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Research Paper

BINARY BLOCK-CODES OF MV-ALGEBRAS AND FIBONACCI SEQUENCES

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Abstract. In this paper, the notion of an M-function and cut function on a set, are introduced and investigated several properties. We use algebraic properties to introduce an algorithm which show that every finite MV-algebras and Fibonacci sequences determines a block-code and presented some connections between Fibonacci sequences, MV-algebras and binary block-codes. Furthermore, an MV-algebra arising from block-codes is established.

1. Introduction

The notion of MV-algebra was introduced by C. C. Chang as an algebraic counterpart for the Lukasiewicz infinite-valued propositional logic [5]. The bounded commutative BCK-algebras are precisely the MV-algebras [10]. A recent application of BCK-algebras and residuated lattices have been given in [2,3,4]. In coding theory, a block-code is any member of the large and important family of error-correcting codes that encode data in blocks. Error-correcting

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codes are used to reliably digital data over unreliable communication channels subject to channel noise. When a sender wants to transmit a possibly very long data stream using a block code, the sender breaks the stream up into pieces of some fixed size. Each such piece is called message and the procedure given by the block-code encodes each message individually into a codeword, also called a block in the context of block codes. The sender then transmits all blocks to the receiver, who can in turn use some decoding mechanism to recover the original messages from the possibly corrupted received blocks. The performance and success of the overall transmission depends on the parameters of the channel and the block code. The Fibonacci sequence is an integer sequence defined by a simple linear recurrence relation. The sequence appears in numerous settings in mathematics and other sciences. In particular, the shape of several naturally occurring biological organisms is governed by the Fibonacci sequence and its close relative, the golden ratio. The Fibonacci number has been studied in different forms for centuries and, consequently, the literature on this subject is incredibly vast. Kim, Neggers, and so introduced the concept of generalized Fibonacci sequences over a groupoid in [8] and discussed it specifically for the case where the groupoid contains idempotents and pre-idempotents. In [1], the authors constructed a Fibonacci sequence over MV-algebras and proved in [3] that, to each $n-$ary block-code V, one can associate a BCK-algebra X such that the $n-$ary block-code generated by X, V_X contains code V as a subset, and the converse was also found to be true in certain circumstances.

In present paper, we introduce the notion of a cut function and investigate its properties. Moreover, the present study will show that every finite MV-algebra and Fibonacci sequence over MV-algebras determines a binary block-code such that these codes are the same and show that, to each binary block-code V, associated an MV-algebra X such that the binary block-code generated by X, V_X contains code V as a subset. Using codes, we can easily obtain orders determining the supplementary properties of these algebras and provide an algorithm which allows us to find an MV-algebra starting from a given binary block-code. This new look will help us to achieve new results and applications of these algebras and sequences. Due to this connection of MV-algebras and Fibonacci sequences with coding Theory, we can consider the above results as a starting point in the study of new applications of these algebras in the coding theory and computer science. It is well known that various classical error-correcting codes are ideals in certain algebras. For example, all cyclic codes are principal ideals in group algebras of cyclic groups. Several other classes of codes have also been shown to be ideals in group algebras, and this additional algebraic structure has been used to develop faster encoding and decoding algorithms for these codes.

2. Preliminaries

For a non-empty set A and "$*$" be a binary operation defined on A and $x \in A$ a fixed element, we have that $(A, *, x)$ is a $(2,0)$ type set.

Definition 2.1. [5] An MV-algebra A is an algebra $A = (A, \oplus, *, 0)$ of type $(2,1,0)$ satisfying the following equations:

(MV_1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(MV_2) $x \oplus y = y \oplus x$,

(MV_3) $x \oplus 0 = x$,

(MV_4) $x^{**} = x$,

(MV_5) $x \oplus 0^* = 0^*$,

(MV_6) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$, for all $x, y, z \in A$.

In MV-algebra A, we define the constant 1 and auxiliary operation $\odot, \ominus$ and $\rightarrow$ as follows:

$$1 = 0^*,$$

$$x \odot y = (x^* \oplus y^*)^*,$$

$$x \ominus y = x \odot y^* = (x^* \oplus y)^*,$$

$$x \rightarrow y = x^* \oplus y^*,$$

for any $x, y \in A$.

Lemma 2.2. [5] *For $x, y \in A$, the following conditions are equivalent:*

(i) $x^* \oplus y = 1$,

(ii) $x \odot y^* = 0$,

(iii) $y = x \oplus (y \ominus x)$,

(iv) *There is an element $z \in A$ such that $x \oplus z = y$.*

For any two elements $x, y \in A$ let us agree to write $x \leq y$ if and only if x and y satisfy the equivalent conditions $(i) - (iv)$ in the above lemma.

So, $\leq$ is an order relation on A (called the natural order on A). We will say that an MV-algebra A is an MV-chain if it is linearly ordered relative to natural order.

