Algebraic Structures and Their Applications Vol. 9 No. 2 (2022) pp 113-120.

Research Paper

A NEW CHARACTERIZATION OF PROJECTIVE SPECIAL UNITARY GROUPS $U_3(3^n)$ BY THE ORDER OF GROUP AND THE NUMBER OF ELEMENTS WITH THE SAME ORDER

BEHNAM EBRAHIMZADEH, ALI IRANMANESH*

Abstract. In this paper, we prove that projective special unitary groups $U_3(3^n)$, where $3^{2n} - 3^n + 1$ is a prime number and $3^n \equiv \pm 2(\mod 5)$, can be uniquely determined by the order of group and the number of elements with the same order.

1. Introduction

Let G be a finite group, $\pi(G)$ be the set of prime divisors of order of G and $\pi_e(G)$ be the set of elements order in G. If $k \in \pi_e(G)$, then we denote the set of the number of elements of order k in G by $m_k(G)$ and the set of the number of elements with the same order in G by $nse(G)$. In other word, $nse(G) = m_k(G)|k \in \pi_e(G)$. Also we denote a Sylow p-subgroup of G by G_p and the number of Sylow p-subgroups of G by $n_p(G)$. The prime graph $\Gamma(G)$ of group G is a graph whose vertex set is $\pi(G)$, and two distinct vertices u and v are adjacent if

DOI: 10.22034/as.2022.2675

MSC(2010): 20D06; 20D60

Keywords: Element order, Prime graph, Projective special unitary group, The number of elements with the same order.

Received: 22 September 2021, Accepted: 26 April 2022.

∗Corresponding author

© 2022 Yazd University.

and only if $uv \in \pi_e(G)$. Moreover, assume that $\Gamma(G)$ has $t(G)$ connected components π_i, for $i = 1, 2, , t(G)$. In the case where G is of even order, we always assume that $2 \in \pi_1$.
One of the important problems in finite groups theory is group characterization by group theoretic properties namely the element order, the set of elements with the same order, etc. In this research, we discuss about group characterization by using the order of the group and $nse(G)$. Next, for example the authors in ([3, 4, 5, 9, 10, 11, 16, 18]), proved that the sporadic groups, symmetric groups S_r, suzuki groups, Ree-groups ${}^2G_2(q)$ where $q \pm \sqrt{3q} + 1$ is prime number, simple K_4-groups $L_2(p)$ and $L_2(2^n)$ where $2^n - 1$ or $2^n + 1$ is prime number, the symplectic group $C_2(3^n)$ and suzuki group $Sz(q)$, where $q \pm \sqrt{2q} + 1$ is prime number by are characterizable by using the order of the group and $nse(G)$. In this paper, we prove that projective special unitary groups $U_3(3^n)$, where $3^{2n} - 3^n + 1$ is a prime number and $3^n \equiv \pm 2 (\mod 5)$, can be uniquely determined by the order of group and the number of elements with the same order. In fact, we prove the following main theorem.

Main Theorem. Let G be a group with $|G| = |U_3(3^n)|$ and $nse(G) = nse(U_3(3^n))$, where $3^{2n} - 3^n + 1$ is a prime number and $3^n \equiv \pm 2(\mod 5)$. Then $G \cong U_3(3^n)$.

In this research, we consider the Projective special unitary group $U_3(3^n)$, where $3^{2n} - 3^n + 1$ is a prime number. In fact we use Lemma 2.8 of [22], where G be a non-abelian simple group such that order of G and number 5 is coprime.

2. Preliminaries

Lemma 2.1. [8] *Let G be a Frobenius group of even order with kernel K and complement H. Then*

(1) *$t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$;*
(2) *$|H|$ divides $|K| - 1$;*
(3) *K is nilpotent.*

Definition 2.2. A group G is called a 2-Frobenius group if there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H respectively.

