widetilde{*}_X\gamma^{\varrho}$, we get $\beta^{\varrho}\widetilde{*}_X\gamma^{\varrho} \in K$. Since $\beta^{\varrho} \in K$, we get $\gamma\widetilde{*}_X 0 = \gamma^{\varrho} \in K$. Since $0\widetilde{*}_X\gamma = 1 \in K$, we get $(\gamma, 0) \in R_K$. Hence $\gamma \in [0]_K$. Therefore $[0]_K$ is a GE-ideal of X. □

Now, we introduce a congruence relation on bordered GE-algebras with respect to GE-ideals and we derive some bordered GE-morphism theorems.

Definition 4.11. Let G be a GE-ideal of a bordered GE-algebra X. For any $\beta, \gamma \in X$, define a relation R_G on X as follows:

$$(\beta, \gamma) \in R_G \text{ if and only if } (\beta \widetilde{*} \gamma)^\varrho \in G \text{ and } (\gamma \widetilde{*} \beta)^\varrho \in G.$$

Theorem 4.12. *If X is a transitive bordered GE-algebra and G a GE-ideal of X, then R_G is a congruence relation on X. Moreover R_G is a unique congruence such that $[0]_G = G$, where $[0]_G$ is the equivalence class of 0 with respect to R_G.*

Proof. Clearly R_G is reflexive and symmetric. Let $(\beta, \gamma), (\gamma, \alpha) \in R_G$. Then $(\beta \widetilde{*} \gamma)^\varrho \in G, (\gamma \widetilde{*} \beta)^\varrho \in G$ and $(\gamma \widetilde{*} \alpha)^\varrho \in G, (\alpha \widetilde{*} \gamma)^\varrho \in G$. By (8), we get

$$\gamma \widetilde{*} \alpha \leq (\beta \widetilde{*} \gamma) \widetilde{*} (\beta \widetilde{*} \alpha) \leq (\beta \widetilde{*} \gamma)^{\varrho\varrho} \widetilde{*} (\beta \widetilde{*} \alpha)^{\varrho\varrho}.$$

Hence $((\beta \widetilde{*} \gamma)^{\varrho\varrho} \widetilde{*} (\beta \widetilde{*} \alpha)^{\varrho\varrho})^\varrho \leq (\gamma \widetilde{*} \alpha)^\varrho$. Since $(\gamma \widetilde{*} \alpha)^\varrho \in G$, we get that $((\beta \widetilde{*} \gamma)^{\varrho\varrho} \widetilde{*} (\beta \widetilde{*} \alpha)^{\varrho\varrho})^\varrho \in G$. Since $(\beta \widetilde{*} \gamma)^\varrho \in G$, we get $(\beta \widetilde{*} \alpha)^\varrho \in G$. Similarly, we can obtain $(\alpha \widetilde{*} \beta)^\varrho \in G$. Hence $(\beta, \alpha) \in R_G$. Therefore R_G is an equivalence relation on X. Let $(\beta, \gamma) \in R_G$ and $(\mu, \nu) \in R_G$. Then $(\beta \widetilde{*} \gamma)^\varrho \in G, (\gamma \widetilde{*} \beta)^\varrho \in G, (\mu \widetilde{*} \nu)^\varrho \in G$ and $(\nu \widetilde{*} \mu)^\varrho \in G$. Since X is transitive, we get $\beta \widetilde{*} \gamma \leq (\mu \widetilde{*} \beta) \widetilde{*} (\mu \widetilde{*} \gamma)$ and so $((\mu \widetilde{*} \beta) \widetilde{*} (\mu \widetilde{*} \gamma))^\varrho \leq (\beta \widetilde{*} \gamma)^\varrho$. Since $(\beta \widetilde{*} \gamma)^\varrho \in G$, we get $((\mu \widetilde{*} \beta) \widetilde{*} (\mu \widetilde{*} \gamma))^\varrho \in G$. Similarly, we can get $((\mu \widetilde{*} \gamma) \widetilde{*} (\mu \widetilde{*} \beta))^\varrho \in G$ since $(\gamma \widetilde{*} \beta)^\varrho \in G$. Hence $(\mu \widetilde{*} \beta, \mu \widetilde{*} \gamma) \in R_G$. Also, $\nu \widetilde{*} \gamma \leq (\mu \widetilde{*} \nu) \widetilde{*} (\mu \widetilde{*} \gamma)$ since X is transitive. Thus

$$\mu \widetilde{*} \nu \leq (\nu \widetilde{*} \gamma) \widetilde{*} (\mu \widetilde{*} \gamma) \leq ((\nu \widetilde{*} \gamma) \widetilde{*} (\mu \widetilde{*} \gamma))^{\varrho\varrho}$$