Let A and B be MV-algebras . A function $f : A \longrightarrow B$ is a morphism of MV-algebras if and only if it satisfies the following conditions, for every $x, y \in A$:

(MV_7) $f(0) = 0$,

(MV_8) $f(x \oplus y) = f(x) \oplus f(y)$,

(MV_9) $f(x^*) = (f(x))^*$.

If A and B are MV-algebras we write $A \approx B$ if and only if there is an isomorphism of MV-algebras from A to B (that is a bijective morphism of MV-algebras).

Definition 2.3. [1] Let A= $(A, \oplus, *, 0)$ be an MV-algebra. If $a, b \in A$, we construct a sequence as follows:

$$[a, b] := \{a, b, u_0, u_1, u_2, ..., u_k, ...\},$$

where $u_0 := a \oplus b$,$u_1 = b \oplus u_0$, $u_2 = u_0 \oplus u_1$, and $u_{k+2} = u_k \oplus u_{k+1}$.
A sequence $[a, b]$ is called a Fibonacci sequence on MV-algebra.

Remark 2.4. [9] In every MV-chain A we have:
(i) $x \oplus y = x$ if and only if $x = 1$ or $y = 0$,
(ii) $x \oplus y = x$ if and only if $x^* \oplus y^* = y^*$.

Definition 2.5. [6] An algebra $(L, \rightarrow, *, 1)$ of type $(2, 1, 0)$ will be called Wajsberg algebra if for every $x, y, z \in L$ the following axioms are verified:
(W_1) $1 \rightarrow x = x$,
(W_2) $(x \rightarrow y) \rightarrow [(y \rightarrow z) \rightarrow (x \rightarrow z)] = 1$,
(W_3) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,
(W_4) $(x^* \rightarrow y^*) \rightarrow (y \rightarrow x) = 1$.
There is a one-to-one correspondence between MV-algebras and Wajsberg algebras.

3. Codes based on MV-algebras and Fibonacci sequences

We aim to achieve binary block-codes from the algebraic properties of MV-algebras and using the notion of MV-chain and Boolean algebras. Several relations on binary block-codes are derived from MV-algebras.
Let A be a non-empty set and X be an MV-algebra.

Definition 3.1. A mapping $\widetilde{A} : A \rightarrow X$ is called an M-function on A. A cut function of $\widetilde{A}$, for $q \in X$, where X is an MV-algebra, is defined by $\widetilde{A}_q : A \rightarrow \{0, 1\}$ such that (for all $x \in A$) $(\widetilde{A}_q(x) = 1 \Leftrightarrow q \oplus \widetilde{A}(x) = q)$.

Remark 3.2. (i) $\widetilde{A}_q$ is the characteristic function of the following subset of A, called a cut subset or an q-cut of $\widetilde{A}$: $A_q := \{x \in A | q \oplus \widetilde{A}(x) = q\}$,
(ii) $A_1 = A$ and $A_0 = \{x \in A | \widetilde{A}(x) = 0\}$.

Definition 3.3. Let $A = \{1, 2, ..., n\}$ and X be an MV-algebra. A codeword in a binary block-code V is $v_x = x_1x_2...x_n$ such that $x_i = j \Leftrightarrow \widetilde{A}_x(i) = j$ for $i \in A$ and $j \in \{0, 1\}$. We denote this code with V_X. In this way, each M-function $\widetilde{A} : A \rightarrow X$ has associated a binary block-code of length n.

Example 3.4. Let $A = \{0, x, y, z\}$ and $X = \{0, a, b, 1\}$ be an MV-algebra with the following operation:

$\oplus$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

The function $\widetilde{A}: A \to X$ given by

$$\widetilde{A}=\begin{pmatrix} 0 & x & y & z \\ 0 & a & b & 1 \end{pmatrix}$$

is an M-function on A. Then

$\widetilde{A}_x$	0	a	b	1
$\widetilde{A}_0$	1	0	0	0
$\widetilde{A}_a$	1	1	0	0
$\widetilde{A}_b$	1	0	1	0
$\widetilde{A}_1$	1	1	1	1

Thus its cut subsets of $\widetilde{A}$ are as follows:
$\widetilde{A}_0 = 0$, $\widetilde{A}_a = \{0, x\}$, $\widetilde{A}_b = \{0, y\}$, $\widetilde{A}_1 = A$.