Lemma 2.3. [2] *Let G be a 2-Frobenius group of even order. Then*

(1) *$t(G) = 2$, $\pi(H) \cup \pi(G/K) = \pi_1$ and $\pi(K/H) = \pi_2$;*
(2) *G/K and K/H are cyclic groups satisfying $|G/K|$ divides $|Aut(K/H)|$.*

Lemma 2.4. [23] *Let G be a finite group with $t(G) \geq 2$. Then one of the following statements holds:*

(1) *G is a Frobenius group;*

(2) *G is a 2-Frobenius group;*

(3) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|Out(K/H)|$.*

Lemma 2.5. [7] *Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G \mid g^m = 1\}$, then $m \mid |L_m(G)|$.*

Lemma 2.6. *Let G be a finite group. Then for every $i \in \pi_e(G)$, $\varphi(i)$ divides $m_i(G)$, and i divides $\sum_{j|i} m_j(G)$. Moreover, if $i > 2$, then $m_i(G)$ is even.*

Proof. By Lemma 2.5, the proof is straightforward. □

Lemma 2.7. [24] *Let q, k, l be natural numbers. Then*

(1) $(q^k - 1, q^l - 1) = q^{(k,l)} - 1$.

(2) $$(q^k + 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if both } \frac{k}{(k,l)} \text{ and } \frac{l}{(k,l)} \text{ are odd,} \\ (2, q+1) & \text{otherwise.} \end{cases}$$

(3) $$(q^k - 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if } \frac{k}{(k,l)} \text{ is even and } \frac{l}{(k,l)} \text{ is odd,} \\ (2, q+1) & \text{otherwise.} \end{cases}$$

In particular, for every $q \geq 2$ and $k \geq 1$, the inequality $(q^k - 1, q^k + 1) \leq 2$ holds.

Lemma 2.8. [22] *Let G be a non-abelian simple group such that $(5, |G|) = 1$. Then G is isomorphic to one of the following groups:*

(1) $A_n(q)$, $n = 1, 2$, $q \equiv \pm 2 \pmod 5$;

(2) $G_2(q)$, $q \equiv \pm 2 \pmod 5$;

(3) ${}^2A_2(q)$, $q \equiv \pm 2 \pmod 5$;

(4) ${}^3D_4(q)$, $q \equiv \pm 2 \pmod 5$;

(5) ${}^2G_2(q)$, $q = 3^{2m+1}$, $m \geq 1$.

3. Proof of the Main Theorem

In this section,we prove that the projective special unitary groups $U_3(3^n)$ are characterizable by order of group and the number of elements with the same order. In fact, we prove that if G is a group with$|G| = |U_3(3^n)|$ and $nse(G) = nse(U_3(3^n))$, where $3^n \equiv \pm 2(\mod 5)$ and $3^{2n} - 3^n + 1$ is a prime number, then $G \cong U_3(3^n)$. We divide the proof to several lemmas. From now on, we denote the projective special unitary group $U_3(3^n)$ by U and the numbers 3^n and $3^{2n} - 3^n + 1$ by q and p, respectively. Recall that G is a group with $|G| = |U|$ and $nse(G) = nse(U)$.

Lemma 3.1. *Let $U := U_3(3^n)$. If $p := 3^{2n} - 3^n + 1$ is a prime number. Then $m_p(U) = (p-1)|U|/(3p)$ and for every $i \in \pi_e(U) - \{1, p\}$, p divides $m_i(U)$.*

Proof. Since $|U_p| = p$, we deduce that U_p is a cyclic group of order p. Thus $m_p(U) = \varphi(p)n_p(U) = (p-1)n_p(U)$. Now it is enough to show $n_p(U) = |U|/(3p)$. By [23], p is an isolated vertex of $\Gamma(G)$. Hence $|C_U(U_p)| = p$ and $|N_U(U_p)| = xp$ for a natural number x. We know that $N_U(U_p)/C_U(U_p)$ embeds into $Aut(U_p)$, which implies $x \mid p-1$. Furthermore, by the Sylow's Theorem, $n_p(U) = |U : N_U(U_p)|$ and $n_p(U) \equiv 1 \pmod p$.Therefore p divides $|U|/(xp) - 1$. Thus $q^2 - q + 1$ divides $q^3(q^3+1)(q^2-1)/(xp) - 1$. It follows that $q^2 - q + 1$ divides $q^6 + q^5 - q^4 - q^3 - x$, so we have $p \mid 3 - x$, since $x \mid p - 1$, we deduce that $x = 3$. Therefore $n_p(U) = |U|/(3p)$. Let $i \in \pi_e(U) - \{1, p\}$. Since p is an isolated vertex of $\Gamma(U)$, we conclude that $p \nmid i$ and $pi \notin \pi_e(U)$. Thus U_p acts fixed point freely on the set of elements of order i by conjugation and hence $|U_p| \mid m_i(U)$. So we conclude that $p \mid m_i(U)$. □