Hence $((\nu \widetilde{*} \gamma) \widetilde{*} (\mu \widetilde{*} \gamma))^\varrho \leq (\mu \widetilde{*} \nu)^\varrho$. Since $(\mu \widetilde{*} \nu)^\varrho \in G$, we get $((\nu \widetilde{*} \gamma) \widetilde{*} (\mu \widetilde{*} \gamma))^\varrho \in G$. Similarly, we get $((\mu \widetilde{*} \gamma) \widetilde{*} (\nu \widetilde{*} \gamma))^\varrho \in G$ since $(\nu \widetilde{*} \mu)^\varrho \in G$. Thus $(\mu \widetilde{*} \gamma, \nu \widetilde{*} \gamma) \in R_G$. Therefore R_G is a congruence on X. Now, let $\beta \in [0]_G$. Then $\beta^{\varrho\varrho} = (\beta \widetilde{*} 0)^\varrho \in G$. Since $\beta \leq \beta^{\varrho\varrho}$, we get $\beta \in G$. Therefore $[0]_G \subseteq G$. Again, let $\beta \in G$. Then $(\beta \widetilde{*} 0)^\varrho = \beta^{\varrho\varrho} \in G$. Clearly $(0 \widetilde{*} \beta)^\varrho = 1^\varrho = 0 \in G$. Hence $(\beta, 0) \in R_G$, which implies $\beta \in [0]_G$. Thus $G \subseteq [0]_G$. Therefore $[0]_G = G$. □

We can observe that $X/R_G = \{[\beta]_G \mid \beta \in X\}$(where $[\beta]_G$ is the equivalence class of β with respect to R_G) is a bordered GE-algebra in which the binary operation $\widetilde{*}_G$ is defined as $[\beta]_G \widetilde{*}_G [\gamma]_G = [\beta \widetilde{*}_X \gamma]_G$ for $\beta, \gamma \in X$. Moreover, X/R_G contains the element $[0]_G$. For any GE-ideal G of a transitive bordered GE-algebra X, we can get the bordered GE-epimorphism $\chi : X \to X/R_G$ given by $\chi(\beta) = [\beta]_G$.

Theorem 4.13. *Let G, M be two GE-ideals of a transitive bordered GE-algebra X. Then*

$$G \vee M = \{\beta \in X \mid \gamma^\varrho \widetilde{*} (\delta^\varrho \widetilde{*} \beta^\varrho) = 1 \text{ for some } \gamma \in G \text{ and } \delta \in M\}$$

is the smallest GE-ideal of X containing G and M.

Proof. Clearly, $0 \in G \vee M$. Let $\beta \in G \vee M$ and $(\beta^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho} \in G \vee M$. Then there exists $\gamma, \nu \in G$ and $\delta, \tau \in M$ such that $\gamma^{\varrho}\widetilde{*}(\delta^{\varrho}\widetilde{*}\beta^{\varrho}) = 1$ and $\nu^{\varrho}\widetilde{*}(\tau^{\varrho}\widetilde{*}(\beta^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho\varrho}) = 1$. Then by Lemma 3.1(iv),(8) and (6), we get

$$1 = \nu^{\varrho}\widetilde{*}(\tau^{\varrho}\widetilde{*}(\beta^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho\varrho}) \leq \nu^{\varrho}\widetilde{*}(\tau^{\varrho}\widetilde{*}(\beta^{\varrho}\widetilde{*}\gamma^{\varrho})) \leq \beta^{\varrho}\widetilde{*}(\nu^{\varrho}\widetilde{*}(\tau^{\varrho}\widetilde{*}\gamma^{\varrho})).$$

Hence $\beta^{\varrho} \leq \nu^{\varrho}\widetilde{*}(\tau^{\varrho}\widetilde{*}\gamma^{\varrho})$. Since X is transitive, we get

$$1 = \gamma^{\varrho}\widetilde{*}(\delta^{\varrho}\widetilde{*}\beta^{\varrho}) \leq \gamma^{\varrho}\widetilde{*}(\delta^{\varrho}\widetilde{*}(\nu^{\varrho}\widetilde{*}(\tau^{\varrho}\widetilde{*}\gamma^{\varrho}))) \leq \gamma^{\varrho}\widetilde{*}(\nu^{\varrho}\widetilde{*}(\delta^{\varrho}\widetilde{*}(\tau^{\varrho}\widetilde{*}\gamma^{\varrho}))).$$

Hence $\gamma^{\varrho}\widetilde{*}(\nu^{\varrho}\widetilde{*}(\delta^{\varrho}\widetilde{*}(\tau^{\varrho}\widetilde{*}\gamma^{\varrho}))) = 1$. Thus by Lemma 3.1(iv), (8) and (6) we get

$$\begin{aligned}(\gamma^{\varrho}\widetilde{*}(\nu^{\varrho}\widetilde{*}(\delta^{\varrho}\widetilde{*}(\tau^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho\varrho})^{\varrho\varrho})^{\varrho\varrho})^{\varrho} &\leq (\gamma^{\varrho}\widetilde{*}(\nu^{\varrho}\widetilde{*}(\delta^{\varrho}\widetilde{*}(\tau^{\varrho}\widetilde{*}\gamma^{\varrho}))))^{\varrho} \\ &= 1^{\varrho} \\ &= 0 \in G\end{aligned}$$