Proposition 3.5. *Let X be an MV-chain and $\widetilde{A}: A \to X$ be an M-function on A. Then*

$$(\forall p, q \in X)\ (p \oplus q = p \Rightarrow A_q \subseteq A_p).$$

Proof. Let $p, q \in X$ be such that $p \oplus q = p$ and $x \in A_q$. Then $q \oplus \widetilde{A}(x) = q$. Using Remark 2.4, we have $q^* \oplus \widetilde{A}(x)^* = \widetilde{A}(x)^*$ and $p^* \oplus q^* = q^*$, so $(p^* \oplus q^*) \oplus \widetilde{A}(x)^* = \widetilde{A}(x)^*$, using (MV_1), it follows that $p^* \oplus (q^* \oplus \widetilde{A}(x)^*) = \widetilde{A}(x)^*$ hence $p^* \oplus \widetilde{A}(x)^* = \widetilde{A}(x)^*$, thus $p \oplus \widetilde{A}(x) = p$. Therefore $x \in A_p$, i.e., $A_q \subseteq A_p$. □

Notice that in above proposition $A_p \nsubseteq A_q$, in Example 3.4, for $0, a, b \in X$, we have

$$(a \oplus 0 = a \Rightarrow A_0 \subseteq A_a) \text{ and } (b \oplus 0 = b \Rightarrow A_0 \subseteq A_b).$$

But $(A_a \nsubseteq A_0)$ and $(A_b \nsubseteq A_0)$.

Let $\widetilde{A}: A \to X$ be an M-function on A and $\sim$ be a binary relation on X defined by $(\forall p, q \in X)\ (p \sim q \Longleftrightarrow A_p = A_q)$. Then $\sim$ is an equivalence relation on X.

Let $\widetilde{A}(A)$:=$\{q \in X | \widetilde{A}(x) = q$, for some $x \in A\}$ and for $q \in X$, $(q] := \{x \in X | q \oplus x = q\}$.

Proposition 3.6. *For an M-function $\widetilde{A}: A \to X$ on A, we have*

$$(\forall p, q \in X)\ (p \sim q \Longleftrightarrow (p] \cap \widetilde{A}(A) = (q] \cap \widetilde{A}(A)).$$

Proof. We have

$$\begin{aligned} p \sim q &\Longleftrightarrow A_p = A_q, \\ &\Longleftrightarrow (\forall x \in A)(p \oplus \widetilde{A}(x) = p \Longleftrightarrow q \oplus \widetilde{A}(x) = q), \\ &\Longleftrightarrow \{x \in A | \widetilde{A}(x) \in (p]\} = \{x \in A | \widetilde{A}(x) \in (q]\}, \\ &\Longleftrightarrow (p] \cap \widetilde{A}(A) = (q] \cap \widetilde{A}(A). \end{aligned}$$

□

For any $x \in X$, let $x/\sim$ denote the equivalence class containing x, that is, $x/\sim := \{y \in X | x \sim y\}$.

Lemma 3.7. *Let $\widetilde{A} : A \to X$ be an M-function on A. For every $x \in A$, we have $\widetilde{A}(x) = inf\{\widetilde{A}(x)/\sim\}$, that is, $\widetilde{A}(x)$ is the smallest element of the $\sim$-class to which it belongs.*

Proof. We have

$$\begin{aligned} \widetilde{A}(x) &= inf\{q \in X | \widetilde{A}_q(x) = 1\}, \\ &= inf\{q \in X | q \oplus \widetilde{A}(x) = q\}, \\ &= inf\{q \in X | q \in \widetilde{A}(x)/\sim\}, \\ &= inf\{\widetilde{A}(x)/\sim\}. \end{aligned}$$

□

Construction of the code: Let $A = \{1, 2, ..., n\}$ and X be a finite MV-algebra. Every M-function $\widetilde{A} : A \to X$ on A determines a binary block-code C of length n in the following way:
to every $x/\sim$, where $x \in X$ corresponds a codeword $w_x = x_1x_2...x_n$ such that $x_i = j$ if and only if $\widetilde{A}_x(i) = j$, for $i \in X$ and $j \in \{0, 1\}$.
Let V be a binary block-code and let $v_x = x_1x_2...x_n$ and $v_y = y_1y_2...y_n$ be two code words belonging to V. We define an order $\preceq_c$ on V as following:
$v_x \preceq_c v_y$ if and only if $y_i \leq x_i$, for all $i \in \{1, 2, ..., n\}$.

Example 3.8. Let $X = \{0, a, b, c, d, 1\}$ be an MV-algebra with the following operation:

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	d	1	d	1
b	b	d	c	c	1	1
c	c	1	c	c	1	1
d	d	d	1	1	1	1
1	1	1	1	1	1	1

Let $\widetilde{A} : X \to X$ be the identity M-function on X. Then

$\widetilde{A}_x$	0	a	b	c	d	1
$\widetilde{A}_0$	1	0	0	0	0	0
$\widetilde{A}_a$	1	1	0	0	0	0
$\widetilde{A}_b$	1	0	0	0	0	0
$\widetilde{A}_c$	1	0	1	1	0	0
$\widetilde{A}_d$	1	1	0	0	0	0
$\widetilde{A}_1$	1	1	1	1	1	1

thus $V_X = \{100000, 110000, 100000, 101100, 110000, 111111\}$ is a code obtained by the MV-algebra X.