Lemma 3.2. *$m_2(G) = m_2(U)$, $m_p(G) = m_p(U)$, $n_p(G) = n_p(U)$, p is an isolate vertex of $\Gamma(G)$ and $p \mid m_k(G)$ for every $k \in \pi_e(G) - \{1, p\}$.*

Proof. By Lemma 2.6, for every $1 \neq r \in \pi_e(G)$, $r = 2$ if and only if $m_r(G)$ is odd. Thus $m_2(G) = m_2(U)$. According to Lemma 2.6, $(m_p(G), p) = 1$.Thus $p \nmid m_p(G)$ and hence Lemma 3.1 implies that $m_p(G) \in \{m_1(U), m_2(U), m_p(U)\}$. Moreover, $m_p(G)$ is even, so $m_p(G) = m_p(U)$. Since G_p and U_p are cyclic groups of order p and $m_p(G) = m_p(U)$, so $m_p(G) = \varphi(p)n_p(G) = \varphi(p)n_p(U) = m_p(U)$, so $n_p(G) = n_p(U)$. Now, we prove that p is an isolated vertex of Γ(G). Assume to the contrary that there is $t \in \pi(G) - \{p\}$ such that $tp \in \pi_e(G)$. So $m_{tp}(G) = \varphi(tp)n_p(G)k$, where k is the number of cyclic subgroups of order t in $C_G(G_p)$ and since $n_p(G) = n_p(U)$, it follows that $m_{tp}(G) = (t-1)(p-1)|U|k/(3p)$. If $m_{tp}(G) = m_p(U)$, then $t = 2$ and $k = 1$. Furthermore, Lemma 2.5 yields $p \mid m_2(G) + m_{2p}(G)$ and since $m_2(G) = m_2(U)$ and $p \mid m_2(U)$, we have $p \mid m_{2p}(G)$ which is a contradiction. So Lemma 3.1 implies that $p \mid m_{tp}(G)$. Hence $p \mid t - 1$ and since $m_{tp}(G) < |G|$, we deduce that $p - 1 \leq 3$. But this is impossible because $(q^2 - q + 1) - 1 \leq 3$ and $q = 3^n$ Let $k \in \pi_e(G) - \{1, p\}$. Since p is an isolated vertex of $\Gamma(G)$, $p \nmid k$ and $pk \notin \pi_e(G)$. Thus G_p acts fixed point freely on the set of elements of order k by conjugation and hence $|G_p| \mid m_k(G)$. So $p \mid m_k(G)$. □