Hence $(\gamma^{\varrho}\widetilde{*}(\nu^{\varrho}\widetilde{*}(\delta^{\varrho}\widetilde{*}(\tau^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho\varrho})^{\varrho\varrho})^{\varrho\varrho})^{\varrho} \in G$ where $\gamma, \nu \in G$ and $\delta, \tau \in M$. Since $\gamma, \nu \in G$, we get $(\delta^{\varrho}\widetilde{*}(\tau^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho\varrho})^{\varrho} \in G$. Put $\mu = (\delta^{\varrho}\widetilde{*}(\tau^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho\varrho})^{\varrho}$. Then $\mu^{\varrho} = (\delta^{\varrho}\widetilde{*}(\tau^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho\varrho})^{\varrho\varrho}$. By Lemma 3.1(iv), (8) and (6), we have

$$\mu^{\varrho} = (\delta^{\varrho}\widetilde{*}(\tau^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho\varrho})^{\varrho\varrho} \leq \delta^{\varrho}\widetilde{*}(\tau^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho\varrho} \leq \delta^{\varrho}\widetilde{*}(\tau^{\varrho}\widetilde{*}\gamma^{\varrho}).$$

Hence $1 = \mu^{\varrho}\widetilde{*}(\delta^{\varrho}\widetilde{*}(\tau^{\varrho}\widetilde{*}\gamma^{\varrho})) \leq \delta^{\varrho}\widetilde{*}(\tau^{\varrho}\widetilde{*}(\mu^{\varrho}\widetilde{*}\gamma^{\varrho}))$. Thus, we get

$$(\delta^{\varrho}\widetilde{*}(\tau^{\varrho}\widetilde{*}(\mu^{\varrho}\widetilde{*}\gamma^{\varrho})))^{\varrho} = 0 \in M.$$

Hence $(\delta^{\varrho}\widetilde{*}(\tau^{\varrho}\widetilde{*}(\mu^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho\varrho})^{\varrho\varrho})^{\varrho} \leq (\delta^{\varrho}\widetilde{*}(\tau^{\varrho}\widetilde{*}(\mu^{\varrho}\widetilde{*}\gamma^{\varrho})))^{\varrho} \in M$. Since $\delta, \tau \in M$, we get $(\mu^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho} \in M$. Put $\nu = (\mu^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho}$. Then $\nu^{\varrho} = (\mu^{\varrho}\widetilde{*}\gamma^{\varrho})^{\varrho\varrho} \leq \mu^{\varrho}\widetilde{*}\gamma^{\varrho}$ and hence

$$1 = \nu^{\varrho}\widetilde{*}\nu^{\varrho} \leq \nu^{\varrho}\widetilde{*}(\mu^{\varrho}\widetilde{*}\gamma^{\varrho}) \leq \mu^{\varrho}\widetilde{*}(\nu^{\varrho}\widetilde{*}\gamma^{\varrho})$$

Since $\mu \in G, \nu \in M$, we get $\gamma \in G \vee M$. Therefore $G \vee M$ is a GE-ideal of X. Let $\beta \in G$. Clearly $\beta^{\varrho}\widetilde{*}(0^{\varrho}\widetilde{*}\beta^{\varrho}) = \beta^{\varrho}\widetilde{*}\beta^{\varrho} = 1$. Since $0 \in M$, we get $\beta \in G \vee M$. Hence $G \subseteq G \vee M$. Similarly, we get $M \subseteq G \vee M$.

Let K be any GE-ideal of X such that $G \subseteq K$ and $M \subseteq K$. Let $\beta \in G \vee M$. Then there exists $\gamma \in G \subseteq K$ and $\delta \in M \subseteq K$ such that $\gamma^{\varrho}\widetilde{*}(\delta^{\varrho}\widetilde{*}\beta^{\varrho}) = 1$. Hence $\gamma^{\varrho}\widetilde{*}(\delta^{\varrho}\widetilde{*}\beta^{\varrho})^{\varrho\varrho} = 1$, which implies $(\gamma^{\varrho}\widetilde{*}(\delta^{\varrho}\widetilde{*}\beta^{\varrho})^{\varrho\varrho})^{\varrho} = 0 \in K$. Since $\gamma \in K$, we get $(\delta^{\varrho}\widetilde{*}\beta^{\varrho})^{\varrho} \in K$. Since $\delta \in K$, we get $\beta \in K$. Hence $G \vee M \subseteq K$. Therefore $G \vee M$ is the smallest GE-ideal which contains both G and M. □

The following example illustrates Theorem 4.13.