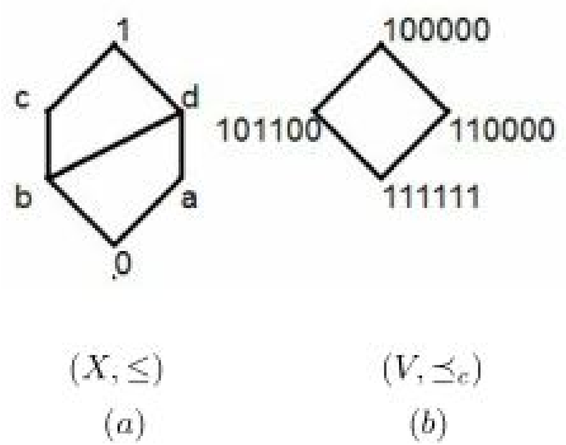


FIGURE 1. a) Partial ordering . b) Order relation$\preceq_c$

Example 3.9. Let $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ and let $X = \{0, a, b, c, d, 1\}$ be an MV-algebra with the following operation:

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	c	c	1	1
b	b	c	d	1	d	1
c	c	c	1	1	1	1
d	d	1	d	1	d	1
1	1	1	1	1	1	1

Let $\widetilde{A} : A \to X$ be an M-function on A given by

$$\widetilde{A}=\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a & c & b & 1 & 0 & d \end{pmatrix}.$$

Then

$\widetilde{A}_x$	a	c	b	1	0	d
$\widetilde{A}_0$	0	0	0	0	1	0
$\widetilde{A}_a$	1	0	0	0	1	0
$\widetilde{A}_b$	0	0	0	0	1	0
$\widetilde{A}_c$	1	0	0	0	1	0
$\widetilde{A}_d$	0	0	1	0	1	1
$\widetilde{A}_1$	1	1	1	1	1	1

thus
$V_X = \{000010, 100010, 000010, 100010, 001011, 111111\}$ is a code obtained by the MV-algebra X.

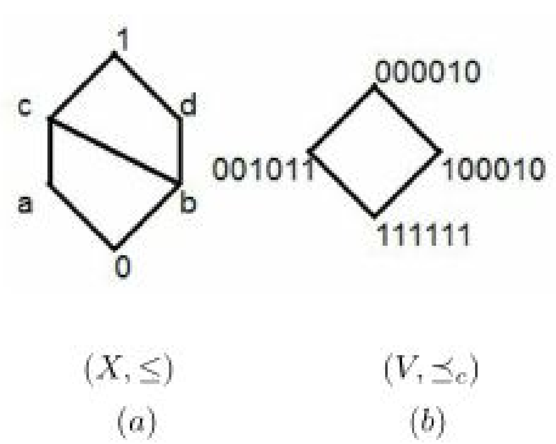


FIGURE 2. a) Partial ordering . b) Order relation$\preceq_c$

cut sets of $\widetilde{A}$ are as follows:

$\widetilde{A}_0 = \{a_5\} = \widetilde{A}_b$, $\widetilde{A}_a = \{a_1, a_5\} = \widetilde{A}_c$, $\widetilde{A}_d = \{a_3, a_5, a_6\}$, $\widetilde{A}_1 = A$.

Example 3.10. Let $L_4 = \{0, 1/3, 2/3, 1\}$ be an MV-chain and $\widetilde{A} : A \to X$ be an M-function on A given by

$\widetilde{A} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ 1/3 & 0 & 1 & 2/3 \end{pmatrix}$. Then

$\widetilde{A}_x$	1/3	0	1	2/3
$\widetilde{A}_0$	0	1	0	0
$\widetilde{A}_{1/3}$	0	1	0	0
$\widetilde{A}_{2/3}$	0	1	0	0
$\widetilde{A}_1$	1	1	1	1

Thus $V_X = \{0100, 0100, 0100, 1111\}$ is a code obtained by the MV-chain X.

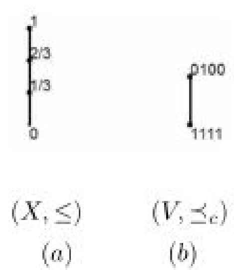


Figure 3. a) Partial ordering . b) Order relation$\preceq_c$

Theorem 3.11. *Every finite MV-chain X determines a block-code C such that $(X, \leq)$ is isomorphic to $(C, \preceq_c)$.*

Proof. Let $X = \{q_1, q_2, ..., q_n\}$ be a finite MV-chain in which q_1 is the least element and let $\widetilde{A} : X \longrightarrow X$ be the identity M-function on X. The decomposition of $\widetilde{A}$ provides a family $\{\widetilde{A}_x | x \in X\}$ which is the desired code under the order $\preceq_c$. Let $g : X \longrightarrow \{\widetilde{A}_x | x \in X\}$ be a function defined by $g(x) = \widetilde{A}_x$, for all $x \in X$. By Lemma 3.7 every $\sim$-class contains exactly one element. Hence g is onto. Let $p, q \in X$ be such that $p \oplus q = p$. Then $A_q \subseteq A_p$ by Proposition 3.5, which means that $\widetilde{A}_q \leq \widetilde{A}_p$. Therefore g is an isomorphism. □