Lemma 3.3. *The group G nor a Frobenius group nor a 2-Frobenius group.*

Proof. Let G be a Frobenius group with kernel K and complement H. Then by Lemma 2.1, $t(G) = 2$ and $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$ and $|H|$ divides $|K| - 1$. Now by Lemma 3.2, p is an isolated vertex of $\Gamma(G)$. Thus we deduce that (i) $|H| = p$ and $|K| = |G|/p$ or (ii) $|H| = |G|/p$ and $|K| = p$. Since $|H|$ divides $|K| - 1$, we conclude that the last case can not occur case(ii). So we consider case(i). In otherwords $|H| = p$ and $|K| = |G|/p$, hence $q^2 - q + 1 \mid q^3(q^3+1)(q^2-1)/(q^2-q+1) - 1$. So, we conclude that $(q^2 - q + 1) \mid ((q^2-q+1)(q^4+2q^3-3q-3)+2$. Therefore $p \mid 2$ where is a impossible. We now show that G is not a 2-Frobenius group. Let G be a 2-Frobenius group.Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups by kernels K/H and H respectively. Since p is an isolated vertex of $\Gamma(G)$, then $\pi_2(G) = p$ in result $|K/H| = p$. Now since $|G/K|$ divides $|Aut(K/H)|$, so $|G/K||(p-1)$. By Lemma 2.8, there is an odd prime divisor t of $q+1$ such that $(p-1, t) = 1$. Now since $|G/K| \mid (p-1)$, so $t \mid |H|$, now since H is nilpotent, $H_t \rtimes K/H$ is a Frobenius group with kernel H_t and complementK/H. So $|K/H|$divides $|H_t| - 1$. It implies that $q^2 - q + 1 \leq (t-1) \leq q$, but this is a contradiction. □

Lemma 3.4. *The group G is isomorphic to the group U.*

Proof. By Lemma 3.2, p is an isolated vertex of $\Gamma(G)$. Thus $t(G) > 1$ and G satisfies one of the cases of Lemma 2.4. Now Lemma 3.3 implies that G is neither a Frobenius group nor a 2-Frobenius group. Thus only the case (c) of Lemma 2.4 occure. So G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group. Since p is an isolated vertex of $\Gamma(G)$, we have $p \mid |K/H|$.By Lemma 3.2, p is an isolated vertex of $\Gamma(G)$. Thus $t(G) > 1$ and G satisfies one of the cases of Lemma 2.4. Now Lemma 3.3 implies that G is neither a Frobenius group nor a 2-Frobenius group. Thus only the case (c) of Lemma 2.4 occure. So G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group. Since p is an isolated vertex of $\Gamma(G)$, we have $p \mid |K/H|$. On the other hand, we know that $5 \nmid |G|$.Thus K/H is isomorphic to one of the groups in Lemma 2.8. Hence the following two cases

(1) If $K/H \cong G_2(q')$, where $q' \equiv \pm 2 (\mod 5)$, then by[23], $\pi(G_2(q')) = q'^2 \pm q' + 1$. Now we consider $q^2 - q + 1 = q'^2 \pm q' + 1$ so $q^2 - q = q'^2 \pm q'$. It follows that $q = q'$ or $q = q' + 1$ because $(q', q'+1) = (q', q'-1) = (q, q-1) = 1$. Next we know that $|G_2(q') \mid |G|$, hence we deduce two the following cases :

(i) $q'^6(q'^6-1)(q'^2-1) \mid (q'^3(q'^3+1)(q'^2-1)$,

or

(ii) $q'^6(q'^6-1)(q'^2-1) \mid (q'+1)^3((q'+1)^3+1)((q'+1)^2-1)$ which is a contradiction.

(2) If $K/H \cong {}^2G_2(q')$, where $q' = 3^{2m+1}$, then by [23], $\pi({}^2G_2(q')) = q' \pm \sqrt{3q'} + 1$. So we consider $q^2 - q + 1 = q' \pm \sqrt(3q') + 1$. It followes that $3^m(3^m - 1) = 3^{2n+1}(3^{2n+1} \pm 1)$ and in result $m = 2n+1$. Now since $|{}^2G_2(q') \nmid |G|$, we deduce a contradiction.

(3) If $K/H \cong {}^3D_4(q')$ then by [23], $\pi({}^3D_4(q') = q'^4 - q'^2 + 1$. Now we consider $q^2 - q + 1 = q'^4 - q'^2 + 1$ in conclude we deduce $q(q-1) = (q'^2(q'^2-1)$ and hence $q = q'^2$ because $(q, q-1) = 1$. Now since $|{}^3D_4(q') \nmid |G|$, we deduce a contradiction.

(4) If $K/H \cong L_2(q')$, where $q' \equiv \pm 2(\mod 5)$, $q' = p'^m$, then by [7,13] $\pi(L_2(q') = q' \pm 1$, where q' be even also q', $q' \pm 1)/2$ where q' be Odd.