TABLE 5. Cayley tables for the binary operation "$\widetilde{*}$"

$\widetilde{*}$	0	1	2	3	4
0	1	1	1	1	1
1	0	1	2	3	4
2	3	1	1	3	3
3	2	1	2	1	1
4	2	1	2	1	1

Example 4.14. Consider the set $X = \{0, 1, 2, 3, 4\}$ with binary operation "$\widetilde{*}$" which is given by the following Table 5. Then $(X, \widetilde{*}, 1)$ is a transitive bordered GE-algebra. Here we can observe that $M_1 = \{0\}, M_2 = \{0, 2\}, M_3 = \{0, 3, 4\}$, and X are the only GE-ideals of X and $M_1 \vee M_2 = M_2$ is the smallest GE-ideal of X containing M_1 and M_2.

Since the intersection of GE-ideals is again a GE-ideal, the following is direct:

Corollary 4.15. *For any transitive bordered GE-algebra X, the set $\mathcal{I}(X)$ of all GE-ideals of X forms a complete lattice.*

Theorem 4.16. *Let G and M be two GE-ideals of a transitive bordered GE-algebra X. Then the mapping $\xi : X \to (X/R_G) \times (X/R_M)$ defined by $\xi(\beta) = ([\beta]_G, [\beta]_M)$ for all $\beta \in X$ is a GE-morphism. Moreover, the following hold:*

(i) If ξ is injective, then $G \cap M = \{0\}$,

(ii) If ξ is surjective, then $G \vee M = X$.

Proof. Clearly ξ is well-defined. Let $\beta, \gamma \in X$. Then

$$\xi(\beta\widetilde{*}\gamma) = ([\beta\widetilde{*}\gamma]_G, [\beta\widetilde{*}\gamma]_M) = ([\beta]_G\widetilde{*}_G[\gamma]_G, [\beta]_M\widetilde{*}_M[\gamma]_M) = ([\beta]_G, [\beta]_M)\widetilde{*}([\gamma]_G, [\gamma]_M) = \xi(\beta)\widetilde{*}\xi(\gamma).$$

Therefore ξ is a GE-morphism.

(i). Suppose ξ is injective. Then clearly $DKer\ (\xi) = \{0\}$. Now

$$\begin{aligned}
\beta \in DKer(\xi) &\Leftrightarrow \xi(\beta) = \overline{0} = ([0]_G, [0]_M) \\
&\Leftrightarrow ([\beta]_G, [\beta]_M) = ([0]_G, [0]_M) \\
&\Leftrightarrow [\beta]_G = [0]_G \text{ and } [\beta]_M = [0]_M \\
&\Leftrightarrow \beta^{\varrho\varrho} \in G \text{ and } \beta^{\varrho\varrho} \in M \\
&\Leftrightarrow \beta \in G \text{ and } \beta \in M \qquad \text{since } \beta \leq \beta^{\varrho\varrho} \\
&\Leftrightarrow \beta \in G \cap M
\end{aligned}$$

Thus $DKer(\xi) = G \cap M$. Therefore $G \cap M = \{0\}$ whenever ξ is injective.

(ii). Assume that ξ is surjective. Clearly $([0]_G, [1]_M) \in (X/G) \times (X/M)$. Since ξ is surjective, there exists $\beta \in X$ such that $\xi(\beta) = ([0]_G, [1]_M)$. Hence

$$\begin{aligned} \xi(\beta) = ([0]_G, [1]_M) &\Leftrightarrow ([\beta]_G, [\beta]_M) = ([0]_G, [1]_M) \\ &\Leftrightarrow [\beta]_G = [0]_G \text{ and } [\beta]_M = [1]_M \\ &\Leftrightarrow \beta^{\varrho\varrho} \in G \text{ and } \beta^{\varrho} \in M \\ &\Leftrightarrow \beta \in G \text{ and } \beta^{\varrho} \in M \end{aligned}$$

Clearly $\beta^{\varrho}\widetilde{*}(\beta^{\varrho\varrho}\widetilde{*}1^{\varrho}) = \beta^{\varrho}\widetilde{*}\beta^{\varrho\varrho\varrho} = 1$. Since $\beta \in G$ and $\beta^{\varrho} \in M$, it imply that $1 \in G \vee M$. Therefore $G \vee M = X$ whenever ξ is surjective. □

Theorem 4.17. *Let $(X, \widetilde{*}_X, 1_X)$, $(Y, \widetilde{*}_Y, 1_Y)$ and $(Z, \widetilde{*}_Z, 1_Z)$ be bordered GE-algebras. If $\xi : X \to Y$ and $\chi : Y \to Z$ are bordered GE-morphisms, then*

$$\chi \circ \xi : X \to Z, \ \beta \mapsto \chi(\xi(\beta))$$

is a bordered GE-morphism.