Theorem 3.12. *Every finite MV-algebras X determines a block-code C such that $(X, \leq)$ is isomorphic to $(C, \preceq_c)$.*

Proof. We use Remark 3.2(i), Proposition 3.6 and Lemma 3.7. □

An element a of an MV-algebra A is called an idempotent or Boolean if $a \oplus a = a$, if a and b are idempotents, then $a \oplus b$ and $a \odot b$ are also idempotents. Boolean algebras are just the MV-algebras obeying the additional identity $a \oplus a = a$ or $a \odot b$. In fact MV-algebras are non-idempotent generalizations of Boolean algebras. We can provide examples of MV-algebras with some properties, in our case, Boolean algebras.

Example 3.13. Let $X = \{0, a, b, 1\}$ be an MV-algebra with the following table obtained from Example 3.4.
Then $V_X = \{1000, 1100, 1010, 1111\}$ is a code obtained by the MV- algebra X.

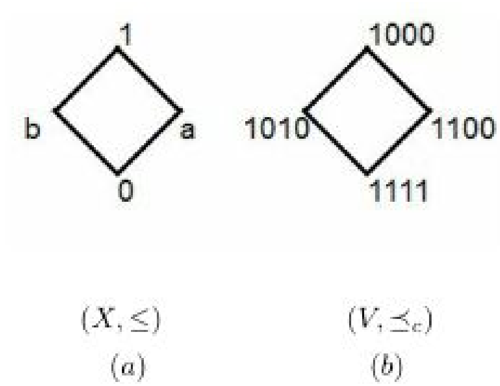


FIGURE 4. a) Partial ordering . b) Order relation$\preceq_c$

Example 3.14. Let $X = \{0, a, b, c, d, e, f, 1\}$ be an MV-algebra with the following operation:

$\oplus$	0	a	b	c	d	e	f	1
0	0	a	b	c	d	e	f	1
a	a	a	c	c	e	e	1	1
b	b	c	b	c	f	1	f	1
c	c	c	c	c	1	1	1	1
d	d	e	f	1	d	e	f	1
e	e	e	1	1	e	e	1	1
f	f	1	f	1	f	1	f	1
1	1	1	1	1	1	1	1	1

Let $\widetilde{A} : X \to X$ be the identity M-function on X. Then

$\widetilde{A}_x$	0	a	b	c	d	e	f	1
$\widetilde{A}_0$	1	0	0	0	0	0	0	0
$\widetilde{A}_a$	1	1	0	0	0	0	0	0
$\widetilde{A}_b$	1	0	1	0	0	0	0	0
$\widetilde{A}_c$	1	1	1	1	0	0	0	0
$\widetilde{A}_d$	1	0	0	0	1	0	0	0
$\widetilde{A}_e$	1	1	0	0	1	1	0	0
$\widetilde{A}_f$	1	0	1	0	1	0	1	0
$\widetilde{A}_1$	1	1	1	1	1	1	1	1

thus
$V_X = \{10000000, 11000000, 10100000, 11110000, 10001000, 11001100, 10101010, 11111111\}$ is a code obtained by the MV-algebra X.

According to Examples 3.13 and 3.14, we see that there will be no duplicate code.

Now, using Definition 2.3, we attempt to obtain binary codes on Fibonacci sequences and compare the results with those presented in the previous content.

Example 3.15. Let $X = \{0, a, b, c, d, 1\}$ be an MV-algebra with the table obtained from Example 3.8. If $a, b \in X$, then $[a, b] := \{a, b, u_0, u_1, u_2, ...\}$, then by Definition 2.3 we have:
$u_0 = a \oplus b = d$, $u_1 = b \oplus d = 1$, $u_3 = d \oplus 1 = 1$, $u_4 = 1 \oplus 1 = 1$,... . Hence
$[a, b] := \{a, b, d, 1, 1, 1, ...\}$, and $[b, a] := \{b, a, d, d, 1, 1, 1, ...\}$,... .
The number of modes that can be checked is 36.
$[0, 0], [0, a], ..., [0, 1], [a, 0], [a, a], ..., [a, 1], ..., [1, 0], [1, a], ..., [1, 1]$.

Notation. Let $A = (A, \oplus, *, 0)$ be an MV-algebra and $u \in A$ be such that $[a, b] := \{a, b, u_0, u_1, ..., u, u, ...\}$ for any $a, b \in A$. Then we will show the general sentence of the sequence with the symbol g[a,b]=u. For other details about Fibonacci sequences on MV-algebras and about some new applications of them, the reader is referred to [1].