Now, we assume q' be even then $p = q' \pm 1$, so we have $q^2 - q + 1 = q' \pm 1$. First we assume $q^2 - q + 1 = q' + 1$ then $q(q-1) = q'$, which is a contradiction because q' is power of p'. Now if $q^2 - q + 1 = q' - 1$ then $q^2 - q + 2 = q'$. Since $|L_2(q') \nmid |G|$, we deduce a contradiction. In the way we assume q' be odd. First we consider $p = q'$, so $q^2 - q + 1 = q'$ now since that $|L_2(q')| \nmid |G|$ where this is a contradiction. Now if $p = (q' \pm 1)/2$, then we have $q^2 - q + 1 = (q' \pm 1)/2$, so $q' = 2q^2 - 2q + 1$ or $q' = 2q^2 - 2q + 3$. Since $|L_2(q') \nmid |G|$, we have a contradiction.

(5) If $K/H \cong L_3(q')$, where $q' \equiv \pm 2(\mod 5)$, then by [23], $\pi(L_3(q')) = q'^2 + q' + 1/(3, q' - 1)$ so we consider two cases. First we assume $(3, q'-1) = 1$ then we have $q^2 - q + 1 = (q'^2 + q' + 1)$, so $q(q-1) = q'(q'+1)$. Now since that $(q, q-1) = (q', q'+1) = 1$ so $q' = q-1$. Now since that $|L_3(q') \nmid |G|$, so we have a contradiction. Now if $(3, q'-1) = 3$ then $q^2 - q + 1 = q'^2 + q' + 1/3$, so $3q^2 - 3q + 3 = q'^2 + q' + 1$. Therefore $3q^2 - 3q = q'^2 + q' - 2$ so $3q(q-1) = (q'-1)(q'+2)$. As a result $3^{n+1}(3^{n+1} - 1) = (q'-1)(q'+2)$. On the otherhand we have $(q'-1, q'+2) = 1$ or 3. Now if $(q'-1, q'+2) = 1$ then $q' - 1 = 3^{n+1} - 1$ and $q' + 2 = 3^{n+1}$. First, if $q' - 1 = 3^{n+1} - 1$ then $q' = 3^{n+1}$, so $3^{n+1}(3^{n+1} - 1) = (3^{n+1} - 1)(3^{n+1} + 2)$. Therefore $3^{n+1} = 3^{n+1} + 2$, where this is a contradiction, also if $q' + 2 = 3^{n+1}$ then we have a contradiction, similarily. Now if $(q'-1, q'+2) = 3$ then $3 \mid q' - 1$, $3 \mid q' + 2$, so we have $q' = 3k + 1$ and $q' = 3k - 2$. Now if $q' = 3k + 1$ then we have $3q(q-1) = 3k(3k+3)$, so we have $3q^2 - 3q - 9k^2 - 9k = 0$. Now, we can see easily this equation has not any solution. Hence this is a contradiction. Now assume $q' = 3k - 2$, where is a contradiction, similarily. Hence, $K/H \cong U$ therefore $|K/H| = |U|$. Now since p is an isolated vertex and also $p \mid |K/H|$, we consider $q^2 - q + 1 = q'^2 - q' + 1$, so $q = q'$. Therefore $n = n'$. On the otherhand $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, so $H = 1$, $G = K \cong U$. □

4. Acknowledgments

The authors wish to sincerely thank the referees for several useful comments.

References

[1] G. Y., Chen, *On the structure of Frobenius groups and 2-Frobenius groups*, J. Southwest China Normal University, **20** No. 5 (1995) 485-487.

[2] J. H., Conway, R. T., Curtis, S. P., Norton, R. A., Parker and R. A., Wilson *Atlas of Finite Groups*, Clarendon Press, New York, 1985.

[3] B., Ebrahimzadeh, A., Iranmanesh, and H. Parvizi Mosaed, *A new characterization of Ree group ${}^2G_2(q)$ by the order of group and number of elements with same order*, Int. J. Group Theory, **6** No. 4 (2017) 1-6.