Proof. Straightforward. □

Theorem 4.18. (Fundamental bordered GE-morphism theorem) *Given two bordered GE-algebras $(X, \widetilde{*}_X, 1_X)$ and $(Y, \widetilde{*}_Y, 1_Y)$ in which $(X, \widetilde{*}_X, 1_X)$ is transitive and $(Y, \widetilde{*}_Y, 1_Y)$ is duplex and antisymmetric, let $\xi : X \to Y$ be a bordered GE-morphism, G a GE-ideal of X and φ the canonical bordered GE-epimorphism $X \to X/R_G$. If G is a subset of* Dker(ξ) *then there exists a unique bordered GE-morphism $\widetilde{\xi} : X/R_G \to Y$ such that the diagram:*

$$\begin{array}{ccc} X & \xrightarrow{\ \xi\ } & Y \\ \downarrow{\scriptstyle\varphi} & & \uparrow{\scriptstyle\widetilde{\xi}} \\ X/R_G & = & X/R_G \end{array} \tag{27}$$

is commutative. Moreover, $\widetilde{\xi}$ is a bordered GE-isomorphism if and only if ξ is a bordered GE-epimorphism and $G =$ Dker(ξ).

Proof. Let G be a subset of Dker(ξ) and define

$$\widetilde{\xi} : X/R_G \to Y, \ [\beta]_G \mapsto \xi(\beta).$$

Let $[\beta]_G, [\gamma]_G \in X/R_G$ be such that $[\beta]_G = [\gamma]_G$. Then $(\beta, \gamma) \in R_G$, and so $(\beta\widetilde{*}_X\gamma)^{\varrho} \in G \subseteq$ Dker(ξ) and $(\gamma\widetilde{*}_X\beta)^{\varrho} \in G \subseteq$ Dker(ξ). Thus

$$\xi((\beta\widetilde{*}_X\gamma)^{\varrho}) = 0_Y \Rightarrow (\xi(\beta\widetilde{*}_X\gamma))^{\varrho} = 0_Y \Rightarrow (\xi(\beta)\widetilde{*}_Y\xi(\gamma))^{\varrho} = 0_Y \Rightarrow \xi(\beta)\widetilde{*}_Y\xi(\gamma) = 1_Y$$

and

$$\xi((\gamma\tilde{*}_X\beta)^{\varrho}) = 0_Y \Rightarrow (\xi(\gamma\tilde{*}_X\beta))^{\varrho} = 0_Y \Rightarrow (\xi(\gamma)\tilde{*}_Y\xi(\beta))^{\varrho} = 0_Y \Rightarrow \xi(\gamma)\tilde{*}_Y\xi(\beta) = 1_Y.$$

Since $(Y, \tilde{*}_Y, 1_Y)$ is antisymmetric, we have

$$\widetilde{\xi}([\beta]_G) = \xi(\beta) = \xi(\gamma) = \widetilde{\xi}([\gamma]_G).$$

Hence $\widetilde{\xi}$ is well-defined. For any $\beta, \gamma \in X$, we can observe that

$$\widetilde{\xi}([\beta]_G\tilde{*}_G[\gamma]_G) = \widetilde{\xi}([\beta\tilde{*}_X\gamma]_G) = \xi(\beta\tilde{*}_X\gamma) = \xi(\beta)\tilde{*}_Y\xi(\gamma) = \widetilde{\xi}([\beta]_G)\tilde{*}_Y\widetilde{\xi}([\gamma]_G),$$

$$\widetilde{\xi}([0_X]_G) = \xi(0_X) = 0_Y.$$

which shows that $\widetilde{\xi}$ is a bordered GE-morphism. Since

$$(\widetilde{\xi} \circ \varphi)(\beta) = \widetilde{\xi}(\varphi(\beta)) = \widetilde{\xi}([\beta]_G) = \xi(\beta)$$

for all $\beta \in X$, we have $\widetilde{\xi} \circ \varphi = \xi$, that is, the diagram in (27) is commutative. Let $\widetilde{\chi} : X/R_G \to Y$ be a GE-morphism such that $\widetilde{\chi} \circ \varphi = \xi$. Then

$$\widetilde{\chi}([x]_G) = \widetilde{\chi}(\varphi(\beta)) = (\widetilde{\chi} \circ \varphi)(\beta) = \xi(\beta) = (\widetilde{\xi} \circ \varphi)(\beta) = \widetilde{\xi}(\varphi(\beta)) = \widetilde{\xi}([x]_G)$$

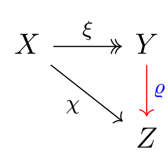
for all $[\beta]_G \in X/R_G$. Hence $\widetilde{\chi} = \widetilde{\xi}$, which means that $\widetilde{\xi}$ is unique. Suppose $\widetilde{\xi}$ is a bordered GE-isomorphism. For every $\gamma \in Y$, there exists $[\beta]_G \in X/R_G$ such that $\widetilde{\xi}([\beta]) = \gamma$. Thus $\xi(\beta) = \widetilde{\xi}([\beta]_G) = \gamma$, and so ξ is a bordered GE-epimorphism. Let $\beta \in D\ker(\xi)$. Then $\widetilde{\xi}([\beta]) = \xi(\beta) = 0_Y = \widetilde{\xi}([0]_G)$ and hence $[\beta]_G = [0]_G$. Therefore $\beta \leq \beta^{\varrho\varrho} = (\beta\tilde{*}_X 0)^{\varrho} \in G$ and hence $\beta \in G$. Hence $G = D\ker(\xi)$. Conversely, assume that ξ is a bordered GE-epimorphism and $G = D\ker(\xi)$. Let $[\beta]_G, [\gamma]_G \in X/R_G$ be such that $\widetilde{\xi}([\beta]_G) = \widetilde{\xi}([\gamma]_G)$. Then $\xi(\beta) = \xi(\gamma)$, and