Definition 3.16. Let $\widetilde{A} : A \to X$ be an M-function on A. We define the following cut function of $\widetilde{A}$, for $q \in X$ on Fibonacci sequences as follows:
$\widetilde{A}_q : A \to \{0, 1\}$ such that $(\forall x \in A)(\widetilde{A}_q(x) = 1 \Leftrightarrow \widetilde{A}(q, x) = g[q, x] = q$.

Example 3.17. Let $X = \{0, a, b, c, d, 1\}$ be an MV-algebra with the following table obtained from Example 3.8.
Then we have following Fibonacci sequences table:

$g[-,-]$	0	a	b	c	d	1
0	0	a	c	c	1	1
a	a	a	1	1	1	1
b	c	1	c	c	1	1
c	c	1	c	c	1	1
d	1	1	1	1	1	1
1	1	1	1	1	1	1

Let $\widetilde{A} : X \to X$ be the identity M-function on X. Then

$\widetilde{A}_x$	0	a	b	c	d	1
$\widetilde{A}_0$	1	0	0	0	0	0
$\widetilde{A}_a$	1	1	0	0	0	0
$\widetilde{A}_b$	0	0	0	0	0	0
$\widetilde{A}_c$	1	0	1	1	0	0
$\widetilde{A}_d$	0	0	0	0	0	0
$\widetilde{A}_1$	1	1	1	1	1	1

Thus $V_X = \{100000, 110000, 000000, 101100, 000000, 111111\}$ is a code obtained by the Fibonacci sequences of X.

Example 3.18. Let $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ and $X = \{0, a, b, c, d, 1\}$ be an MV-algebra with the following table obtained from Example 3.9.
Then we have following Fibonacci sequences table:

$g[-,-]$	0	a	b	c	d	1
0	0	a	d	1	d	1
a	a	a	1	1	1	1
b	d	1	d	1	d	1
c	1	1	1	1	1	1
d	d	1	d	1	d	1
1	1	1	1	1	1	1

Let $\widetilde{A} : A \to X$ be an M-function on A given by
$\widetilde{A} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a & c & b & 1 & 0 & d \end{pmatrix}$. Then

$\widetilde{A}_x$	a	c	b	1	0	d
$\widetilde{A}_0$	0	0	0	0	1	0
$\widetilde{A}_a$	1	0	0	0	1	0
$\widetilde{A}_b$	0	0	0	0	0	0
$\widetilde{A}_c$	0	0	0	0	0	0
$\widetilde{A}_d$	0	0	1	0	1	1
$\widetilde{A}_1$	1	1	1	1	1	1

Thus $V_X = \{000010, 100010, 000000, 000000, 001011, 111111\}$ is a code obtained by the Fibonacci sequences of X. cut sets of $\widetilde{A}$ are as follows:
$\widetilde{A}_0 = \{a_5\}$, $\widetilde{A}_a = \{a_1, a_5\}$, $\widetilde{A}_b = \widetilde{A}_c = \emptyset$, $\widetilde{A}_d = \{a_3, a_5, a_6\}$, $\widetilde{A}_1 = A$.

Example 3.19. Let $X = \{0, a, b, 1\}$ be an MV-algebra with the following table obtained from Example 3.13.
Then we have following Fibonacci sequences table:

$g[-,-]$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

Let $\widetilde{A} : X \to X$ be the identity M-function on X. Thus $V_X = \{1000, 1100, 1010, 1111\}$ is a code obtained by the Fibonacci sequences of X.

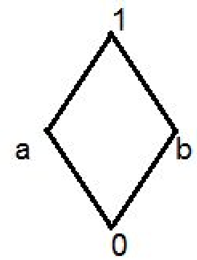
Remark 3.20. From the block-code obtained by the aforesaid methods, it is obvious that the code attained in Example 3.8 is the same as that obtained in Example 3.17, and the codes obtained in Examples 3.9 and 3.13 are the same as those attained in Examples 3.18 and 3.19. The explanation is that we use only algebraic properties, not its order of MV-algebra. In [7], the authors constructed a binary block-codes of MV-algebras and Wajsberg algebras, but the problem is that the properties of MV-algebras have not been used. But this method codes based on the properties of MV-algebras and is a more comprehensive method.

4. MV- algebras arising from block-codes

Definition 4.1. Let $(X, \leq, 0, 1)$ be a finite partially ordered set, which is bounded. We define the following binary $\to$ on X as follows:
$x = 1 \to x$, $x \to (y \to z) = y \to (x \to z)$ and $(x \to y) \to y = (y \to x) \to x$
for all $x, y, z \in X$.
We define the operation $\odot$ such that $(\odot, \to)$ forms an adjoint pair, i.e., $z \leq x \to y$ if and only if $x \odot z \leq y$.