[4] B. Ebrahimzadeh and R. Mohammadyari, *A new characterization of Suzuki groups*, Arch. Math., **55** No. 1 (2019) 17-21.

[5] B. Ebrahimzadeh and R. Mohammadyari, *A new characterization of symplectic groups* $C_2(3^n)$, Acta Comment. Univ. Tartu. Math., **23** No. 1 (2019) 117-124.

[6] B. Ebrahimzadeh, *A new characterization of projective special linear group* $l_3(q)$, Algebra Discrete Math., **31** No. 2 (2021) 212-218.

[7] G. Frobenius, *Verallgemeinerung des sylowschen satze*, Berliner sitz., (1895) 981-983.

[8] D. Gorenstein, *Finite Groups*, Harper and Row, New York, 1980.

[9] A. Khalili Asboei and A. Iranmanesh, *Characterization of the linear groups* $L_2(p)$, Czechoslov. Math. J., **64** No. 139 (2014) 459-464.

[10] A. Khalili Asboei and A. Iranmanesh, *A characterization of Symmetric group* S_r*, where r is prime number*, Ann. Math. Inform., **40** (2012) 13-23.

[11] A. Khalili Asboei, S. S. S. Amiri, A. Iranmanesh and A. Tehranian, *A new characterization of sporadic simple groups by NSE and order*, J. Algebra Appl., **12** No. 2 (2013) 1250158.

[12] M. Khatami,B. Khosravi and Z. Akhlaghi, *A new characterization for some linear groups*, Monatsh. Math., **163** (2011) 39-50.

[13] A. Khosravi and B. Khosravi, *A new characterization of some alternating and symmetric groups (II)*, Houston J. Math., **30** No. 4 (2004) 465-478.

[14] A. S. Kondrat'ev, *Prime graph components of finite simple groups*, Math. USSR-Sb., **67** No. 1 (1990) 235-247.

[15] V. D. Mazurov and E. I. Khukhro, *Unsolved Problems in Group Theory*, The Kourovka Notebook, 16 ed., Inst. Mat. Sibirsk. Otdel. Akad. Novosibirsk, 2006.

[16] H. Parvizi Mosaed, A. Iranmanesh and A. Tehranian, *Characterization of suzuki group by nse and order of group*, Bull. Korean Math. Soc., **53** No. 3 (2016) 651-656.

[17] C. G. Shao, W. J. Shi and Q. H. Jiang, *Characterization of simple* K_4*-groups*, Front. Math. China, **3** No. 3 (2008) 355-370.

[18] C. G. Shao, W. J. Shi and Q. H. Jiang, *A new characterization of some linear groups by nse*, J. Algebra Appl., **13** No. 2 (2014) 1350094.

[19] W. J. Shi, *Pure quantitative characterization of each finite simple groups*, Front. Math. China, **2** No. 1 (2007) 123-125.

[20] W. J. Shi, *A new characterization of the sporadic simple groups*, Group Theory-Porc. Singapore Group Theory Conf., (1989) 531-540.

[21] W. J. Shi, *A characterization of* $U_3(2^n)$ *by their element orders*, J. Southwest China Normal University, **25** No. 4 (2000) 353-360.

[22] A. V. Vasilev, M. A. Grechkoseeva and V. D. Mazurov, *Characterization of finite simple groups by spectrum and order*, J. Algebra and Logic, **48** No. 6 (2009) 385-409.

[23] J. S. Williams, *Prime graph components of finite groups*, J. Algebra, **69** No. 2 (1981) 487-513.

[24] A. V. Zavarnitsine, *Recognition of the simple groups* $L_3(q)$ *by element orders*, J. Group Theory, **7** No. 1 (2004) 81-97.

Behnam Ebrahimzadeh

Department of Mathematics, Persian Gulf University, Bushehr, Iran

behnam.ebrahimzadeh@gmail.com

Ali Iranmanesh

Department of Mathematics, Tarbiat Modares University, Tehran, Iran

iranmana@modares.ac.ir