$$\xi(\beta\tilde{*}_X\gamma) = \xi(\beta)\tilde{*}_Y\xi(\gamma) = \xi(\gamma)\tilde{*}_Y\xi(\gamma) = 1_Y \Rightarrow (\xi(\beta\tilde{*}_X\gamma))^{\varrho} = 0_Y \Rightarrow \xi((\beta\tilde{*}_X\gamma)^{\varrho}) = 0_Y.$$

Hence $(\beta\tilde{*}_X\gamma)^{\varrho} \in D\ker(\xi) = G$. Similarly, $(\gamma\tilde{*}_X\beta)^{\varrho} \in G$. Therefore $(\beta, \gamma) \in R_G$ and $[\beta]_G = [\gamma]_G$. Hence $\widetilde{\xi}$ is injective. Let $\gamma \in Y$. Then there exists $\beta \in X$ such that $\xi(\beta) = \gamma$. Thus $\gamma = \xi(\beta) = \widetilde{\xi}([\beta]_G)$, so $\widetilde{\xi}$ is surjective. Therefore $\widetilde{\xi}$ is a bordered GE-isomorphism. □

Theorem 4.19. *Given three bordered GE-algebras $(X, \tilde{*}_X, 1_X)$, $(Y, \tilde{*}_Y, 1_Y)$ and $(Z, \tilde{*}_Z, 1_Z)$ in which $(Z, \tilde{*}_Z, 1_Z)$ is duplex and antisymmetric, let $\xi : X \to Y$ and $\chi : X \to Z$ be bordered GE-morphisms. If $\mathrm{D}\ker(\xi) \subseteq \mathrm{D}\ker(\chi)$ and ξ is a bordered GE-epimorphism, then there exists a unique bordered GE-morphism $\varrho : Y \to Z$ such that the diagram*

(28)



is commutative.

Proof. Assume that ξ is a bordered GE-epimorphism and $\mathrm{D}\ker(\xi) \subseteq \mathrm{D}\ker(\chi)$. For every $\gamma \in Y$, there exists $\beta \in X$ such that $\xi(\beta) = \gamma$. For the element $\beta \in X$, put $\alpha := \chi(\beta)$ and define

$$\varrho : Y \to Z,\ \gamma \mapsto \alpha = \chi(\beta).$$

We first show that ϱ is well-defined. Let $\gamma_1, \gamma_2 \in Y$ be such that $\gamma_1 = \gamma_2$, $\gamma_1 = \xi(\beta_1)$ and $\gamma_2 = \xi(\beta_2)$ for some $\beta_1, \beta_2 \in X$. Then $\xi(\beta_1 \widetilde{*}_X \beta_2) = \xi(\beta_1)\widetilde{*}_Y \xi(\beta_2) = 1_Y$ and hence $\xi((\beta_1\widetilde{*}_X\beta_2)^{\varrho}) = (\xi(\beta_1\widetilde{*}_X\beta_2))^{\varrho} = 0_Y$. Therefore $(\beta_1\widetilde{*}_X\beta_2)^{\varrho} \in \ker(\xi) \subseteq \ker(\chi)$. Thus $0_Z = \chi((\beta_1\widetilde{*}_X\beta_2)^{\varrho}) = (\chi(\beta_1)\widetilde{*}_Z\chi(\beta_2))^{\varrho} \Rightarrow 1_Z = \chi(\beta_1)\widetilde{*}_Z\chi(\beta_2)$ since Z is duplex. The similarly way induces $\chi(\beta_2)\widetilde{*}_Z\chi(\beta_1) = 1_Z$, and thus $\chi(\beta_1) = \chi(\beta_2)$ Since Z is antisymmetric. Hence ϱ is well-defined. Also, we have $\chi(\beta) = \alpha = \varrho(\gamma) = \varrho(\xi(\beta))$ for all $\beta \in X$, which shows that the diagram in (28) is commutative. Let $\gamma_1, \gamma_2 \in Y$. For every $\beta_1, \beta_2 \in X$ with $\gamma_1 = \xi(\beta_1)$ and $\gamma_2 = \xi(\beta_2)$, we have

$$\begin{aligned}
\varrho(\gamma_1\widetilde{*}_Y\gamma_2) &= \varrho(\xi(\beta_1)\widetilde{*}_Y\xi(\beta_2)) \\
&= \varrho(\xi(\beta_1\widetilde{*}_X\beta_2)) = \chi(\beta_1\widetilde{*}_X\beta_2) \\
&= \chi(\beta_1)\widetilde{*}_Z\chi(\beta_2) = \varrho(\xi(\beta_1))\widetilde{*}_Z\varrho(\xi(\beta_2)) \\
&= \varrho(\gamma_1)\widetilde{*}_Z\varrho(\gamma_2).
\end{aligned}$$