Proposition 4.2. *With the above operations on X, the lattice $(X, \rightarrow, *, 1 = 0^*)$ is an $Wajsberg$-algebra and $(X, \oplus, \odot, *, 0, 1)$ is an MV-algebra, we denote $x \oplus y = x^* \rightarrow y$ and $x \rightarrow y = x^* \oplus y$, where $(x^* = x \rightarrow 0)$, for all $x, y \in X$.*

Example 4.3. Let $X = \{0, a, b, 1\}$ be a set with partial ordering. Define a unary operation $*$ and $\rightarrow$ on X as follows:



$(X, \leq)$

(a)

FIGURE 5. a) Partial order

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

$*$	0	a	b	1
x^*	1	b	a	0

Then $(X, \rightarrow, *, 1)$ is an $Wajsberg$-algebra.

Let C be a binary block-code with n codewords of length n. We consider the matrix $M_C = (m_{ij})_{i,j \in \{1,2,...,n\}} \in M_n(\{0, 1\})$ with the rows consisting of the codeword of C. This matrix associated to the code C.

Proposition 4.4. *With the above notation, if the codeword $\underbrace{11...1}_{n-times}$ is in C and the matrix M_C is upper triangular with $m_{ii} = 1$ for all $i \in \{1, 2, ..., n\}$, there are a set A with n element, an MV-algebra X and an M-function $\widetilde{A} : A \rightarrow X$ on A such that $\widetilde{A}$ determines C.*

Proof. We consider the lexicographic order, denote by $\leq_{lex}$ on C. It is clear that $(C, \leq_{lex})$ is a totally ordered set. Let $C = \{w_1, w_2, ..., w_n\}$, with $w_1 \geq_{lex} w_2 \geq_{lex} ... \geq_{lex} w_n$. This

implies that $w_1 = \underbrace{11...1}_{n-times}$ and $w_n = \underbrace{00...0}_{(n-1)-times}1$. On C, we define a partial order $\preceq_C$ as in construction of the code by the M-function. Now, $(C, \preceq_C)$ is a partially ordered set with $w_1 \preceq_C w_i \preceq_C w_n$, $i \in \{1, 2, ..., n\}$. Note that w_1 correspond to 0 and w_n correspond to 1 in X. Hence $(C, \preceq_C, 0, 1)$ is a bounded lattice. We define on $(C, \preceq_C, 0, 1)$ a binary relation $\oplus$ and the operation $\odot$ as Proposition 4.2 and Definition 4.1. Then $X = (C, \preceq_C, 0, 1, \oplus, \odot, *)$ is an MV-algebra and C is isomorphic to X. We consider $A = C$ and the identity map $\widetilde{A} : A \to X$, $w \mapsto w$, as an M-function on A. The decomposition of $\widetilde{A}$ provides a family
$C_X = \{\widetilde{A}_q : A \to \{0,1\} | \widetilde{A}_q(x) = 1 \Leftrightarrow q \oplus \widetilde{A}(x) = q, \forall x \in A, q \in X\}$.
This family is the binary block-code C relative to the order relation $\preceq_C$. □

Proposition 4.5. *Let $A = (a_{i,j})_{i \in \{1,2,...,n\}, j \in \{1,2,...,m\}} \in M_{n,m}(\{0,1\})$ be a matrix with rows lexicographic ordered in the descending sense. Starting from this matrix, we can find a matrix $B = (b_{i,j})_{i,j \in \{1,2,...,k\}} \in M_k(\{0,1\})$, $k = n + m$, such that B is an upper triangular matrix, with $b_{ii} = 1$, $\forall i \in \{1, 2, ..., k\}$ and A becomes a sub matrix of the matrix B.*

Proof. We can extend the matrix A to a square matrix B, such that B is an upper triangular matrix. For this purpose, we insert in the left side of the matrix A (from the right to the left) the following n new columns of the form $\underbrace{00...01}_{n}, \underbrace{00...10}_{n}, ..., \underbrace{10...00}_{n}$.
It results a new matrix U with n rows and n+m columns. Now, we insert the bottom of the matrix U the following m rows:$\underbrace{00...00}_{n}\underbrace{10...00}_{m}, \underbrace{00...0}_{n+1}\underbrace{01...00}_{m-1}, ..., \underbrace{000}_{n+m-1}1$. We obtained the desired matrix B. □

Proposition 4.6. *With the above notations, we consider C a binary block-code with n codewords of length m, $n \neq m$, or a block-code with n codewords of length n such that the codeword $\underbrace{11...1}_{n-times}$ is not in C, or a block-code with n codewords of length n such that the matrix M_C is not upper triangular. There are a natural number $k \geqslant max\{m, n\}$, a set A with m elements and an M-function $\widetilde{A} : A \to C_k$, where C_k, denote the MV-algebra with k elements, such that the obtained block-code C_{C_n} contains the block-code C as a subset.*