We know that $\xi(0_X) = 0_Y \in Y$. Hence $0_X \in D\ker(\xi) \subseteq D\ker(\chi)$. Therefore $\chi(0_X) = 0_Z$. Now $\varrho(0_Y) = \varrho(\xi(0_X)) = \varrho\circ\xi(0_X) = \chi(0_X) = 0_Z$. Hence ϱ is a bordered GE-morphism. The uniqueness of ϱ is straightforward since ξ is a bordered GE-epimorphism. □

Theorem 4.20. *Given two bordered GE-algebras $(X, \widetilde{*}_X, 1_X)$ and $(Y, \widetilde{*}_Y, 1_Y)$, let $\xi : X \to Y$ be a bordered GE-epimorphism. If $(X, \widetilde{*}_X, 1_X)$ is transitive and $(Y, \widetilde{*}_Y, 1_Y)$ is duplex and antisymmetric, then $X/R_{\mathrm{D}\ker(\xi)}$ is bordered GE-isomorphic to Y.*

Proof. Note from Corollary 4.8 that $\mathrm{D}\ker(\xi)$ is a GE-ideal of X, and so $X/R_{\mathrm{D}\ker(\xi)}$ is a bordered GE-algebra with the special element $[0_X]_{D\ker(\xi)}$. Define a mapping

$$\chi : X/R_{\mathrm{Dker}(\xi)} \to Y,\ [\beta]_{\mathrm{Dker}(\xi)} \mapsto \xi(\beta).$$

If $[\beta_1]_{\mathrm{Dker}(\xi)} = [\beta_2]_{\mathrm{Dker}(\xi)}$ in $X/R_{\mathrm{Dker}(\xi)}$, then $(\beta_1\widetilde{*}_X\beta_2)^{\varrho} \in \mathrm{D}\ker(\xi)$ and $(\beta_2\widetilde{*}_X\beta_1)^{\varrho} \in \mathrm{D}\ker(\xi)$. Hence

$$\xi((\beta_1\widetilde{*}_X\beta_2)^{\varrho}) = 0_Y \Rightarrow (\xi(\beta_1\widetilde{*}_X\beta_2))^{\varrho} = 0_Y \Rightarrow (\xi(\beta_1)\widetilde{*}_Y\xi(\beta_2))^{\varrho} = 0_Y \Rightarrow \xi(\beta_1)\widetilde{*}_Y\xi(\beta_2) = 1_Y$$

and

$$\xi((\beta_2\widetilde{*}_X\beta_1)^\varrho) = 0_Y \Rightarrow (\xi(\beta_2\widetilde{*}_X\beta_1))^\varrho = 0_Y \Rightarrow (\xi(\beta_2)\widetilde{*}_Y\xi(\beta_1))^\varrho = 0_Y \Rightarrow \xi(\beta_2)\widetilde{*}_Y\xi(\beta_1) = 1_Y,$$

and thus $\chi([\beta_1]_{\mathrm{D}\ker(\xi)}) = \xi(\beta_1) = \xi(\beta_2) = \chi([\beta_2]_{\mathrm{D}\ker(\xi)})$. This shows that χ is a well-defined mapping. For each $\gamma \in Y$, there exists $\beta \in X$ such that $\xi(\beta) = \gamma$ since ξ is onto. Thus $\chi([\beta]_{\mathrm{D}\ker(\xi)}) = \xi(\beta) = \gamma$ which shows that χ is onto. Suppose that $\chi([\beta]_{\mathrm{D}\ker(\xi)}) = \chi([\gamma]_{\mathrm{D}\ker(\xi)})$ in $X/R_{\mathrm{D}\ker}(\xi)$. Then $\xi(\beta) = \xi(\gamma)$ and hence $\xi(\beta)\widetilde{*}_X\xi(\gamma) = 1_Y$ which implies that $\xi((\beta\widetilde{*}_X\gamma)^\varrho) = 0_Y$. Hence $(\beta\widetilde{*}_X\gamma)^\varrho \in \mathrm{D}\ker(\xi)$ and similarly $(\gamma\widetilde{*}_X\beta)^\varrho \in \mathrm{D}\ker(\xi)$. Therefore $(\beta,\gamma) \in R_{D\ker(\xi)}$. Hence $[\beta]_{\mathrm{D}\ker(\xi)} = [\gamma]_{\mathrm{D}\ker(\xi)}$. Hence χ is injective. Let $[\beta]_{\mathrm{D}\ker(\xi)} \in X/R_{\mathrm{D}\ker(\xi)}$ and $[\gamma]_{\mathrm{D}\ker(\xi)} \in X/R_{\mathrm{D}\ker(\xi)}$. Then