Proof. Let C be a binary block-code, $C = \{w_1, w_2, ..., w_n\}$, with codewords of length m. We consider the codewords $w_1, w_2, ..., w_n$ lexicographic ordered, $w_1 \leqslant_{lex} w_2 \leqslant_{lex} ... \leqslant_{lex} w_n$. Let $M \in M_{n,m}(\{0,1\})$ be the associated matrix with the rows $w_1, w_2, ..., w_n$ in this order. Using Proposition 4.4, we can extend the matrix M to a square matrix $M^{'} \in M_p(\{0,1\})$, p=m+n, such that $M^{'} = (m^{'}_{i,j})_{i,j \in \{1,2,...,p\}}$ is an upper triangular matrix with $m^{'}_{ii} = 1$, for all $i \in \{1, 2, ..., p\}$. If the first line of the matrix $M^{'}$ is not $\underbrace{11...1}_{p-times}$ then we insert the row $\underbrace{11...1}_{p+1-times}$ as a first row and the 1 $\underbrace{00...0}_{p-times}$ as a first column. Let k=p+1, applying Proposition 4.4 for the matrix M',we obtain a MV-algebra $C_k = \{x_1, x_2, ..., x_k\}$, with x_1 correspond to 0 and x_k correspond to 1, and a binary block-code C_{C_k}. Assuming that the initial column of the matrix M have in the new matrix M' positions $i_{j1}, i_{j2}, ..., i_{jn} \in \{1, 2, ..., k\}$, let $X = \{x_{j1}, x_{j2}, ..., x_{jn}\} \subseteq C_k$. The M-function $\widetilde{A} : A \to C_k$ is such that $\widetilde{A}(x_{ji}) = x_{ji}$, $i \in \{1, 2, ..., m\}$, determines the binary block-code C_k such that $C \subseteq C_{C_k}$ as restriction of the M-function $\widetilde{A} : C_k \to C_k$ on A such that $\widetilde{A}(x_i) = x_i$. □

Remark 4.7. Propositions 4.4, 4.5 and 4.6 generalized Proposition 4.4, 4.5 and 4.6 in [2] to MV-algebras.

Example 4.8. Let $V = \{100000, 110000, 100000, 101100, 110000, 111111\}$ be a binary block-code. Using the lexicographic order, the code V can be written $V = \{111111, 110000, 110000, 101100, 100000, 100000\}$=$\{w_1, w_2, w_3, w_4, w_5, w_6\}$. From defining the partial order $\preceq$ on V, Note that $w_1 \preceq w_i$, $i \in \{2, 3, 4, 5, 6\}$, $w_2, w_3 \preceq w_5, w_6$ and w_4 cant be compared with w_2, w_3. Let $M_V \in M_{6,6}(\{0, 1\})$ be the associative matrix,

$$M_V = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Using Proposition 4.5 , we construct an upper triangular matrix, starting from the matrix M_V. It result the following matrices:

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 1 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 1 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 1 & 0 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 0 & 1 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & 1 & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 1 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 1 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 1 & 0 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 0 & 1 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & 1 & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Since the first row is not $\underbrace{11...1}_{12-times}$ using Proposition 4.6 , we insert a new row $\underbrace{11...1}_{13-times}$ as a first row and a new column $\underbrace{10...0}_{13-times}$ as a first column. We obtain the following matrix:

$$B^{'} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The binary block-code $W = \{w_1, ..., w_{13}\}$, whose codewords are the rows of the matrix $B^{'}$, determines an MV-algebra $(X, \oplus, *, w_1)$.

Let $A = \{w_7, w_8, w_9, w_{10}, w_{11}, w_{12}, w_{13}\}$ and $f : A \to X$, $f(w_i) = w_i$, $i \in \{7, 8, 9, 10, 11, 12, 13\}$ be an M-function which determines the binary block-code U=$\{111111, 111111, 110000, 110000, 101100, 100000, 100000, 100000, 010000,$ $001000, 000100, 000010, 000001\}$. The code V is a subset of the code U.

CONCLUSIONS

We first introduced the notion of M-functions and investigated their properties. Using this concept, we established block-codes. To this end, we use only algebraic properties of MV-algebra and show that a binary block-code exists without using order relation and proved that, to each binary block-code V, we can associate an MV-algebra X such that the binary block-code generated by X, V_X contains code V as a subset. They have particular properties that are discussed in this article. On the other hand, the MV -chain and Boolean algebra properties can be used to obtain a non-duplicate binary block-code. In this paper, an algorithm for construction binary block-codes based on Fibonacci sequences is proposed and make some connections between MV-algebras and Fibonacci sequences. These connections will also help us to achieve new results and applications of these algebras and sequences. In future research, we will look for the answer to how to obtain Fibonacci sequences from binary block-codes.

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