$$\begin{aligned}
\chi([\beta]_{\mathrm{D}\ker(\xi)}\widetilde{*}_{\mathrm{D}\ker(\xi)}[\gamma]_{\mathrm{Dker}(\xi)}) &= \chi([\beta\widetilde{*}_X\gamma]_{\mathrm{Dker}(\xi)}) \\
&= \xi(\beta\widetilde{*}_X\gamma) \\
&= \xi(\beta)\widetilde{*}_Y\xi(\gamma) \\
&= \chi([\beta]_{\mathrm{Dker}(\xi)})\widetilde{*}_Y\chi([\gamma]_{\mathrm{D}\ker(\xi)}).
\end{aligned}$$

Also, $\chi([0_X]_{D\ker(\xi)}) = \xi(0_X) = 0_Y$. Thus $X/R_{D\ker(\xi)}$ is bordered GE-isomorphic to Y. □

Theorem 4.21. *Given two transitive bordered GE-algebras $(X, \widetilde{*}_X, 1_X)$ and $(Y, \widetilde{*}_Y, 1_Y)$, let $\xi : X \to Y$ be a bordered GE-epimorphism. If $(Y, \widetilde{*}_Y, 1_Y)$ is antisymmetric and K is a GE-ideal of Y, then $X/R_{\xi^{-1}(K)}$ is bordered GE-isomorphic to Y/R_K.*

Proof. We know that $\xi^{-1}(K)$ is a GE-ideal of X. Hence we can make the quotient GE-algebra $X/R_{\xi^{-1}(K)}$. Let $\pi : Y \to Y/R_K$ be the canonical GE-morphism. Then $\chi := \pi \circ \xi : X \to Y/R_K$ is a GE-epimorphism and Y/R_K is antisymmetric since Y is antisymmetric. For any $\beta \in X$, we get $\chi(\beta) = (\pi \circ \xi)(\beta) = \pi(\xi(\beta)) = [\xi(\beta)]_K$ where $[\xi(\beta)]_K$ is the equivalence class containing $\xi(\beta)$ in Y/R_K. If $\beta \in \xi^{-1}(K)$, then $\xi(\beta) \in K$ and so $[\xi(\beta)]_K = K$ which says $\chi(\beta) = K$. Hence $\beta \in D\ker(\chi)$, and thus $\xi^{-1}(K) \subseteq D\ker(\chi)$. If $\beta \in D\ker(\chi)$, then $K = \chi(\beta) = [\xi(\beta)]_K$. Hence $\xi(\beta) \in K$, i.e., $\beta \in \xi^{-1}(K)$, and so $D\ker(\chi) \subseteq \xi^{-1}(K)$. Therefore $D\ker(\chi) = \xi^{-1}(K)$. It follows from Theorem 4.20 that there exists a bijective bordered GE-morphism $\xi : X/R_{\xi^{-1}(K)} \to Y/R_K$, and so $X/R_{\xi^{-1}(K)}$ is bordered GE-isomorphic to Y/R_K.
□

Proposition 4.22. *Given two bordered GE-algebras $(X, \widetilde{*}_X, 1_X)$ and $(Y, \widetilde{*}_Y, 1_Y)$, let $\xi : X \to Y$ be a bordered GE-epimorphism. If G is a GE-ideal of X which contains* $\mathrm{D}\ker(\xi)$*, then $\xi^{-1}(\xi(G)) = G$.*

Proof. It is clear that $G \subseteq \xi^{-1}(\xi(G))$. If $\beta \in \xi^{-1}(\xi(G))$, then $\xi(\beta) \in \xi(G)$ and hence there exists $\gamma \in G$ such that $\xi(\beta) = \xi(\gamma)$. Hence

$$\xi(\beta \widetilde{*}_X \gamma) = \xi(\beta) \widetilde{*}_Y \xi(\gamma) = 1_Y \Rightarrow (\xi(\beta \widetilde{*}_X \gamma))^{\varrho} = 0_Y \Rightarrow \xi((\beta \widetilde{*}_X \gamma)^{\varrho}) = 0_Y.$$

which implies that $(\beta \widetilde{*}_X \gamma)^{\varrho} \in D\ker(\xi) \subseteq G$. Thus $\beta \in G$ since G is a GE-ideal of X. Therefore $\xi^{-1}(\xi(G)) = G$. □

5. Conclusion

In this paper, we have studied the properties of GE-ideals of a transitive bordered GE-algebra and given the characterization of GE-ideals. We have observed that the set of all GE-ideals of a transitive bordered GE-algebra forms a complete lattice. We have introduced the notion of bordered GE-morphism and established fundamental bordered GE-morphism theorem. We have introduced a congruence relation on a bordered GE-algebra with respect to GE-ideal and derived some bordered GE-morphism theorems.

6. Acknowledgment

The authors wish to thank the anonymous reviewers for their valuable suggestions.

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